Theoretical Computer Science 475 (2013) 21–33

Contents lists available at SciVerse ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

On weight function methods in Chooser–Picker games

Małgorzata Bednarska-Bzdęga *

Department of Discrete Mathematics, Adam Mickiewicz University, Poznań, Poland

A R T I C L E I N F O

Article history:
Received 16 March 2012
Received in revised form 17 November 2012
Accepted 30 December 2012
Communicated by A. Fraenkel

Keywords:
Combinatorial games
Chooser–Picker games
Picker–Chooser games
Maker–Breaker games

A B S T R A C T

We study NP-hard Chooser–Picker biased games on hypergraphs and their connections to classic Maker–Breaker games. We prove two weight-function-based winning criteria for Picker and show that the Erdős–Selfridge winning criterion for Breaker's win is also the winning criterion for Picker in $(1 : 1)$ games. Thereby we improve previous results by Beck and by Csernenszky, Mándity and Pluhár. Moreover we estimate the critical bias for Picker in Chooser–Picker $(1 : q)$ games in which the aim of Chooser is to build a copy of a fixed size graph G in K_n.

1. Introduction

Let $\mathcal{H} = (V, E)$ be a hypergraph, i.e. V is a finite set and E is a collection of subsets of V. The elements of V and E are called the vertices and edges of the hypergraph, respectively. We allow multi-edges and empty edges in $\mathcal{H}$.

We study the following class of positional games played on $\mathcal{H}$. Two players claim in turns previously unselected elements of V, until all vertices are occupied. One of the players, let us call him the builder, wins when he claims all elements in at least one edge of the hypergraph; otherwise the other player, the spoiler, is the winner. In a well-known Maker–Breaker version of the game, Maker, who is the builder, selects at most $p \geqslant 1$ elements and Breaker claims at most $q \geqslant 1$ elements per turn. Every player has to select at least one element in his move. We denote such games by $\mathcal{MB}(\mathcal{H}, p, q)_M$ and $\mathcal{MB}(\mathcal{H}, p, q)_B$, where the subscript letter indicates who starts the game.

Beck [4,5] introduced another two versions of the builder–spoiler games played on $\mathcal{H}$, Picker–Chooser and Chooser–Picker games, which we will denote by $\mathcal{PC}(\mathcal{H}, p, q)$ and $\mathcal{CP}(\mathcal{H}, p, q)$, respectively. In $\mathcal{PC}(\mathcal{H}, p, q)$ Picker is the builder and Chooser is the spoiler. At every turn Picker selects at most $p + q$ (but not less than $q + 1$) unoccupied vertices. Chooser keeps q of the vertices selected by Picker and the remaining elements go to Picker. We have a special rule for the last turn, if there are $t < p + q$ unoccupied vertices left: then Chooser takes $\min\{t, q\}$ of them and the remaining elements (if any) belong to Picker. In $\mathcal{CP}(\mathcal{H}, p, q)$ Chooser is the builder, Picker is the spoiler and the rules are analogous to these in the previous game. At every turn Picker selects at most $p + q$ (but not less than $p + 1$) unoccupied vertices. Chooser takes p of the vertices offered him by Picker and the remaining elements go to Picker.

Let us emphasize that in $\mathcal{MB}(\mathcal{H}, p, q)_M$, $\mathcal{MB}(\mathcal{H}, p, q)_B$, $\mathcal{PC}(\mathcal{H}, p, q)$, $\mathcal{CP}(\mathcal{H}, p, q)$ and in the phrase "a $(p : q)$ game" the first number always refers to the number of vertices selected by the builder.

While studying Maker–Breaker games, the authors usually consider them in *the strict version*: it is assumed that Maker and Breaker claim *exactly* p and q elements, respectively. It is natural since the players can take no advantage of selecting less elements than the maximum allowed. Throughout the paper we pay attention to this assumption every time it may influence the results.

* Tel.: +48 618295389.
E-mail address: mbed@amu.edu.pl.

0304-3975/$ – see front matter
doi:10.1016/j.tcs.2012.12.037

It is well known that Maker–Breaker games in the strict version, as well as $\mathcal{MB}(\mathcal{H}, p, q)_M$ and $\mathcal{MB}(\mathcal{H}, p, q)_B$, possess two nice monotonicity properties: with respect to subhypergraphs of $\mathcal{H}$ and with respect to the bias q (or p). More precisely, if Breaker wins for instance $\mathcal{MB}(\mathcal{H}, p, q)_M$, then he wins $\mathcal{MB}(\mathcal{H}', p, q)_M$ for every subhypergraph $\mathcal{H}' \subseteq \mathcal{H}$. Also, if Breaker can win $\mathcal{MB}(\mathcal{H}, p, q)_M$, then he can win $\mathcal{MB}(\mathcal{H}, p, q')_M$ for every $q' \geqslant q$.

On the contrary, in Chooser–Picker and Picker–Chooser games, the assumption that at every turn Picker can offer less than $p + q$ vertices is quite important. Otherwise it might have happened that $\mathcal{CP}(\mathcal{H}, 1, 1)$ is won for Picker, while Picker loses in $\mathcal{CP}(\mathcal{H}, 1, q)$ with $q > 1$; consider for example $\mathcal{H}$ consisting of $q/2 + 1$ disjoint edges, each of size 2, where q is even.

The above assumption guarantees the subhypergraph monotonicity as well. One can check that for $q = 1$, if Picker has a winning strategy in $\mathcal{CP}(\mathcal{H}, p, q)$, then he can win $\mathcal{CP}(\mathcal{H}', p, q)$ for every subhypergraph $\mathcal{H}' \subseteq \mathcal{H}$ (see [10] for a rigorous proof). The same holds for $q > 1$, provided we allow Picker to select less than $p + q$ vertices per turn.

Beck observed that sometimes the results of Maker–Breaker, Picker–Chooser, and Chooser–Picker (1 : 1) games surprisingly resonate. As an illustration, consider the following clique game, described in graph language. Suppose that the players select edges of the complete graph K_N and we ask about the greatest n such that the builder can create K_n in K_N. A remarkable result of Beck is that in the Maker–Breaker and Chooser–Picker versions the greatest such n equals $\lfloor 2\log_2 N - 2\log_2\log_2 N + 2\log_2 e - 3 + o(1)\rfloor$, and in the Picker–Chooser version the corresponding number is $\lfloor 2\log_2 N - 2\log_2\log_2 N + 2\log_2 e - 1 + o(1)\rfloor$. The proofs are long and difficult and we refer the reader to [5] for details. Observe that the latest number is also the clique number of the random graph $G(n, 1/2)$. Nonetheless, the interesting connection between random graphs and positional games is not the subject of this note.

Beck remarked [4,5] that seemingly Picker in $\mathcal{PC}(\mathcal{H}, 1, 1)$ has an easier job than Maker in $\mathcal{MB}(\mathcal{H}, 1, 1)_B$. Similarly, it is easier to win as Picker in $\mathcal{CP}(\mathcal{H}, 1, 1)$ than as Breaker in $\mathcal{MB}(\mathcal{H}, 1, 1)_M$. This remark has been formalized in [10] as the following conjecture.

Conjecture. *If Breaker wins $\mathcal{MB}(\mathcal{H}, 1, 1)_M$, then the spoiler Picker wins $\mathcal{CP}(\mathcal{H}, 1, 1)$. If Maker wins $\mathcal{MB}(\mathcal{H}, 1, 1)_B$, then the builder Picker wins $\mathcal{PC}(\mathcal{H}, 1, 1)$.*

In fact both parts of the conjecture are equivalent since every game $\mathcal{PC}(\mathcal{H}, p, q)$ is equivalent to the game $\mathcal{CP}(\mathcal{H}^*, q, p)$ played on the so-called transversal hypergraph [10].

Recently the above conjecture has been disproved by Knox [16] by constructing a 3-uniform hypergraph $\mathcal{H}$ on 15 vertices such that Breaker wins $\mathcal{MB}(\mathcal{H}, 1, 1)_M$ but the spoiler Picker loses $\mathcal{CP}(\mathcal{H}, 1, 1)$. Though the conjecture is not true in general, it holds, as we will prove in this paper, for a class of games in which winning strategies of Breaker (or Maker) are determined by natural weight functions.

The weight function method is a standard tool in combinatorial game theory. Since the problem of deciding who has, a winning strategy in Picker–Chooser and Chooser–Picker games is NP-hard [11], and the analogous problem for Maker–Breaker games is PSPACE-complete [18], it is natural to look for efficient winning criteria at least for some classes of the hypergraphs $\mathcal{H}$. A well-known Erdős-Selfridge theorem [12] says that if

$$w(\mathcal{H}) = \sum_{A \in E(\mathcal{H})} 2^{-|A|} < \frac{1}{2}, \tag{1}$$

then Breaker has an explicit winning strategy in $\mathcal{MB}(\mathcal{H}, 1, 1)_M$. This result was generalized by Beck [2], who proved that the inequality

$$w(\mathcal{H}) = \sum_{A \in E(\mathcal{H})} (q+1)^{-|A|/p} < \frac{1}{q+1} \tag{2}$$

is a winning criterion for Breaker in $\mathcal{MB}(\mathcal{H}, p, q)_M$. As noted by Beck [5], this is also a winning criterion for the spoiler Chooser in $\mathcal{PC}(\mathcal{H}, p, q)$.

