

Small deterministic Turing machines

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1. Introduction

In this paper small deterministic Turing machines with one bi-infinite tape and one scanning head are considered. Let $DTM(m, n)$ denote the class of all such machines with m tape symbols (including the blank) and n states (excluding the halting state). It is known that there exist universal Turing machines in $DTM(5, 6)$ [10] and in $DTM(4, 7)$ [4]. It is also known that the halting problem for all machines from $DTM(1, n)$ and $DTM(m, 1)$ is decidable. The first fact is trivial whereas the second one has been shown in [2]. In [4, 5] it is mentioned that the halting problem for $DTM(2, 2)$ is decidable too; however without giving any proof and stating only [5] that this result is unpublished and unpublishable. In such a state it remained since 1961 or 1972, respectively, until 1988. By a reduction to few cases it was possible to solve the problem and bring it into a publishable form. A first version can be found in [1]. At that time the results from 1975 [6] stating the halting problem to be decidable for $DTM(2, 2)$ and $DTM(2, 3)$ have not been known to the authors. In that paper a completely different method was used. Neither were known the results by [7] stating that there are universal Turing machines in $DTM(2, 24)$, $DTM(3, 11)$, $DTM(5, 5)$, $DTM(6, 4)$, $DTM(10, 3)$, and $DTM(21, 2)$. All these results were only little known in Western countries before 1991.

It is also shown that the sets accepted by machines from $DTM(2, 1)$ are regular, and that those accepted by machines from $DTM(3, 1)$ are regular too, except one case giving a deterministic linear context-free language which is essentially the nonregular language $L = \{a^n b^n \mid 1 \leq n\}$.

The languages accepted by machines from $DTM(2, 2)$ are also regular, except one case (up to symmetries) giving essentially the same language as in the exceptional case of $DTM(3, 1)$, namely a deterministic linear context-free language. Thus, the halting problem is decidable for $DTM(2, 2)$. To obtain this result several symmetries are used to reduce the number of machines to consider. Finally, also a machine from $DTM(4, 4)$ is presented accepting a context-sensitive language, essentially $L = \{a^k \mid k = 2^n, 0 \leq n\}$.

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Possibly, similar methods can be applied also to $DTM(3,2)$, $DTM(2,3)$, and perhaps to $DTM(3,3)$. But for the last case, probably the aid of an automatic system should be used.

2. Definitions

Let T be any Turing machine with one bi-infinite tape, one scanning head, an alphabet V of m symbols including the blank, and a set S of n states (excluding the halting state H). The set of movements is given by $\{L, M, R\}$, standing for movement to the left, no movement, and movement to the right, respectively. The set of instructions, or program P of T , is denoted by $P \subseteq S \times V \times V \times \{L, M, R\} \times S$, with its elements (p, x, y, I, q) written in the form $pxyIq$. For shortness, y, I, q if $y = x$, $I = M$, $q = p$, is omitted, respectively. If the halting state H appears, this is always understood as pxH , i.e. $pxxMH$. If T is a deterministic Turing machine P represents a (total) function.

The class of all such deterministic Turing machines will be denoted by $DTM(m, n)$. Only such machines will be considered here.

As usual, V^* denotes the free monoid generated by V , λ its neutral element (the empty word), $lg(w)$ the length of a word $w \in V^*$, are $rev(w)$ its reversal (mirror image).

Let ${}^\infty V$ and V^∞ stand for the set of all left-infinite and right-infinite words over V , respectively.

A configuration of T is any bi-infinite word $\alpha \binom{x}{p} \beta \in {}^\infty V \cdot (V \times S) \cdot V^\infty$ denoting the fact that the head is scanning x and T is in state p . As usually, $\alpha \binom{x}{p} \beta \xrightarrow{*} \gamma \binom{y}{q} \delta$ means that there exists a finite sequence of Turing steps from $\alpha \binom{x}{p} \beta$ to $\gamma \binom{y}{q} \delta$, including 0 steps and $q = H$. It is assumed that there is no continuation of any configuration $\gamma \binom{y}{H} \delta$.

Finally, let \mathbb{N}_k denote the set $\mathbb{N}_k := \{i \in \mathbb{N} \mid 0 \leq i \leq k\}$.

As for acceptance, there exist several possibilities to define sets of words accepted by Turing machines. The first one just gives the set of all such bi-infinite words being configurations which lead to acceptance.

Definition 1. $L_\infty(T, p) := \{\alpha \binom{x}{p} \beta \in {}^\infty V \cdot (V \times S) \cdot V^\infty \mid \exists \gamma \in {}^\infty V \exists \delta \in V^\infty \exists y \in V : \alpha \binom{x}{p} \beta \xrightarrow{*} \gamma \binom{y}{H} \delta\}$.

A second possibility is the use of the work space for acceptance. Note, that by this definition one gets only context-sensitive languages.

Definition 2.

$$L_s(T, p) := \left\{ u \binom{x}{p} v \in V^* \cdot (V \times S) \cdot V^* \mid \exists k \in \mathbb{N} \forall i \in \mathbb{N}_k \exists u_i, v_i \in V^* \exists x_i \in V \exists p_i \in S : \right. \\ \left. u_0 = u \wedge v_0 = v \wedge x_0 = x \wedge p_0 = p \wedge p_k = H \wedge lg(u_i v_i) = lg(uv) \right\}$$

$$\left. \begin{aligned} \bigwedge \forall i \in \mathbb{N}_k - \{k\} : \left(p_i \neq H \wedge u_i \begin{pmatrix} xi \\ pi \end{pmatrix} v_i \rightarrow u_{i+1} \begin{pmatrix} x_{i+1} \\ p_{i+1} \end{pmatrix} v_{i+1} \right) \\ \bigwedge \exists j \in \mathbb{N}_k : u_j = \lambda \wedge \exists l \in \mathbb{N}_k : v_l = \lambda \end{aligned} \right\}.$$

Thus, $lg(uv) + 1$ is exactly the work space for accepting $u \begin{pmatrix} x \\ p \end{pmatrix} v$, and this word is the relevant part of the initial configuration being scanned by the head.

From this definition trivially follows that

Lemma 1. $L_\infty(T, p) = {}^\infty V \cdot L_s(T, p) \cdot V^\infty$.

A third possibility is not to look onto the relevant part of one initial configuration only but to consider them in some global sense, namely that if w is a word of the set then no proper subword of it is also contained in the set.

Definition 3.

$$\begin{aligned} L_c(T, p) := \left\{ u \begin{pmatrix} x \\ p \end{pmatrix} v \in V^* \cdot (V \times S) \cdot V^* \mid \forall \alpha \in {}^\infty V \forall \beta \in V^\infty : \right. \\ \left. \alpha u \begin{pmatrix} x \\ p \end{pmatrix} v \beta \in L_\infty(T, p) \wedge (u = yu' \Rightarrow \exists y' \in V : y'u' \notin L_c(T, p)) \right. \\ \left. \wedge (v = v'z \Rightarrow \exists z' \in V : v'z' \notin L_c(T, p)) \right\}. \end{aligned}$$

This set is called the *core* of $L_\infty(T, p)$. Trivially again, it follows that

Lemma 2. $L_\infty(T, p) = {}^\infty V \cdot L_c(T, p) \cdot V^\infty$.

Finally, the normal definition of an accepted set can be defined by cutting off all 0's at left and right ends from words in $L_s(T, p)$.

Definition 4. $L_0(T, p) = \{0\}^* \setminus L_s(T, p) / \{0\}^*$

In the sequel it is assumed that $V = \mathbb{N}_k - \{k\}$, and that 0 represents the blank. It is easy to establish the following lemmas.

Lemma 3. *If there is no occurrence of pxH in the set of instructions of a Turing machine T , then $L_\infty(T, p) = L_s(T, p) = L_c(T, p) = L_0(T, p) = \emptyset$.*

Lemma 4. *For deterministic Turing machines it suffices that there is at most 1 occurrence of halting qxH in the set of instructions.*

Proof. Among all occurrences of halting in P choose one, e.g. qxH . Replace all other pyH by $pyxMq$. Then it is trivial to see that the accepted languages remain the same. \square

Definition 5. Let $T = (S, V, P)$ be any deterministic Turing machine. Let $\pi : S \rightarrow S$ be any permutation of the set of states S , $\sigma : V \rightarrow V$ be any permutation of the alphabet V , and $\mu : \{L, M, R\} \rightarrow \{L, M, R\}$ be defined by $\mu(L) = R$, $\mu(M) = M$, $\mu(R) = L$.