In [5] a reader can find other winning criteria for Picker–Chooser and Chooser–Picker games. They proved useful in studying the clique game, the Van der Waerden game, discrepancy games and games played on the lattice grid [5]. Results in that direction were obtained also by Csernenszky [9] and Csernenszky, Mándity and Pluhár [10]. Let us mention two winning criteria for the spoiler Picker in biased Chooser–Picker games.

The following one was proved by Csernenszky.

Theorem 1.1 (*[9]*). *If*

$$w(\mathcal{H}) = \sum_{A \in E(\mathcal{H})} 2^{-|A|/p} < \frac{p+1}{|V(\mathcal{H})|},$$

then Picker has a winning strategy in $\mathcal{CP}(\mathcal{H}, p, 1)$.

The next criterion, due to Beck, applies to games in which Picker's bias is larger than 1. Here *the rank* of the hypergraph $\mathcal{H}$ is $\operatorname{rank}(\mathcal{H}) = \max_{A \in E(\mathcal{H})} |A|$.

Theorem 1.2 (*[5], Thm. 47.1*). *Let $q \geqslant 2$ and $\mathcal{H}$ be a hypergraph of rank a_0. Suppose that there exists a natural number t, with $1 \leqslant t \leqslant a_0$, such that*

$$w(\mathcal{H}) = \sum_{A \in E(\mathcal{H})} (q+1)^{-|A|+t} \leqslant 1.$$

Moreover assume that for every $X \subseteq V(\mathcal{H})$ such that $|V(\mathcal{H}) \setminus X| \leqslant q^{24} a_0^{12}$ there are at most $2^{t-3}/(t+1)$ different edges in the set $\{A \setminus X : A \in E(\mathcal{H})\}$. Then Picker has a winning strategy in $\mathcal{CP}(\mathcal{H}, 1, q)$.

In the proofs of all presented above winning criteria the winning strategy of the spoiler is based, roughly speaking, on minimizing the weight function w. The weight w is defined on the set of all positions which may arise in the course of the play, and at every turn a player makes a move leading to one of the positions with minimal possible weight. To be more precise, we introduce some notation.

If $X \subseteq V(\mathcal{H})$, then $\mathcal{H} \setminus X$ denotes the hypergraph with the vertex-set $V(\mathcal{H}) \setminus X$ and the edge-multiset $\{A \setminus X : A \in E(\mathcal{H})\}$. By $\mathcal{H} - X$ we denote the hypergraph with the vertex-set $V(\mathcal{H}) \setminus X$ and the edge-multiset $\{A \in E(\mathcal{H}) : A \cap X = \emptyset\}$. We write $\mathcal{H} - x$ and $\mathcal{H} \setminus x$ instead of $\mathcal{H} - \{x\}$ and $\mathcal{H} \setminus \{x\}$, respectively. Suppose that at a given moment of a game played on a hypergraph $\mathcal{H}$, X is the set of all vertices the builder has selected so far and Y is the set of vertices which belong to the spoiler. Then we call the hypergraph $(\mathcal{H} \setminus X) - Y$ *the position* at the given moment.

Let us consider a Maker–Breaker $(1 : q)$ game played on a hypergraph $\mathcal{H}$. Given a real-valued function w, the weight, defined on the class of all hypergraphs with vertex-sets contained in $V(\mathcal{H})$, we say that Maker in a Maker–Breaker $(1 : q)$ game uses *a max-weight strategy*, if at every turn, for the resulting position $\mathcal{H}'$, he selects a vertex for which $w(\mathcal{H}' \setminus x)$ is maximal over all $x \in V(\mathcal{H}')$. Similarly, by *a min-weight strategy* of Breaker in a Maker–Breaker $(p : 1)$ game we mean selecting in every position $\mathcal{H}'$ a vertex for which $w(\mathcal{H}' - x)$ is minimal over all $x \in V(\mathcal{H}')$.

It turns out that if the game $\mathcal{MB}(\mathcal{H}, p, 1)_M$ satisfies criterion (2), then Breaker has a winning min-weight strategy, which can be easily transformed into a winning strategy of the spoiler Chooser in $\mathcal{PC}(\mathcal{H}, p, 1)$. The only difference between the strategies of Breaker and Chooser is that Chooser keeps the vertex for which the weight $w(\mathcal{H}' - x)$ is minimal over all vertices offered him by Picker. Similarly, we have winning criteria for Maker in $\mathcal{MB}(\mathcal{H}, 1, q)_M$ which guarantee the existence of his max-weight winning strategy (see [5]), and based on this strategy we can obtain a winning strategy of the builder Chooser in $\mathcal{CP}(\mathcal{H}, 1, q)$. It is an interesting problem to find a simple rule which relates the existence of a winning weight-function-based strategy of Maker or Breaker with a winning strategy of the corresponding Picker. The theorem below, which is our main result, establishes a relationship of this kind, for a restricted but still wide class of games. We need the following definition.

Given a set V_0 and a weight function w on the class of all hypergraphs with vertex-sets contained in V_0, we say that w *has property (GW)* if for every hypergraph $\mathcal{H}$ with $V(\mathcal{H}) \subseteq V_0$ and distinct $x_1, \ldots, x_n \in V(\mathcal{H})$, the following condition is satisfied:

$$\sum_{i=1}^{n} w((\mathcal{H} \setminus x_i) - x_{i+1}) = \sum_{i=1}^{n} w((\mathcal{H} \setminus x_{i+1}) - x_i), \tag{GW}$$

where $x_{n+1} = x_1$. Throughout the paper, when considering a game on a hypergraph $\mathcal{H}$, by default we assume that every weigh function we talk about is defined on the class of all hypergraphs with vertex-sets contained in $V(\mathcal{H})$.

Theorem 1.3. *Suppose that Breaker has a winning min-weight strategy in $\mathcal{MB}(\mathcal{H}, 2p-1, 1)_M$, for some weight w which has property (GW). Then the spoiler Picker has an explicit winning strategy in $\mathcal{CP}(\mathcal{H}, p, 1)$.*

Suppose that Maker has a winning max-weight strategy in $\mathcal{MB}(\mathcal{H}, 1, 2q-1)_B$, for some weight w which has property (GW). Then the builder Picker has an explicit winning strategy in $\mathcal{PC}(\mathcal{H}, 1, q)$.

The assumption that the weight function satisfies (GW) is not so restrictive as it may seem. For instance the Erdős-Selfridge weight function (1) has property (GW). As a consequence, we will obtain the following perfect analog of the Erdős-Selfridge winning criterion (1).

Corollary 1.4. *If*

$$\sum_{A \in E(\mathcal{H})} 2^{-|A|} < \frac{1}{2},$$

then the spoiler Picker has a winning strategy in $\mathcal{CP}(\mathcal{H}, 1, 1)$.

The above corollary improves previous bounds: $1/(8\mathrm{rank}(\mathcal{H}) + 1)$ obtained by Beck [4] and $1/(3(\mathrm{rank}(\mathcal{H}) + 1/2)^{1/2})$ proved by Csernenszky, Mándity and Pluhár [10]. The bound $1/2$ is optimal, which we discuss in Section 6.

Unfortunately, with Theorem 1.3 we are only partially successful in adapting Beck's winning criterion (2) to the Chooser–Picker purpose. As we will show, the result of Beck, together with Theorem 1.3, gives the following Picker's winning criterion.

Corollary 1.5. *If*

$$\sum_{A \in E(\mathcal{H})} 2^{-|A|/(2p-1)} < \frac{1}{2},$$

then Picker has an explicit winning strategy in $\mathcal{CP}(\mathcal{H}, p, 1)$.

If we compare it with Theorem 1.1, we see that none of the two criteria is stronger than the other. Corollary 1.5 is stronger for example in the case of uniform hypergraphs on n vertices with rank of order $O(\ln n)$, provided p is fixed.

The second main result of our paper is the following winning criterion, for Picker in $\mathcal{CP}(\mathcal{H}, 1, q)$.

Theorem 1.6. *Let $q_0 \geqslant 2$ and $\mathcal{H}$ be a hypergraph for which*

$$\sum_{A \in E(\mathcal{H})} (q_0 + 1)^{-|A|} < \frac{1}{2(q_0 + 1)}.$$

Then Picker (the spoiler) has a winning strategy in $\mathcal{CP}(\mathcal{H}, 1, q)$ for every $q \geqslant 100 a_0 q_0 \ln(a_0 q_0)$, where $a_0 = \mathrm{rank}(\mathcal{H})$.

We are going to use Theorem 1.6 in Chooser–Picker $(1 : q)$ games in which the rank of the hypergraph is fixed and q tends to infinity with $|V(\mathcal{H})|$. On the contrary, Theorem 1.2 is more useful in games played on hypergraphs $\mathcal{H}$ with rank tending to infinity with $|V(\mathcal{H})|$. For example Theorem 1.2 was applied by Beck to study asymptotic aspects of the Chooser–Picker clique game in biased $(1 : q)$ version, with q fixed [5].

The problem of estimating the greatest q such that the builder has a winning strategy in a $(1 : q)$ game is one of the most extensively studied problems of Maker–Breaker games. There are several papers ([3,6,8,13,14,17], to mention just a few) which study this problem for games played on the complete graph K_n or, more precisely, games $\mathcal{MB}(\mathcal{H}, 1, q)_M$ and $\mathcal{MB}(\mathcal{H}, 1, q)_B$ in which $V(\mathcal{H}) = E(K_n)$ and $E(\mathcal{H})$ corresponds to some family of subgraphs of K_n. We are not going to consider in detail the same problem for Chooser–Picker games, but we would like to illustrate the weight function methods by estimating the spoiler "threshold bias" of games in which the builder wants to create a copy of a small graph G in K_n. We consider such games in the last section.

Our paper is organized in the following way. We prove Theorem 1.3 in Section 3 and Theorem 1.6 in Section 4. In Section 5 we formulate a winning criterion for Chooser in $\mathcal{CP}(\mathcal{H}, 1, q)$, used later in Section 6. The last section contains some applications of Theorems 1.3 and 1.6.

2. Preliminaries

Apart from the definitions formulated in the Introduction, we will use the following notation.

For a hypergraph $\mathcal{H}$ and $X \subseteq V(\mathcal{H})$, we denote by $\mathcal{H}_X$ the subhypergraph of $\mathcal{H}$, induced by all edges $A \in E(\mathcal{H})$ such that $X \subseteq A$. For simplicity we write $\mathcal{H}_{x_1, x_2, \ldots, x_t}$ instead of $\mathcal{H}_{\{x_1, x_2, \ldots, x_t\}}$.

For a natural number s and a multiset X let $\binom{X}{s}$ be the family of all s-element subsets of X. We assume that in an s-element subset of X an element can appear with multiplicity not greater than in X. Given a hypergraph $\mathcal{H}$, we denote by $\mathcal{H}_2^s$ the hypergraph with the vertex-set $V(\mathcal{H})$ and the following edge-multiset:

$$E(\mathcal{H}_2^s) = \left\{ \bigcup_{i=1}^{s} A_i : \{A_1, \ldots, A_s\} \in \binom{E(\mathcal{H})}{s} \text{ and } \left| \bigcap_{i=1}^{s} A_i \right| \geqslant 2 \right\}.$$

Even if all $A_1, \ldots, A_s$ are distinct, the multiple edges in $E(\mathcal{H}_2^s)$ are possible: for example if $\bigcup_{i=1}^{s} A_i = \bigcup_{i=1}^{s} A'_i$ for two different sets $\{A_1, \ldots, A_s\}$ and $\{A'_1, \ldots, A'_s\}$, then we have two copies of the edge $\bigcup_{i=1}^{s} A_i$ in $E(\mathcal{H}_2^s)$.

Let us recall that by a position at a moment of a game played on a hypergraph $\mathcal{H}$ we mean the hypergraph $(\mathcal{H} \setminus X) - Y$, where X is the set of all vertices the builder has selected so far and Y is the set of vertices which belong to the spoiler. By $\mathcal{H}(i)$ we denote the position after the i-th turn and put $\mathcal{H}(0) = \mathcal{H}$. Clearly, if a player uses his winning strategy in a game played on $\mathcal{H}$ and we assume that after the i-th turn the new game, played on $\mathcal{H}(i)$, starts, then the player has a winning strategy in this new game as well. Moreover, if $\mathcal{H}(i)$ is a hypergraph after the last turn of the game, then the builder wins if and only if $E(\mathcal{H}(i))$ contains an empty edge.

Consider a hypergraph $\mathcal{H}$ and a weight function w. We say that a vertex $x \in V(\mathcal{H})$ is *Breaker's w-best response* in $\mathcal{H}$ if

$$w(\mathcal{H} - x) \leqslant w(\mathcal{H} - x') \quad \text{for all } x' \in V(\mathcal{H}).$$

A vertex $x \in V(\mathcal{H})$ is *Maker's w-best response* in $\mathcal{H}$ if

$$w(\mathcal{H} \setminus x) \geqslant w(\mathcal{H} \setminus x') \quad \text{for all } x' \in V(\mathcal{H}).$$

By *Maker's w-best response to y* in $\mathcal{H}$ ($y \in V(\mathcal{H})$) we mean any vertex which is Maker's best response in $\mathcal{H} - y$. A vertex is *Breaker's w-best response to x* in $\mathcal{H}$ if it is Breaker's best response in $\mathcal{H} \setminus x$. Thus, if x is Maker's w-best response to y in $\mathcal{H}$, then for every $x' \neq y$

$$w((\mathcal{H} \setminus x') - y) \leqslant w((\mathcal{H} \setminus x) - y),$$

while if y is Breaker's w-best response to x in $\mathcal{H}$, then for every $y' \neq x$

$$w((\mathcal{H} \setminus x) - y') \geqslant w((\mathcal{H} \setminus x) - y).$$

We will use the above best response definitions for hypergraphs $\mathcal{H}$ in general, not only in Maker–Breaker games.

Notice that, given a weight function w, Maker in a Maker–Breaker $(1 : q)$ game uses a max-weight strategy if and only if at every turn, for the resulting position $\mathcal{H}'$, he selects Maker's best response in $\mathcal{H}'$. Similarly, Breaker in a Maker–Breaker $(p : 1)$ game carries out his min-weight strategy if and only if in every position $\mathcal{H}'$ he selects Breaker's best response in $\mathcal{H}'$.

Usually in applications $w(\mathcal{H})$ is the sum of some weights of edges of the hypergraph $\mathcal{H}$ and these edge-weights depend linearly or exponentially on the size of an edge. We define the weight functions T_δ of the latter kind, with $\delta > 0$, as follows:

$$T_\delta(\mathcal{H}) = \sum_{A \in E(\mathcal{H})} \delta^{|A|}.$$

If $E(\mathcal{H}) = \emptyset$, then $T_\delta(\mathcal{H}) = 0$, so the final position of a game is won by the builder if and only if its weight T_δ is positive.

Fact 2.1. *For every $\delta > 0$ the weight function T_δ has property (GW).*

Proof. Indeed, we have

$$\begin{aligned} T_\delta((\mathcal{H} \setminus x_i) - x_{i+1}) - T_\delta((\mathcal{H} \setminus x_{i+1}) - x_i) &= \delta^{-1}(T_\delta(\mathcal{H}_{x_i}) - T_\delta(\mathcal{H}_{x_i,x_{i+1}})) - T_\delta(\mathcal{H}_{x_{i+1}}) \\ &\quad -\delta^{-1}(T_\delta(\mathcal{H}_{x_{i+1}}) - T_\delta(\mathcal{H}_{x_i,x_{i+1}})) + T_\delta(\mathcal{H}_{x_i}) \\ &= (\delta^{-1} + 1)(T_\delta(\mathcal{H}_{x_i}) - T_\delta(\mathcal{H}_{x_{i+1}})), \end{aligned}$$

and summation over i leads to the conclusion that T_δ has property (GW). $\square$

Throughout the paper we will utilize weight functions T_δ very often, as well as the fact that they satisfy (GW). As the reader may have noticed, in almost every theorem mentioned in the Introduction, a function T_δ is involved.

3. Proof of Theorem 1.3

We begin by the key lemma.

Lemma 3.1. *Suppose that a weight function w has property (GW). Then for every hypergraph $\mathcal{H}$ with at least two vertices there exist distinct vertices $v, v' \in V(\mathcal{H})$ such that they are Breaker's w-best responses to each other. Analogously, we can find two vertices which are Maker's w-best responses to each other.*

Proof. We will prove only the first part of the lemma since an analogous argument applies to the second one.

Given a hypergraph $\mathcal{H}$, let us define an auxiliary directed graph G. The vertex-set of G is $V(\mathcal{H})$, and (v, v') is an edge of G if and only if $v \neq v'$ and v' is Breaker's w-best response to v in $\mathcal{H}$. Then every vertex in G has the non-zero out-degree so for some $k \geqslant 2$ there exists an oriented cycle $v_1 v_2 \ldots v_k$ in G. By the definition of a w-best response, the following k inequalities are true, with $v_0 = v_k$ and $v_{k+1} = v_1$:

$$w((\mathcal{H} \setminus v_i) - v_{i+1}) \leqslant w\left((\mathcal{H} \setminus v_i) - v_{i-1}\right) \quad \text{for } i = 1, 2, \ldots, k.$$

We sum them up and, in view of (GW), we conclude that all of the inequalities must be equalities. Thus, for all $i = 1, 2, \ldots, k$, the vertices v_i and v_{i+1} are Breaker's w-best responses to each other. $\square$

Proof of Theorem 1.3. To prove the first part of the theorem, consider a game $\mathcal{CP}(\mathcal{H}, p, 1)$ and a weight function w which satisfies (GW).

We assume that the spoiler Picker records the history of the game by creating a sequence (a) of vertices occupied in the game: before the first move the sequence (a) is empty and after each turn Picker adds at the end of (a) all $p+1$ vertices selected at this turn, in the order we will define later.