Then define also $\pi(T) := (S, V, \pi(P))$ with $\pi(p)xyI\pi(q) \in \pi(P) \Leftrightarrow pxyIq \in P$, $\sigma(T) := (S, V, \sigma(PP))$ with $p\sigma(x)\sigma(y)Iq \in \sigma(P) \Leftrightarrow pxyIq \in P$, and $\mu(T) := (S, V, \mu(P))$ with $pxy\mu(I)q \in \mu(P) \Leftrightarrow pxyIq \in P$.

By the next lemma all Turing machines with movements of the head in one direction only can be eliminated from further consideration.

Lemma 5. Let T be any DTM with movements only $\{M, R\}$ or $\{L, M\}$ in the set of instructions, respectively. Then $L_s(T, p)$ is regular.

Proof. Let the movements be $\{M, R\}$, and the initial configuration $\alpha \binom{x}{p} \beta$. Then, from T , a finite automaton is constructed: $F_x := (V, S \times V; \{ \binom{x}{p} \}, \{ \binom{y}{q} \mid qyH \in P \}, R)$ with $R \subseteq (S \times V) \times (V \cup \{\lambda\}) \times (S \times V)$ given by $(\binom{y}{p}, \lambda, \binom{z}{q}) \in R$ if $pyzMq \in P$, and $(\binom{y}{p}, z, \binom{z}{q}) \in R$ if $pyzRq \in P$. By this construction it is obvious that $L_s(T, p) = \binom{x}{p} L(F_x)$.

Trivially, in the case of $\{L, M\}$ one gets $L_s(T, p) = rev(L(F_x)) \cdot \{ \binom{x}{p} \}$. \square

A stronger lemma is the following one.

Lemma 6. If either

$$\forall x \in V \forall p \in S \exists m, n \in \mathbb{N} \forall u_1 \in V^m \forall u_2 \in V^n \exists y \in V \exists q \in S \cup \{H\} \exists v \in V^{m+n} :$$

$$u_1 \binom{x}{p} u_2 \xrightarrow{*} v \binom{y}{q}$$

or

$$\forall x \in V \forall p \in S \exists m, n \in \mathbb{N} \forall u_1 \in V^m \forall u_2 \in V^n \exists y \in V \exists q \in S \cup \{H\} \exists v \in V^{m+n} :$$

$$u_1 \binom{x}{p} u_2 \xrightarrow{*} \binom{y}{q} v$$

where during the computation the head never leaves the workspace of length $m+n+1$, then $L_s(T, p)$ is regular.

Proof. By constructing a finite automaton with $(\binom{x}{p}, u_2, \binom{y}{q}) \in R$ or $(\binom{x}{p}, \bar{u}_1, \binom{y}{q}) \in R$ and final states $\binom{z}{H}$ where $\bar{v} = rev(v)$, respecting the initial part in front of or behind $\binom{x}{p}$ for the initial configuration. \square

Although there exist deterministic Turing machines with all accepted sets $L_s(T, p)$, $L_c(T, p)$, $L_0(T, P)$ being different, only the first possibility will be considered in the sequel.

3. DTM(1, n)

Any Turing machine T from $DTM(1, n)$ can be considered as a finite directed graph with $n + 1$ nodes from $S \cup \{H\}$. For each $p \in S$ it is decidable whether there exists a directed path from p to H , and this path has a length of at most n . Thus one gets

Theorem 1. *All sets accepted by Turing machines from $DTM(1, n)$ are finite, and the halting problem for $DTM(1, n)$ is decidable.*

4. DTM(2, 1)

Generally, it has been shown by Herman [2] that the halting problem of arbitrary 1 state DTMs with 1 head and 1 k -dimensional tape is decidable. He also proved that there exist accepted sets which are not regular.

Let $S = \{p\}$ and $V = \{0, 1\}$. Then

Theorem 2. *All sets accepted by machines from $DTM(2, 1)$ are regular, and the halting problem for $DTM(2, 1)$ is decidable.*

Proof. Let σ be the symmetry defined by $\sigma(0) = 1$, $\sigma(1) = 0$, and μ that one defined by $\mu(L) = R$, $\mu(M) = M$, $\mu(R) = L$. By Lemma 4 it suffices to have at most 1 possibility of halting in the program. Let there be exactly one. It also suffices that this is $p1H$ for if it is $p0H$ then this may be treated using the symmetry σ . The case $p0L$ may be ruled out using the symmetry μ . Thus, only 2 cases remain:

p	and	p
0 · R		0 · M
1 · H		1 · H

The first one gives a regular set, the second one finite sets. □

5. DTM(3, 1)

Let $S = \{p\}$, $V = \{0, 1, 2\}$, and $f(p, x) \in V$, $g(p, x) \in \{L, M, R\}$, $h(p, x) \in S \cup \{H\}$ denote the new symbol, the head movement, and the new state, respectively, if T is in state p and x the scanned symbol.