The key realization is that we can consider a sequence (a) also as a history of the game $\mathcal{MB}(\mathcal{H}, p', 1)_M$, where Chooser's elements of (a) correspond to Maker's choices in $\mathcal{MB}(\mathcal{H}, p', 1)_M$, Picker's elements of (a) correspond to vertices selected by Breaker, and p' is the maximal length of a subsequence of consecutive Chooser's elements in (a). Here we use the assumption that Maker in $\mathcal{MB}(\mathcal{H}, p', 1)_M$ is allowed to select less than p' vertices per turn. Clearly, if at the end of the Maker–Breaker game, played according to the history (a), no edge of $\mathcal{H}$ is occupied entirely by Maker, then also the builder Chooser loses the Chooser–Picker game played according to (a).

We will show that at every turn of $\mathcal{CP}(\mathcal{H}, p, 1)$ Picker can select the vertices so that no matter which ones go to Chooser, the remaining element (which goes to Picker) will form Breaker's w-best response to one of the elements of Chooser.

Now let us describe the strategy of Picker and define the order of the vertices in (a). Suppose that $\mathcal{H}'$ is the position after some turn (at the start $\mathcal{H}' = \mathcal{H}$). Picker selects vertices $v_1, v_2, \ldots, v_{p+1}$ recursively: v_1 and v_2 are any two Breaker's w-best responses to each other in $\mathcal{H}'$. Notice that such two elements exist by Lemma 3.1, since w satisfies (GW). For $i > 2$,

v_i is Breaker's w-best response in the hypergraph $\mathcal{H}' \setminus \{v_1, \ldots, v_{i-1}\}$. Then, if Chooser decides that the vertex v_1 goes to Picker, Picker adds the vertices to the sequence (a) in the order: $v_2, v_1, v_3, \ldots, v_{p+1}$; otherwise he adds them in the order: $v_1, v_2, \ldots, v_{p+1}$.

Suppose that at the end of the game $(a) = (a_1, a_2, \ldots, a_t)$ and $a_{k_1}, a_{k_2}, \ldots, a_{k_j}$ is the subsequence of all Picker's vertices in (a). By a straightforward analysis of the strategy of Picker we conclude that $k_1 \geqslant 2$ and a_{k_1} is Breaker's best w-response in $\mathcal{H} \setminus \{a_1, \ldots, a_{k_1-1}\}$. Similarly, for $i = 1, 2, \ldots, j-1$, if $A = \{a_{k_1}, a_{k_2}, \ldots, a_{k_i}\}$ and A' is the set of Chooser vertices in the sequence $a_1, a_2, \ldots, a_{k_i}$, then $a_{k_{i+1}}$ is Breaker's best w-response in the hypergraph $(\mathcal{H} \setminus A') - A$. Finally, it is not difficult to check that the maximal length of a subsequence of consecutive Chooser's elements in (a) is not greater than $2p - 1$.

Thereby we can view a sequence (a) as a history of the game $\mathcal{MB}(\mathcal{H}, 2p-1, 1)_M$, in which Maker starts by selecting $a_1, a_2, \ldots, a_{k_1-1}$ and then for $i = 1, 2, \ldots, j$, at the i-th turn, in a position $\mathcal{H}'$, Breaker selects Breaker's best response a_{k_i} in $\mathcal{H}'$, followed by the choice of $\{a_m : k_i < m < k_{i+1}\}$ by Maker in the next turn (we put $k_{j+1} = t + 1$). Thus Breaker in $\mathcal{MB}(\mathcal{H}, 2p-1, 1)_M$ carries out his min-weight strategy. By the assumption of our theorem this is his winning strategy so Maker cannot win this game. It implies that also Chooser in $\mathcal{CP}(\mathcal{H}, p, 1)$ cannot win, which completes the proof of the first part of the thesis.

The argument for the second part is analogous, with the only difference that the builder Picker in $\mathcal{PC}(\mathcal{H}, 1, q)$ creates the sequence (a) so that his every element in (a) is Maker's w-best response to the preceding (Chooser's) element, and we treat (a) as a history of the game $\mathcal{MB}(\mathcal{H}, 1, 2q-1)_B$. $\square$

4. Proof of Theorem 1.6

The idea of the proof is based on the technique used by Beck for proving Theorem 1.2. We analyze the game in stages, depending on the size of the position. If it has many edges, we use the argument by Beck, stated below as a lemma. When the hypergraph is smaller, our argument is different.

Lemma 4.1 *([5], proof of Thm. 47.1).* *Let $q_0 \geqslant 2$, $\mathcal{H}$ be a hypergraph of rank a_0 and let $T(\mathcal{H}) = T_{1/(q_0+1)}(\mathcal{H}) = \sum_{A \in E(\mathcal{H})} (q_0+1)^{-|A|}$. Then Picker in $\mathcal{CP}(\mathcal{H}, 1, q_0)$ has a strategy such that if after the i-th turn $|V(\mathcal{H}(i))| > a_0^{12} q_0^{24}$, then*

$$T(\mathcal{H}(i+1)) < T(\mathcal{H}) \exp\left(\frac{8}{|V(\mathcal{H}(i))|^{1/8}}\right).$$

Proof of Theorem 1.6. Let $\mathcal{H}$ be a hypergraph of rank a_0, $q_0 \geqslant 2$, $T = T_{1/(q_0+1)}$ and $q > 2q_0$. The size of the position $\mathcal{H}(i)$ is, obviously, a monotonically decreasing function of i and it is natural to divide the game $\mathcal{CP}(\mathcal{H}, 1, q)$ into consecutive stages depending on $|V(\mathcal{H}(i))|$. Not all the stages must appear during the game but it does not influence our argument.

Stage 1: $|V(\mathcal{H}(i))| > a_0^{12} q_0^{40}$.

As long as the above condition is satisfied, the spoiler Picker selects $q_0 + 1$ vertices of $\mathcal{H}(i)$, according to Picker's strategy in $\mathcal{CP}(\mathcal{H}, 1, q_0)$, described in Lemma 4.1. Note that in Stage 1 Picker selects less than $q + 1$ vertices per turn and it is permissible by the rules of the game $\mathcal{CP}(\mathcal{H}, 1, q)$.

Let t_1 be the last turn of Stage 1, i.e.

$$|V(\mathcal{H}(t_1 - 1))| > a_0^{12} q_0^{40} \quad \text{and} \quad |V(\mathcal{H}(t_1))| \leqslant a_0^{12} q_0^{40}.$$

(If $|V(\mathcal{H})| < a_0^{12} q_0^{40}$, then we assume that $t_1 = 0$.) Then, by Lemma 4.1 and the assumption that $q_0 \geqslant 2$,

$$T(\mathcal{H}(t_1)) < T(\mathcal{H}) \exp\left(\frac{8}{(a_0^{12} q_0^{40})^{1/8}}\right) \leqslant e^{1/4} T(\mathcal{H}). \tag{3}$$

Stage 2: $2q + 1 \leqslant |V(\mathcal{H}(i))| \leqslant a_0^{12} q_0^{40}$.

In every position for which the above condition holds, Picker selects consecutively $q + 1$ vertices $x_1, x_2, \ldots, x_{q+1} \in V(\mathcal{H}(i))$ for which $T(\mathcal{H}(i)_x)$ is as small as possible. Because of the minimality of $T(\mathcal{H}(i)_{x_1}), \ldots, T(\mathcal{H}(i)_{x_{q+1}})$, we have

$$T(\mathcal{H}(i)_{x_j}) \leqslant \frac{1}{|V(\mathcal{H}(i))| - q} \sum_{x \in V(\mathcal{H}(i))} T(\mathcal{H}(i)_x) \leqslant \frac{a_0 T(\mathcal{H}(i))}{|V(\mathcal{H}(i))| - q}$$

for every $j \leqslant q + 1$.

Now Chooser chooses $x_j \in \{x_1, \ldots, x_{q+1}\}$ and we see that the function T cannot increase by more than

$$(q_0 + 1) T(\mathcal{H}(i)_{x_j}) - T(\mathcal{H}(i)_{x_j}) = q_0 T(\mathcal{H}(i)_{x_j}) \leqslant \frac{q_0 a_0}{|V(\mathcal{H}(i))| - q} T(\mathcal{H}(i)).$$

Therefore, if t_2 is the last turn of Stage 2 and $m_1 = |V(\mathcal{H}(t_1))|$, the following inequalities hold:

$$\begin{aligned}
T(\mathcal{H}(t_2)) &\leqslant T(\mathcal{H}(t_1)) \prod_{i:\, 2q+1 \leqslant |V(\mathcal{H}(i))| \leqslant m_1} \left(1 + \frac{q_0 a_0}{|V(\mathcal{H}(i))| - q}\right) \\
&= T(\mathcal{H}(t_1)) \prod_{i=1}^{\lceil \frac{m_1+1}{q+1} \rceil - 1} \left(1 + \frac{q_0 a_0}{m_1 + 1 - (q+1)i}\right) \\
&< T(\mathcal{H}(t_1)) \exp\left(\frac{q_0 a_0}{q+1} \sum_{i=1}^{\lceil \frac{m_1+1}{q+1} \rceil - 1} \left(\frac{m_1+1}{q+1} - i\right)^{-1}\right) \\
&< T(\mathcal{H}(t_1)) \exp\left(\frac{q_0 a_0}{q+1} \ln \frac{m_1+1}{q+1}\right) \\
&< T(\mathcal{H}(t_1)) \left(\frac{m_1}{q}\right)^{\frac{q_0 a_0}{q}}.
\end{aligned}$$

In view of the above estimation, inequality (3), and the fact that $m_1 \leqslant a_0^{12} q_0^{40}$,

$$T(\mathcal{H}(t_2)) \leqslant \left(\frac{m_1}{q}\right)^{\frac{q_0 a_0}{q}} e^{1/4} T(\mathcal{H}) \leqslant \left(\frac{a_0^{12} q_0^{40}}{q}\right)^{\frac{q_0 a_0}{q}} e^{1/4} T(\mathcal{H}).$$

Hence, for $T(\mathcal{H}) < 1/(2(q_0 + 1))$ and $q \geqslant 100 a_0 q_0 \ln(a_0 q_0)$, simple calculations show that

$$T(\mathcal{H}(t_2)) < \frac{1}{q_0 + 1}. \tag{4}$$

Stage 3: the ending.

Notice that (4) guarantees that in $\mathcal{H}(t_2)$ there is no edge of size 0 or 1. Thus Chooser has not won yet. Picker is going to finish the game within 2 turns ($|V(\mathcal{H}(t_2))| \leqslant 2q$) so all edges of size larger than 2 are irrelevant for him and for the final result of the game. Thereby without loss of generality we can assume that $|A| = 2$ for every $A \in E(\mathcal{H}(t_2))$ and the condition (4) still holds.

Thus $|E(\mathcal{H}(t_2))| = T(\mathcal{H}(t_2))(q_0 + 1)^2 < q_0 + 1$, so $\mathcal{H}(t_2)$ consists of less than $q/2$ edges, all of size 2. At the $(t_2 + 1)$-st turn Picker selects two vertices in every edge and thereby wins the game. □

Let us add that we did not make an effort to optimize the constant 100 in Theorem 1.6, since our proof relies on the lemma by Beck, in which the constants are not optimal as well. In our applications of Theorem 1.6 this constant is not relevant.

5. Advanced building criterion

Given a hypergraph $\mathcal{H}$ of small rank, we are going to find, based on Theorem 1.6, an upper bound for q such that Picker wins $\mathcal{CP}(\mathcal{H}, 1, q)$. On the other hand, we would be interested to what extent the obtained results could be improved. Thus we need also a winning criterion for Chooser in $\mathcal{CP}(\mathcal{H}, 1, q)$.

For Maker–Breaker games we have the so-called Advanced Building Criterion, due to Beck. Before we present it, we need an additional definition.

Fix $\lambda > 0$ and a natural number $s \geqslant 2$. For a given weight function T we define a corresponding weight function

$$L_\lambda^s(\mathcal{H}) = T(\mathcal{H}) - \lambda T(\mathcal{H}_2^s) \quad \text{for every hypergraph } \mathcal{H}.$$

Let us recall that $\mathcal{H}_2^s$ is the hypergraph with the vertex-set $V(\mathcal{H})$ and the edge-multiset

$$E(\mathcal{H}_2^s) = \left\{ \bigcup_{i=1}^{s} A_i : \{A_1, \ldots, A_s\} \in \binom{E(\mathcal{H})}{s} \text{ and } \left|\bigcap_{i=1}^{s} A_i\right| \geqslant 2 \right\}.$$

Here is the Advanced Building Criterion for Maker–Breaker $(1 : 1)$ games.

Theorem 5.1 ([5], Thm. 24.2)**.** *Consider a game $\mathcal{MB}(\mathcal{H}, 1, 1)_M$ with the weight function $T = T_{1/2}$ and suppose that for some $s \geqslant 2$*

$$T(\mathcal{H}) > |V(\mathcal{H})| s \left(1 + 4T(\mathcal{H}_2^s)^{1/s}\right).$$

Then Maker has a winning strategy.

As noted by Beck [5], this is also a winning criterion for Chooser in $\mathcal{CP}(\mathcal{H}, 1, 1)$. We have no such criterion for biased games $\mathcal{MB}(\mathcal{H}, 1, q)_M$ and thus no immediate result for $\mathcal{CP}(\mathcal{H}, 1, q)$. Fortunately, the techniques in [5] allow us to solve partially this problem. Let us start with the following two lemmata.

Lemma 5.2 (*[5], Thm. 24.1*)**.** *Let $\delta \in (0, 1)$ and $T = T_\delta$. Then for every $s \geqslant 2$ and two distinct vertices x_1, x_2 of the hypergraph $\mathcal{H}$ we have*

$$T(\mathcal{H}_{x_1,x_2}) \leqslant s\left(1 + T(\mathcal{H}_2^s)^{1/s}\right).$$

Lemma 5.3 (*[5], Proof of Thm. 33.4*)**.** *Let $s \geqslant 2$ be a natural number and let $\lambda > 0$. Consider a game $\mathcal{CP}(\mathcal{H}, 1, q)$ with the weight function $T = T_{1/(q+1)}$ and the corresponding function $L_\lambda = L_\lambda^s$. If at turn $i + 1$ ($i \geqslant 0$) Picker offers $q + 1$ vertices $x_1, \dots, x_{q+1}$ to Chooser, then Chooser can take one of them so that*

$$L_\lambda(\mathcal{H}(i+1)) - L_\lambda(\mathcal{H}(i)) \geqslant - \sum_{1 \leqslant j < k \leqslant q+1} T(\mathcal{H}(i)_{x_j,x_k}).$$

In future considerations we will make some, artificial at first glance, assumption that at any turn Chooser can refuse accepting any of the vertices offered him by Picker. It is important only for evaluating the weight function during the game and in a real play can be carried out without breaking the rule of accepting a vertex by Chooser (Chooser, while evaluating the weight function, can simply "forget" about some of his vertices). The reason for such an assumption is explained in the remark after the following corollary.

Corollary 5.4. *Let T, L_λ be the weight functions as in the previous lemma. If at turn $i + 1$ ($i \geqslant 0$) of a game $\mathcal{CP}(\mathcal{H}, 1, q)$ Picker offers b vertices $x_1, \dots, x_b$ ($b \leqslant q + 1$), then Chooser can make a choice after which*

$$L_\lambda(\mathcal{H}(i+1)) - L_\lambda(\mathcal{H}(i)) \geqslant -\binom{b}{2} s\left(1 + T(\mathcal{H}(i)_2^s)^{1/s}\right). \tag{5}$$

Remark. The corollary would follow from Lemmas 5.2 and 5.3, but we must be careful if $b < q + 1$. Suppose for instance that $q \geqslant 2$ and in the first turn Picker offers two vertices x and y such that $V(\mathcal{H}_x)$ and $V(\mathcal{H}_y)$ are disjoint. Moreover assume that $L_\lambda(\mathcal{H}_x) = L_\lambda(\mathcal{H}_y) < 0$. Then, if one vertex, say x, goes to Chooser, we have

$$L_\lambda(\mathcal{H}(1)) - L_\lambda(\mathcal{H}) = qL_\lambda(\mathcal{H}_x) - L_\lambda(\mathcal{H}_y) < 0 = -T(\mathcal{H}_{x,y})$$

and we do not know how to obtain (5). In this particular case we can deal with the problem if we allow Chooser not to take any of x, y. If x and y go to Picker, then the function L_λ changes by $-L_\lambda(\mathcal{H}_x) - L_\lambda(\mathcal{H}_y) \geqslant 0$ and (5) holds.

Proof of Corollary 5.4. Suppose that at turn $i + 1$ Picker offers vertices $x_1, x_2, \dots, x_b$ to Chooser. If $b = q + 1$ then the assertion is an immediate consequence of Lemmas 5.2 and 5.3.

If $b < q + 1$, we assume that Picker offers additionally isolated vertices $x_{b+1}, x_{b+2}, \dots, x_{q+1}$ (that is, we formally add $q + 1 - b$ isolated vertices to the hypergraph $\mathcal{H}$). Obviously $T(\mathcal{H}(i)_X) = 0$ and $T((\mathcal{H}(i)_2^s)_X) = 0$ for every set X containing an isolated vertex so, by Lemma 5.3, Chooser can choose one of $x_1, \dots, x_{q+1}$ such that

$$L_\lambda(\mathcal{H}(i+1)) - L_\lambda(\mathcal{H}(i)) \geqslant - \sum_{1 \leqslant j < k \leqslant q+1} T(\mathcal{H}(i)_{x_j,x_k}) = - \sum_{1 \leqslant j < k \leqslant b} T(\mathcal{H}(i)_{x_j,x_k}).$$

If a vertex chosen by Chooser is one of the (artificial) isolated vertices $x_{b+1}, \dots, x_{q+1}$, we interpret that fact as giving all vertices $x_1, \dots, x_b$ to Picker.