By Lemma 5, and using the symmetry μ , similarly to $DTM(2, 1)$, it suffices to consider only the case

p
0 · L
1 · R
2 · H

In the case $f(p, 0) = f(p, 1) = 0$ one gets as accepted set

$$L_s(T, p) = 2(0 \cup 1)^* \binom{0}{p} \cup \binom{1}{p} 1^* 2 \cup 2(0 \cup 1)^* \binom{1}{p} 1^* 0 \cup \binom{2}{p},$$

in the case $f(p, 0) = 0, f(p, 1) = 1$

$$L_s(T, p) = 20^* \binom{0}{p} \cup \binom{1}{p} 1^* 2 \cup \binom{2}{p},$$

in the case $f(p, 0) = f(p, 1) = 1$

$$L_s(T, p) = 20^* \binom{0}{p} \cup 10^* \binom{0}{p} (0 \cup 1)^* 2 \cup \binom{1}{p} (0 \cup 1)^* 2 \cup \binom{2}{p},$$

and in the last case $f(p, 0) = 1, f(p, 1) = 0$ the following accepted set:

$$\begin{aligned} L_s(T, p) = & \left\{ 10^{m_k} 1 \dots 10^{m_1} \binom{0}{p} 1^{n_1} 0 \dots 0 1^{n_k} 2, \right. \\ & 20^{m_{k+1}} 10^{m_k} 1 \dots 10^{m_1} \binom{0}{p} 1^{n_1} 0 \dots 0 1^{n_k} 0, \\ & 20^{m_k} 1 \dots 10^{m_1} \binom{1}{p} 1^{n_1} 0 \dots 0 1^{n_k} 0, \\ & 10^{m_k} 1 \dots 10^{m_1} \binom{1}{p} 1^{n_1} 0 \dots 0 1^{n_k} 0 1^{n_{k+1}} 2 \mid \\ & \left. k \geq 0, m_i \in \mathbb{N}, n_i \in \mathbb{N} \right\} \\ & \cup \binom{2}{p}. \end{aligned}$$

Clearly, this is a linear deterministic context-free language not being regular. Thus:

Theorem 3. *All sets accepted by machines from $DTM(3, 1)$ are regular except one case (up to symmetries) yielding a linear deterministic context-free language. The halting problem for $DTM(3, 1)$ is decidable.*

6. $DTM(2, 2)$

At first some methods reducing the number of machines to be treated are considered. Let $S = \{p, q\}$ and $V = \{0, 1\}$. In the program of a Turing machine T let $f(s, x) \in V$, $g(s, x) \in \{L, M, R\}$, $h(s, x) \in S \cup \{H\}$ denote the new symbol to be written, the movement of the head, and the new state, respectively, if T is in state s and the head is scanning x . By Lemma 4 let there be exactly 1 possibility of H . Thus, from originally $13^4 = 28561$ only $4 \cdot 12^3 = 6912$ machines are left. Using Lemma 5, this number can be reduced to 3072.

Using the symmetries σ and μ from DTM(2, 1) and π defined by $\pi(p) = q, \pi(q) = p$, the following reductions decrease the number of Turing machines to be considered.

Lemma 7. *It suffices to have only $h(q, 1) = H$ as halting.*

Proof. By using the symmetries σ, π , and $\sigma\pi = \pi\sigma$. \square

By the next lemma $g(q, 0) = L$ is eliminated.

Lemma 8. *It suffices to have only $g(q, 0) = M$ or $g(q, 0) = R$.*

Proof. By using the symmetry μ . \square

The following one fixes p as initial state.

Lemma 9. *It suffices to have p as initial state.*

Proof. Let q be initial state. Then there are 2 cases to be considered, namely $h(q, 0) = q$ and $h(q, 0) = p$. The first case yields machines from DTM(2, 1) which have been considered already. Let therefore $h(q, 0) = p$ with $f(q, 0) = z$.

If $g(q, 0) = M$ then let γ be the deterministic *gsm*-mapping defined by the finite automaton

$$\left(V \cup (V \times S), \{A, B\}, \{A\}, \{B\}, \left\{ AxxA, A \begin{pmatrix} z \\ p \end{pmatrix} \begin{pmatrix} 0 \\ q \end{pmatrix} B, BxxB \mid x \in V \right\} \right)$$

with initial state A and final state B . Then

$$L_s(T, q) = \gamma \left(L_s(T, p) \cap V^* \begin{pmatrix} z \\ p \end{pmatrix} V^* \right) \cup \begin{pmatrix} 1 \\ q \end{pmatrix}.$$

If $g(q, 0) = R$ then let δ and ε be deterministic *gsm*-mappings defined by the finite automata

$$\left(V \cup (V \times S), \{A, B, C\}, \{A\}, \{C\}, \left\{ AxxA, A \begin{pmatrix} x \\ p \end{pmatrix} xB, Bz \begin{pmatrix} 0 \\ q \end{pmatrix} C, CxxC \mid x \in V \right\} \right)$$

with initial state A and final state C , and

$$\left(V \cup (V \times \{p\}), \{A, B\}, \{A\}, \{B\}, \left\{ A \begin{pmatrix} x \\ p \end{pmatrix} xB, BxxB \mid x \in V \right\} \right)$$

with initial state A and final state B . Then

$$L_s(T, q) = \begin{pmatrix} 0 \\ q \end{pmatrix} \in (L_s(T, p) \cap (V \times \{p\}) \cdot V^*) \\ \cup \text{rev}(\delta(\text{rev}(L_s(T, p) \cap V^*z(V \times \{p\})V^*))) \cup \begin{pmatrix} 1 \\ q \end{pmatrix}$$

where *rev* denotes the mirror image. \square

Thus, the following cases remain to be considered: $h(q, 0) = p$ with $g(q, 0) = M$ or $g(q, 0) = R$, and with $h(p, 0) = h(p, 1) = q$ or $h(p, 0) = p, h(p, 1) = q$ or $h(p, 0) = q, h(p, 1) = p$. $h(p, 0) = h(p, 1) = p$ trivially can be ruled out.

The next reduction eliminates $h(q, 0) = q$.

Lemma 10. *It suffices to have $h(q, 0) = p$ only.*

Proof. If $h(q, 0) = q$ then, starting with p , all configurations ending in q yield regular sets since this is effectively a machine from $DTM(2, 1)$. After reaching q this is again such a machine yielding regular sets only. \square

Thus, after eliminating the case $g(p, 0) = R, g(p, 1) = L, g(q, 0) = M$ by the symmetry μ , the following cases remain (omitting f and h):

	p	q
0	$\cdot L \cdot$	$\cdot Mp$
1	$\cdot R \cdot$	H

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with the possibilities for $h(p, 0)$ and $h(p, 1)$ mentioned above. This leaves $6 \cdot 3 \cdot 4 \cdot 2 = 144$ machines from originally $4 \cdot 12^3 = 6912$.

The first case to consider is

Case 1: $g(q, 0) = M$ by the possibilities for $h(p, 0)$ and $h(p, 1)$.

Case 1.1: $h(p, 0) = h(p, 1) = q$. Then

$$0 \binom{0}{p} \rightarrow \binom{0}{q} f(p, 0) \rightarrow \binom{f(q, 0)}{p} f(p, 0),$$

$$1 \binom{0}{p} \rightarrow \binom{1}{q} f(p, 0) \rightarrow H,$$

$$\binom{1}{p} 0 \rightarrow f(p, 1) \binom{0}{q} \rightarrow f(p, 1) \binom{f(q, 0)}{p},$$

$$\binom{1}{p} 1 \rightarrow f(p, 1) \binom{1}{q} \rightarrow H.$$

Case 1.1.1: $f(q, 0) = 0$. Then

$$0 \binom{0}{p} \xrightarrow{*} \binom{0}{p} f(p, 0),$$

$$\binom{1}{p} 0 \xrightarrow{*} f(p, 1) \binom{0}{p} \xrightarrow{*} \binom{0}{p} f(p, 0) \quad \text{if } f(p, 1) = 0,$$

$$\begin{pmatrix} 1 \\ p \end{pmatrix} 0 \xrightarrow{*} f(p, 1) \begin{pmatrix} 0 \\ p \end{pmatrix} \xrightarrow{*} H \quad \text{if } f(p, 1) = 1.$$

which shows no effective movement to the right, thus yielding a regular set.