Finally, we apply Lemma 5.2 and receive the desired inequality. □

A straightforward consequence of the above three lemmata and the proof technique of Theorem 5.1 is the following winning criterion for Chooser in $\mathcal{CP}(\mathcal{H}, 1, q)$:

$$T(\mathcal{H}) > |V(\mathcal{H})| s \binom{q+1}{2} \left(1 + 4T(\mathcal{H}_2^s)^{1/s}\right),$$

where $T = T_{1/(q+1)}$.

However, it is not sufficient for our purpose since we are going to deal with small rank hypergraphs and large q, and in this situation the exponent of q is important. Thus we will strengthen the above inequality a little bit.

Theorem 5.5. *Consider a game $\mathcal{CP}(\mathcal{H}, 1, q)$ with the weight function $T = T_{1/(q+1)}$ and suppose that for some $s \geqslant 2$*

$$T(\mathcal{H}) > |V(\mathcal{H})| qs\left(1 + 2T(\mathcal{H}_2^s)^{1/s}\right). \tag{6}$$

Then the builder Chooser has a winning strategy.

Proof. The proof is very similar to the proof of Theorem 5.1, with slight modifications.

Let $s \geqslant 2$, $\beta_0 = 2$ and $\beta_{k+1} = \beta_k^s/2$ for $k = 0, 1, \ldots$. Notice that $\beta_k > 1$ for $k = 0, 1, \ldots$. Based on this fact, we can divide the game into the following stages.

The first stage consists of all consecutive $i \geqslant 0$ for which

$$T(\mathcal{H}(i)_2^s) \leqslant \beta_0^s\, T(\mathcal{H}_2^s).$$

It may happen that the above condition is satisfied until the end of the game; otherwise by t_1 we denote the smallest $i \geqslant 0$ violating it, which means that

$$T(\mathcal{H}(t_1)_2^s) > \beta_0^s\, T(\mathcal{H}_2^s).$$

In such a case we define γ_1 for which the following condition holds:

$$T(\mathcal{H}(t_1)_2^s) = (\gamma_1\beta_0)^s\, T(\mathcal{H}_2^s).$$

Of course $\gamma_1 > 1$.

Inductively we define the $(k+1)$-st stage for $k = 1, 2, \ldots$, provided the game has not ended yet. It consists of all consecutive $i \geqslant t_k$ for which

$$T(\mathcal{H}(i)_2^s) \leqslant \beta_k^s\, T(\mathcal{H}(t_k)_2^s).$$

By t_{k+1} we denote the next turn (if any) after the end of the $(k+1)$-st stage, i.e. the smallest $i > t_k$ such that

$$T(\mathcal{H}(i)_2^s) > \beta_k^s\, T(\mathcal{H}(t_k)_2^s).$$

Additionally we define γ_{k+1} such that the condition

$$T(\mathcal{H}(t_{k+1})_2^s) = (\gamma_{k+1}\beta_k)^s T(\mathcal{H}(t_k)_2^s)$$

is satisfied. Obviously $\gamma_{k+1} > 1$.

Let $\lambda = T(\mathcal{H})/(2T(\mathcal{H}_2^s))$. We will show that if in every stage $k+1$ $(k \geqslant 0)$ the builder considers the weight function $L_{\lambda/2^k} = L^s_{\lambda/2^k}$ and plays so that the condition (5) is satisfied, then he wins the game, provided (6) holds.

We are going to prove by induction that the following properties hold for every $k \geqslant 0$ and for every j of the $(k+1)$-st stage:

(i) $L_{\lambda/2^k}(\mathcal{H}(t_k)) \geqslant T(\mathcal{H}(t_k))/2$.
(ii) $T(\mathcal{H}(t_k)_2^s) = (\prod_{i=0}^{k-1}\beta_i \prod_{i=1}^{k}\gamma_i)^s\, T(\mathcal{H}_2^s)$.
(iii) $T(\mathcal{H}(t_k)) \geqslant \prod_{i=1}^{k}\beta_i(\prod_{i=1}^{k}\gamma_i)^s\, T(\mathcal{H})$.
(iv) $L_{\lambda/2^k}(\mathcal{H}(j+1)) > 0$ and thus $T(\mathcal{H}(j+1)) > 0$.

(For $k = 0$ we put $t_0 = 0$ and assume that the empty products equal 1.)

Checking (i)–(iii) in the base case of $k = 0$ and the induction argument to prove (i)–(iii) for $k \geqslant 1$ is fairly standard. Moreover the details can be found in [5] in the proof of Thm. 24.2 so we omit them. Let us remark that in [5] the initial condition for β_0 is different (4 instead of 2) but it is not used in (i)–(iii).

It remains to prove part (iv). Suppose that (i)–(iii) are true for some $k \geqslant 0$ and fix j with $t_k \leqslant j < t_{k+1}$. Moreover assume that in every position $\mathcal{H}(i)$ of the $(k+1)$-st stage Chooser plays according to the strategy defined in Corollary 5.4, with the weight function $L_{\lambda/2^k}$. Then, after turn $i+1$, the weight function $L_{\lambda/2^k}$ does not decrease more than by $\binom{b_i}{2}s\left(1 + T(\mathcal{H}(i)_2^s)^{1/s}\right)$, where $b_i \leqslant q+1$ is the number of vertices offered by Picker in the position $\mathcal{H}(i)$. Hence

$$L_{\lambda/2^k}(\mathcal{H}(j+1)) \geqslant L_{\lambda/2^k}(\mathcal{H}(t_k)) - \sum_{i=t_k}^{j}\binom{b_i}{2}s\left(1 + T(\mathcal{H}(i)_2^s)^{1/s}\right). \tag{7}$$

By (i), (iii), and the property that $\gamma_i > 1$ for every i, we obtain

$$L_{\lambda/2^k}(\mathcal{H}(t_k)) \geqslant \frac{1}{2}T(\mathcal{H}(t_k)) \geqslant \frac{1}{2}T(\mathcal{H})\prod_{i=1}^{k}\beta_i\gamma_i^s \geqslant \frac{1}{2}T(\mathcal{H})\prod_{i=1}^{k}\beta_i\gamma_i.$$

The definition of the $(k+1)$-st stage and property (ii) imply that

$$T(\mathcal{H}(i)_2^s) \leqslant \beta_k^s\, T(\mathcal{H}(t_k)_2^s) \leqslant T(\mathcal{H}_2^s)\left(\beta_0\prod_{l=1}^{k}\beta_l\gamma_l\right)^s \quad \text{for } i = t_k, t_{k+1}, \ldots, j.$$

Moreover it is easy to check that if $b_i \leqslant q+1$ for $i \in J$ and $\sum_{i\in J} b_i = S$, then

$$\sum_{i\in J}\binom{b_i}{2} \leqslant \frac{S}{q+1}\binom{q+1}{2} = \frac{1}{2}Sq.$$

With the above three estimations we return to (7) and calculate that

$$\begin{aligned} L_{\lambda/2^k}(\mathcal{H}(j+1)) &\geqslant \frac{1}{2}T(\mathcal{H})\prod_{i=1}^{k}\beta_i\gamma_i - s\left(1+T(\mathcal{H}_2^s)^{1/s}\beta_0\prod_{i=1}^{k}\beta_i\gamma_i\right)\sum_{i=t_k}^{j}\binom{b_i}{2} \\ &\geqslant \frac{1}{2}T(\mathcal{H})\prod_{i=1}^{k}\beta_i\gamma_i - \frac{1}{2}|V(\mathcal{H})|qs\left(1+T(\mathcal{H}_2^s)^{1/s}\beta_0\prod_{i=1}^{k}\beta_i\gamma_i\right) \\ &\geqslant \frac{1}{2}\prod_{i=1}^{k}\beta_i\gamma_i\left[T(\mathcal{H}) - |V(\mathcal{H})|qs\left(1+\beta_0 T(\mathcal{H}_2^s)^{1/s}\right)\right]. \end{aligned}$$

Thus, by the assumption (6), for $\beta_0 = 2$ we have $L_{\lambda/2^k}(\mathcal{H}(j+1)) > 0$, which also implies that $T(\mathcal{H}(j+1)) > \frac{\lambda}{2^k}T(\mathcal{H}(j+1)_2^s) \geqslant 0$ and hence the proof of (iv) is complete.

Thereby we showed that the builder Chooser can keep the function T positive until the end, which means he wins the game. □

6. Applications

In this section we are primarily concerned with consequences of Theorems 1.3 and 1.6. We begin by proving Corollary 1.5 and showing that in case of $p = 1$ the upper bound $1/2$ in the assumption of the corollary is optimal.