Case 1.1.2: $f(q, 0) = 1$. Then

$$0 \begin{pmatrix} 0 \\ p \end{pmatrix} \xrightarrow{*} \begin{pmatrix} 1 \\ p \end{pmatrix} f(p, 0) \xrightarrow{*} f(p, 1) \begin{pmatrix} 1 \\ p \end{pmatrix} \quad \text{if } f(p, 0) = 0,$$

$$0 \begin{pmatrix} 0 \\ p \end{pmatrix} \xrightarrow{*} \begin{pmatrix} 1 \\ p \end{pmatrix} f(p, 0) \xrightarrow{*} H \quad \text{if } f(p, 0) = 1,$$

$$\begin{pmatrix} 1 \\ p \end{pmatrix} 0 \xrightarrow{*} f(p, 1) \begin{pmatrix} 1 \\ p \end{pmatrix}.$$

which shows no effective movement to the left, thus also yielding a regular set.

Case 1.2: $h(p, 0) = q, h(p, 1) = p$. Then

$$0 \begin{pmatrix} 0 \\ p \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ q \end{pmatrix} f(p, 0) \rightarrow \begin{pmatrix} f(q, 0) \\ p \end{pmatrix} f(p, 0),$$

$$1 \begin{pmatrix} 0 \\ p \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ q \end{pmatrix} f(p, 0) \rightarrow H,$$

$$\begin{pmatrix} 1 \\ p \end{pmatrix} 0 \rightarrow f(p, 1) \begin{pmatrix} 0 \\ p \end{pmatrix},$$

$$\begin{pmatrix} 1 \\ p \end{pmatrix} 1 \rightarrow f(p, 1) \begin{pmatrix} 1 \\ p \end{pmatrix}.$$

Case 1.2.1: $f(q, 0) = 0$. Then

$$0 \begin{pmatrix} 0 \\ p \end{pmatrix} \xrightarrow{*} \begin{pmatrix} 0 \\ p \end{pmatrix} f(p, 0)$$

and

$$L_s(T, p) = 10^* \begin{pmatrix} 0 \\ p \end{pmatrix} \cup 10^* \begin{pmatrix} 1 \\ p \end{pmatrix} 1^* 0 \quad \text{if } f(p, 1) = 0,$$

$$L_s(T, p) = 10^* \begin{pmatrix} 0 \\ p \end{pmatrix} \cup \begin{pmatrix} 1 \\ p \end{pmatrix} 1^* 0 \quad \text{if } f(p, 1) = 1.$$

Case 1.2.2: $f(q, 0) = 1$. Then

$$0 \begin{pmatrix} 0 \\ p \end{pmatrix} \xrightarrow{*} \begin{pmatrix} 1 \\ p \end{pmatrix} f(p, 0) \rightarrow f(p, 1) \begin{pmatrix} f(p, 0) \\ p \end{pmatrix}$$

and the machine has no effective movement to the left, thus yielding a regular set.

Case 1.3: $h(p, 0) = p$, $h(p, 1) = q$. Then

$$0 \begin{pmatrix} 0 \\ p \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ p \end{pmatrix} f(p, 0),$$

$$1 \begin{pmatrix} 0 \\ p \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ p \end{pmatrix} f(p, 0),$$

$$\begin{pmatrix} 1 \\ p \end{pmatrix} 0 \rightarrow f(p, 1) \begin{pmatrix} 0 \\ q \end{pmatrix} \rightarrow f(p, 1) \begin{pmatrix} f(q, 0) \\ p \end{pmatrix},$$

$$\begin{pmatrix} 1 \\ p \end{pmatrix} 1 \rightarrow f(p, 1) \begin{pmatrix} 1 \\ q \end{pmatrix} \rightarrow H.$$

Case 1.3.1: $f(q, 0) = 0$. Then

$$\begin{pmatrix} 1 \\ p \end{pmatrix} 0 \xrightarrow{*} f(p, 1) \begin{pmatrix} 0 \\ p \end{pmatrix} \rightarrow \begin{pmatrix} f(p, 1) \\ p \end{pmatrix} f(p, 0).$$

This gives no effective movement to the right, thus yielding regular sets.