Proof of Corollary 1.5. In [2] the author shows that if $\delta = (q+1)^{-1/p'}$ and $T_\delta(\mathcal{H}) = \sum_{A\in E(\mathcal{H})}\delta^{-|A|} < 1/(q+1)$, then Breaker has a winning min-weight strategy in $\mathcal{MB}(\mathcal{H}, p', q)_M$. Though the author assumes that Maker always selects exactly p' elements per turn, the proof, without any modifications, applies also to the game in which Maker can select p' or less elements. This fact, for $q = 1$ and $p' = 2p - 1$, together with Theorem 1.3, and the fact that T_δ has property (GW), implies Corollary 1.5. □

For $p = 1$ the upper bound is $1/2$ and it is optimal since $\mathcal{H}$ consisting of one single-element edge is won for Chooser and $T_{1/2}(\mathcal{H}) = 1/2$. It is not the only example and we may construct an infinite class of "extremal" hypergraphs. Let $V(\mathcal{H}_n) = \{1, 2, \ldots, 2n+1\}$, and $E(\mathcal{H}_n) = \{A_0, A_1, \ldots, A_n\}$ with $A_0 = \{1, 3, 5, \ldots, 2n+1\}, A_i = \{1, 3, 5, \ldots, 2i-1\} \cup \{2i\}$ for $i = 1, 2, \ldots, n$. Then $T_{1/2}(\mathcal{H}) = 1/2$ and it is not hard to check that the following strategy of Chooser guarantees him a win in $\mathcal{CP}(\mathcal{H}_n, 1, 1)$: at every turn he chooses the smaller number of the two offered him by Picker.

Theorem 1.6 can be applied to games in which the builder wants to create a copy of a small graph in K_n. Given a nonempty graph G with at least three vertices, by $\mathcal{H}^{G,n}$ we denote the hypergraph with the vertex-set $E(K_n)$, such that every hyper-edge consists of all edges of a copy of G in K_n. In [6] it was proved that in $\mathcal{MB}(\mathcal{H}^{G,n}, 1, q)_M$ and $\mathcal{MB}(\mathcal{H}^{G,n}, 1, q)_B$ the greatest q for which Maker can win the game is of order $\Theta(n^{1/m_2(G)})$, where $m_2(G) = \max\left\{(e(F)-1)/(v(F)-2) : \; F \subseteq G, \; v(F) \geqslant 3\right\}$. Here $v(G)$ and $e(G)$ denote the number of the vertices and edges of G, respectively. In the theorem below we estimate the corresponding threshold bias for the Chooser–Picker version. For that purpose we define two graph parameters. Let

$$m'(G) = \max_{F\subseteq G:\, v(F)\geqslant 1}\frac{e(F)-1}{v(F)} \quad \text{and} \quad m''(G) = \max_{F\subseteq G:\, v(F)\geqslant 3}\frac{e(F)+1}{v(F)-2}.$$

Theorem 6.1. *Let G be a graph with at least two edges. Then for every $\varepsilon \in (0, 1)$ there exists n_0 such that for every $n \geqslant n_0$ the following holds.*

(i) *If $q \geqslant n^{1/m'(G)+\varepsilon}$ then the spoiler Picker has a winning strategy in $\mathcal{CP}(\mathcal{H}^{G,n}, 1, q)$.*
(ii) *If $q \leqslant n^{1/m''(G)-\varepsilon}$ then the builder Chooser wins $\mathcal{CP}(\mathcal{H}^{G,n}, 1, q)$.*

Proof. Let us begin the proof of part (i) by a trivial observation. If the spoiler Picker can prevent a subgraph of G in Chooser's graph, then he prevents creating the graph G as well. Thus, without loss of generality, we can assume that the maximum $m'(G)$ is attained by G.

Observe that there are $O(n^{v(G)})$ copies of G in K_n so we can find some constant $c > 0$ (which depends on G) such that if $q_0 = \lceil cn^{v(G)/(e(G)-1)}\rceil$, then

$$\sum_{A\in E(\mathcal{H}^{G,n})}(q_0+1)^{-e(G)+1} < \frac{1}{2}.$$

Obviously rank$(\mathcal{H}^{G,n}) = e(G)$ so, by Theorem 1.6, Picker has a winning strategy in $\mathcal{CP}(\mathcal{H}^{G,n}, 1, q)$ for $q \geqslant 100e(G)q_0 \ln(e(G)q_0) = O(n^{v(G)/(e(G)-1)}\ln n)$. Hence for every $\varepsilon > 0$, $q \geqslant n^{v(G)/(e(G)-1)+\varepsilon}$, and sufficiently large n Picker can block a copy of G.

To prove part (ii), fix $\varepsilon > 0$, put $v_0 = v(G)$ and $e_0 = e(G)$, and suppose that $q < n^{1/m''(G)-\varepsilon}$. We will check that if n is big enough, condition (6) in Theorem 5.5 holds for some $s \geqslant 2$.

We need to estimate the weight $T(\mathcal{H}_2^s)$ with $T = T_{1/(q+1)}$. Every hyper-edge of $\mathcal{H}_2^s$ corresponds to s different copies $G_1, \ldots, G_s$ of G in K_n which have 2 edges in common so they intersect in at least 3 vertices. Hence

$$T(\mathcal{H}_2^s) = \sum_{\substack{S \subset K_n \\ S \text{ corresp. to a hyper-edge}}} (q+1)^{-e(S)}$$

$$\leqslant \binom{n}{3} (v_0!)^s \sum_{\substack{v:\, v<sv_0, \\ \binom{v-3}{v_0-3} v_0! \geqslant s}} \left(\sum_{\substack{F \subseteq G: \\ v(F) \geqslant 3}} \binom{v-3}{v(F)-3} \binom{n-3}{v_0 - v(F)} (q+1)^{e(F)-e_0} \right)^s .$$

Indeed, the factor $\binom{n}{3}$ counts the possible choices for the vertices in the common intersection of $G_1, \ldots, G_s$, $v = \left|\bigcup_{i=1}^s V(G_i)\right|$, the sum in the parenthesis stands for the number of ways one can add G_j to the copies $G_1, \ldots, G_{j-1}$ we have chosen so far and here by F we denote all possible intersections $E(G_j) \cap \bigcup_{i=1}^{j-1} E(G_i)$. Thus, if by F' we denote a subgraph of G which maximizes the terms of the interior sum, then for some constant $c_1 > 1$, which depends only on G,

$$T(\mathcal{H}_2^s) \leqslant n^3 c_1^s \sum_{v=v_0}^{sv_0} v^{v_0 s} \left(n^{v_0 - v(F')} q^{e(F')-e_0} \right)^s \leqslant n^3 c_1^s s v_0 (s v_0)^{v_0 s} \left(n^{v_0 - v(F')} q^{e(F')-e_0} \right)^s .$$

Hence for $q < n^{1/m''(G)}$ and some constant c_2 (which depends on G) we have

$$\begin{aligned} |V(\mathcal{H})| qs \left(1 + 2(T(\mathcal{H}_2^s))^{1/s}\right) / T(\mathcal{H}) &< c_2 n^2 qs \left(n^{-v_0} q^{e_0} + n^{3/s} s^{v_0+1} n^{-v(F')} q^{e(F')} \right) \\ &< c_2 s^{v_0+1} n^{3/s} \left(\left(n^{-\frac{v_0-2}{e_0+1}} q \right)^{e_0+1} + \left(n^{-\frac{v(F')-2}{e(F')+1}} q \right)^{e(F')+1} \right) \\ &\leqslant 2 c_2 s^{v_0+1} n^{3/s} (n^{-1/m''(G)} q)^{e(F')+1} . \end{aligned}$$

For $q < n^{1/m''(G)-\varepsilon}$, $s = \lceil 6/\varepsilon \rceil$ and sufficiently large n, the right hand side of the last inequality is less than 1 so (6) is satisfied. Thus by Theorem 5.5 the builder Chooser wins $\mathcal{CP}(\mathcal{H}^{G,n}, 1, q)$. □

Finally, we present four examples, in which by Theorem 1.3 we can transform immediately the results for Maker–Breaker graph games into the similar results for Picker–Chooser games. We only sketch the methods of the proofs and for the details we refer the reader to the cited papers.

The first game, considered by Alon, Hefetz and Krivelevich [1], is played on a graph G and the players select edges of G.

Lemma 6.2 ([1]). *Suppose that G is a graph on n vertices, $q \geqslant 2$ and $k = k(n) \geqslant \log_2 n$ are integers. If G is $(100kq \log_2 q)$-edge-connected, then Maker as the second player in a Maker–Breaker $(1 : q)$ game can build a spanning k-edge-connected subgraph of G.*

The authors consider the strict version of the above k-connectivity Maker–Breaker game, i.e. Breaker selects exactly q edges per turn, but the assumption of selecting at most q edges by Breaker does not change their proof, which is a straightforward application of Beck's winning criterion (2).

In order to win the game, Maker defines a hypergraph $\mathcal{H}$ such that $V(\mathcal{H}) = E(G)$, and $E(\mathcal{H})$ consists of all subsets of $E(G)$ of the form $C \setminus A$, where C is a cut in G, $A \subseteq C$ and $|A| = k-1$. It is clear that Maker in the k-connectivity game on G can build a spanning k-edge-connected subgraph iff Breaker in $\mathcal{MB}(\mathcal{H}, q, 1)_M$ has a winning strategy. The assumptions of Lemma 6.2 guarantee that Breaker in $\mathcal{MB}(\mathcal{H}, q, 1)_M$ wins by a min-weight strategy for the standard weight function T_δ with $\delta = 2^{-1/q}$. Thus, by Theorem 1.3, we can state the analogous result for the Picker–Chooser game.

Corollary 6.3. *Suppose that G is a graph on n vertices, $q \geqslant 4$ and $k = k(n) \geqslant \log_2 n$ are integers. If G is $(201kq \log_2 q)$-edge-connected, then Picker in a Picker–Chooser $(1 : q)$ game can build a spanning k-edge-connected subgraph of G.* □

Several authors [7,19] consider Maker–Breaker games played on the random graph $G(n, p)$. The random graph $G(n, p)$ is obtained from K_n by deleting independently every edge of K_n with probability $1 - p$. For games on $G(n, p)$ it is natural to ask for $p = p(n)$ such that a given player has a winning strategy *asymptotically almost surely* (a.a.s.), which means with probability tending to 1 with $n \to \infty$.

Consider the following *Δ-tree-universality $(1 : q)$ game*. This is a Maker–Breaker $(1 : q)$ game in which the players select edges of the random graph $G(n, p)$. The goal of Maker is to build a Δ-tree-universal graph, i.e. a subgraph of $G(n, p)$ containing all spanning trees of maximal degree not greater than Δ. We assume that q, p and Δ depend on n. Johannsen, Krivelevich and Samotij [15] proved the following result.

Lemma 6.4 ([15]). *Suppose that $\Delta = \Delta(n) \geqslant \ln n$ and $p = p(n) \geqslant 840 q \Delta n^{-1/3} \ln^2 n$. Then a.a.s. Maker wins the Δ-tree-universality $(1 : q)$ game.*

We point out that in [15] Maker starts the game and Breaker selects exactly q edges per turn, but the proof would be the same under the assumptions that Breaker is the first player and can select at most q edges. The core of the proof is to study an auxiliary game on some hypergraph $\mathcal{H}$. Universality-game-Maker plays in the auxiliary game a role of Breaker and wins by a min-weight strategy with the weight function T_δ for $\delta = 2^{-1/q}$. Based on Theorem 1.3, we obtain the following corollary for Picker and Chooser.

Corollary 6.5. *Suppose that $\Delta = \Delta(n) \geqslant \ln n$ and $p = p(n) \geqslant 1680q\Delta n^{-1/3}\ln^2 n$. Then a.a.s. the builder Picker wins the Δ-tree-universality Picker–Chooser $(1 : q)$ game.* □

The connectivity game is another example of Maker–Breaker game played on random graph $G(n, p)$. Here Maker, the second player, aims to build a spanning tree in $G(n, p)$. The lemma below is a special case of the more general result of Stojaković and Szabó [19].

Lemma 6.6 ([19]). *If $p = p(n) = \omega(\ln n/n)$ then a.a.s. Maker wins the connectivity $(1 : 1)$ game on $G(n, p)$.*

In the proof of Lemma 6.6 the authors define the hypergraph $\mathcal{H}$ such that $V(\mathcal{H}) = E(G(n, p))$ and $E(\mathcal{H})$ consists of all cuts in $G(n, p)$. Then connectivity-Maker wins (a.a.s.) playing as Breaker in $\mathcal{MB}(\mathcal{H}, 1, 1)_M$ by a min-weight strategy, with the standard weight function $T_{1/2}$. Thus we get the corollary for the Picker–Chooser connectivity game.

Corollary 6.7. *If $p = p(n) = \omega(\ln n/n)$ then a.a.s. the builder Picker wins the Picker–Chooser $(1 : 1)$ connectivity game on $G(n, p)$.* □

In the next example probability is also involved. Consider the random graph process, in which we build a graph on n vertices step by step. Starting with the empty graph, at every step we add to the graph an edge chosen uniformly at random from all edges not selected before. Given a monotone increasing graph property $\mathcal{P}$, we define *the hitting time* $\tau(\mathcal{P})$ as the first moment the graph obtained in the process has the property $\mathcal{P}$. Let δ_2 denote the graph property of having minimum degree at least 2. By $\mathcal{M}_{\text{perf}}$ we denote the graph property that Maker, playing a Maker–Breaker $(1 : 1)$ game on a graph with $2n$ vertices, can obtain a perfect matching in the graph. One of the theorems by Ben-Shimon et al. [7] says that the moment we get the minimum degree 2 in the random graph process is also the moment Maker can build (a.a.s.) a perfect matching in the obtained graph.

Lemma 6.8 ([7]). *In the random graph process on an even number of vertices a.a.s.*

$$\tau(\mathcal{M}_{\text{perf}}) = \tau(\delta_2).$$

Here is the idea of the proof. It is easy to see that $\tau(\mathcal{M}_{\text{perf}}) \geqslant \tau(\delta_2)$ so it is enough to verify that at the moment the random process reaches minimum degree 2, Maker has a winning strategy in the perfect matching game.

The winning strategy of Maker presented in [7] has two ingredients. Maker splits the board G (a graph obtained in the random process) into two edge-disjoint subgraphs, and at the first subgraph he uses a simple pairing strategy. For the second subgraph he defines an auxiliary hypergraph and applies a min-weight strategy with the weight function $T_{1/2}$.

Consider the corresponding Picker–Chooser hitting time problem: by $\mathcal{P}_{\text{perf}}$ we denote the property that Picker wins the perfect matching game. Observe that the above arguments for the Maker–Breaker version can be adapted to the Picker–Chooser $(1 : 1)$ version as well. Indeed, the inequality $\tau(\mathcal{P}_{\text{perf}}) \geqslant \tau(\delta_2)$ is true since if a graph G obtained in the random graph process has a vertex x of degree one, then the spoiler Chooser sooner or later will isolate x. Secondly, any pairing strategy can be easily carried out by Picker. Finally, by Theorem 1.3 we can transform the min-weight strategy of Maker into an effective strategy of the builder Picker. Thus we obtain a perfect analog of Lemma 6.8.

Corollary 6.9. *In the random graph process on an even number of vertices a.a.s.*

$$\tau(\mathcal{P}_{\text{perf}}) = \tau(\delta_2). \quad \square$$

Acknowledgments

I am very grateful to Fiachra Knox for pointing out that the conjecture that $(1 : 1)$ games are always more favorable to the spoiler Picker than to Breaker, is not true in general. I would also like to thank the referees for their very thorough and helpful comments.

References

[1] N. Alon, D. Hefetz, M. Krivelevich, Playing to retain the advantage, Combin. Probab. Comput. 19 (2010) 481–491.
[2] J. Beck, Remarks on positional games. I, Acta Math. Acad. Sci. Hungar. 40 (1982) 65–71.
[3] J. Beck, Random graphs and positional games on the complete graph, Ann. Discrete Math. 28 (1985) 7–13.
[4] J. Beck, Positional games and the second moment method, Combinatorica 22 (2002) 169–216.
[5] J. Beck, Combinatorial games: tic-tac-toe theory, in: Encyclopedia of Mathematics and its Applications, vol. 114, Cambridge University Press, Cambridge, 2008, xiv+732 pp.
[6] M. Bednarska, T. Łuczak, Biased positional games for which random strategies are nearly optimal, Combinatorica 20 (2000) 477–488.

[7] S. Ben-Shimon, A. Ferber, D. Hefetz, M. Krivelevich, Hitting time results for Maker–Breaker games, in: Proceedings of the 22nd Symposium on Discrete Algorithms (SODA'11), pp. 900–912.
[8] V. Chvátal, P. Erdős, Biased positional games, Ann. Discrete Math. 2 (1978) 221–228.
[9] A. Csernenszky, The Picker–Chooser diameter game, Theor. Comput. Sci. 411 (2010) 3757–3762.
[10] A. Csernenszky, C. I. Mándity, A. Pluhár, On Chooser–Picker positional games, Discrete Math. 309 (2009) 5141–5146.
[11] A. Csernenszky, R. Martin, A. Pluhár, On the complexity of Chooser–Picker positional games, Integers 11 (2011).
[12] P. Erdős, J.L. Selfridge, On a combinatorial game, J. Combin. Theory Ser. A 14 (1973) 298–301.
[13] H. Gebauer, T. Szabó, Asymptotic random graph intuition for the biased connectivity game, Random Structures Algorithms 35 (2009) 431–443.
[14] D. Hefetz, M. Krivelevich, M. Stojaković, T. Szabó, Planarity, colorability and minor games, SIAM J. Discrete Math. 22 (2008) 194–212.
[15] D. Johannsen, M. Krivelevich, W. Samotij, Expanders are universal for the class of all spanning trees, in: Proceedings of the 23rd Symposium on Discrete Algorithms, SODA'12, pp. 1539–1551.
[16] F. Knox, Two constructions relating to conjectures of Beck on positional games, arXiv:math/1212.3345.
[17] M. Krivelevich, The critical bias for the Hamiltonicity game is $(1 + o(1))n/\ln n$, J. Amer. Math. Soc. 24 (2011) 125–131.
[18] T.J. Schaefer, On the complexity of some two-person perfect-information games, J. Comput. Syst. Sci. 16 (1978) 185–225.
[19] M. Stojaković, T. Szabó, Positional games on random graphs, Random Structures Algorithms 26 (2005) 204–223.