Case 1.3.2: $f(q, 0) = 1$. Then

$$\begin{pmatrix} 1 \\ p \end{pmatrix} 0 \xrightarrow{*} f(p, 1) \begin{pmatrix} 1 \\ p \end{pmatrix}$$

and

$$L_s(T, p) = 10^* \begin{pmatrix} 0 \\ p \end{pmatrix} 0^* 1 \cup \begin{pmatrix} 1 \\ p \end{pmatrix} 0^* 1 \quad \text{if } f(p, 0) = 0,$$

$$L_s(T, p) = 10^* \begin{pmatrix} 0 \\ p \end{pmatrix} \cup \begin{pmatrix} 1 \\ p \end{pmatrix} 0^* 1 \quad \text{if } f(p, 0) = 1.$$

Now the 120 remaining machines are considered with

Case 2: $g(q, 0) = R$.

Case 2.1: $g(p, 0) = g(p, 1) = L$.

Case 2.1.1: $h(p, 0) = h(p, 1) = p$:

$$x \begin{pmatrix} y \\ p \end{pmatrix} \rightarrow \begin{pmatrix} x \\ p \end{pmatrix} f(p, y)$$

yielding no halt at all.

Case 2.1.2: $h(p, 0) = p$, $h(p, 1) = q$ or $h(p, 0) = q$, $h(p, 1) = p$. Let $m \in \{0, 1\}$ and $n = 1 - m$ such that $h(p, m) = p$ and $h(p, n) = q$. Then

$$x \begin{pmatrix} m \\ p \end{pmatrix} \rightarrow \begin{pmatrix} x \\ p \end{pmatrix} f(p, m),$$

$$0 \begin{pmatrix} n \\ p \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ q \end{pmatrix} f(p, n) \rightarrow f(q, 0) \begin{pmatrix} f(p, n) \\ p \end{pmatrix},$$

$$1 \binom{n}{p} \rightarrow \binom{1}{q} f(p, n) \rightarrow H$$

giving no effective movement to the right, and thus only regular sets.

Case 2.1.3: $h(p, 0) = h(p, 1) = q$:

$$0 \binom{y}{p} \rightarrow \binom{0}{q} f(p, y) \rightarrow f(q, 0) \binom{f(p, y)}{p},$$

$$1 \binom{y}{p} \rightarrow \binom{1}{q} f(p, y) \rightarrow H$$

giving no effective movement at all, and thus only regular sets.

For the remaining 2 subcases the following considerations give a further reduction of the number of cases.

Let $j \in \{0, 1\}$ be such that $g(p, j) = L$ and $k = 1 - j$. If $h(p, j) = q$ then there is no effective movement to the left since

$$x \binom{j}{p} \rightarrow \binom{x}{q} f(p, j) \rightarrow f(q, x) \binom{f(p, j)}{p},$$

$$\binom{k}{p} \rightarrow \binom{f(p, k)}{h(p, k)} \quad \text{if } g(p, k) = M,$$

$$\binom{k}{p} y \rightarrow f(p, k) \binom{y}{h(p, k)} \quad \text{if } g(p, k) = R,$$

thus giving only regular sets. Therefore, $h(p, j) = p$ has to be considered only.

If $h(p, k) = p$ then the machine will never halt. Thus assume $h(p, k) = q$.

Case 2.2: $g(p, k) = M$. Then

$$j \binom{j}{p} \rightarrow \binom{j}{p} f(p, j),$$

$$k \binom{j}{p} \rightarrow \binom{k}{p} f(p, j) \rightarrow \binom{0}{q} f(p, j) \rightarrow f(q, 0) \binom{f(p, j)}{p} \quad \text{if } f(p, k) = 0,$$

$$\binom{k}{p} y \rightarrow \binom{0}{q} y \rightarrow f(q, 0) \binom{y}{p},$$

$$k \binom{j}{p} \rightarrow \binom{k}{p} f(p, j) \rightarrow \binom{1}{q} f(p, j) \rightarrow H \quad \text{if } f(p, k) = 1,$$

$$\binom{k}{p} y \rightarrow \binom{1}{q} y \rightarrow H$$

yielding effectively a machine from $DTM(2,1)$ with only regular sets.

Thus

Case 2.3: $g(p, k) = R$ may be assumed such that the following cases remain: $g(p, j) = L, h(p, j) = p, g(p, k) = R, h(p, k) = q, g(q, 0) = R, h(q, 0) = p$ with $j, f(p, j), f(p, k), f(q, 0) \in \{0, 1\}$ and $k \neq j$. This leaves 16 Turing machines to consider.

At first

Case 2.3.1: $f(p, j) = 1$ one has

$$j \binom{j}{p} \rightarrow \binom{j}{p} 1,$$

$$k \binom{j}{p} \rightarrow \binom{k}{p} 1 \rightarrow f(p, k) \binom{1}{q} \rightarrow H,$$

$$\binom{k}{p} 0j \rightarrow f(p, k) \binom{0}{q} j \rightarrow f(p, k) f(q, 0) \binom{j}{p},$$

$$\binom{k}{p} 0k \rightarrow f(p, k) \binom{0}{q} k \rightarrow f(p, k) f(q, 0) \binom{k}{p},$$

$$\binom{k}{p} 1 \rightarrow f(p, k) \binom{1}{q} \rightarrow H$$

yielding as accepted sets

$$L_s(T, p) = kj^* \binom{j}{p} \cup \binom{k}{p} (0k)^* 1 \cup L'_s(T, p)$$

with

$$L'_s(T, p) = \binom{k}{p} (0k)^* 0j \quad \text{if } f(q, 0) = k \text{ or } (f(q, 0) = j \text{ and } f(p, k) = k),$$

$$L'_s(T, p) = kj^* \binom{k}{p} (0k)^* 0j \quad \text{if } f(q, 0) = f(p, k) = j$$

being all regular.

Case 2.3.2: $f(p, j) = 0$. Here one has

$$j \binom{j}{p} \rightarrow \binom{j}{p} 0,$$

$$k \binom{j}{p} j \rightarrow \binom{k}{p} 0j \rightarrow f(p, k) \binom{0}{q} j \rightarrow f(p, k) f(q, 0) \binom{j}{p},$$

$$k \binom{j}{p} k \rightarrow \binom{k}{p} 0k \rightarrow f(p, k) \binom{0}{q} k \rightarrow f(p, k) f(q, 0) \binom{k}{p},$$

$$\binom{k}{p} 0j \rightarrow f(p, k) \binom{0}{q} j \rightarrow f(p, k) f(q, 0) \binom{j}{p},$$

$$\binom{k}{p} 0k \rightarrow f(p,k) \binom{0}{q} k \rightarrow f(p,k) f(q,0) \binom{k}{p},$$

$$\binom{k}{p} 1 \rightarrow f(p,k) \binom{1}{q} \rightarrow H.$$

Case 2.3.2.1: $f(q,0) = k$. This yields as accepted sets

$$L_s(T,p) = \left(10^* \binom{0}{p} 0^* 1 \cup \binom{1}{p} \right) (0^+ 1)^* 1 \quad \text{if } k = 1,$$

$$L_s(T,p) = \left(\left(\binom{0}{p} \right) \cup 0 \binom{1}{p} 1^* 0 \cup 01^+ \binom{1}{p} \right) (01^* 0)^* 1 \quad \text{if } k = 0$$

being all regular.

Case 2.3.2.2: $f(q,0) = j$. Here all 4 remaining machines have to be considered.

$$j = 0 : L_s(T,p) = \left(1 \binom{0}{p} 1 \cup \binom{1}{p} \right) (01)^* 1$$

$$j = 1, f(p,0) = 0 :$$

$$L_s(T,p) = \left(\left(\binom{0}{p} \right) \cup 0(11)^* \left(\binom{1}{p} (0 \cup 1) \cup 1 \binom{1}{p} \right) \right) (0(0 \cup 1))^* 1$$

all being regular.

The only remaining machine, with $j = 1$, $f(p,0) = 1$ however, does not yield a regular set but

$$L_s(T,p) = \left\{ \begin{aligned} & 0(11)^{m_k} u_{k-1} \cdots u_1 (11)^{m_1} \binom{0}{p} 0(00)^{n_0} 1(00)^{n_1} v_1 \cdots v_{k-1} (00)^{n_k} 01, \\ & 0(11)^{m_k} u_{k-1} \cdots u_1 (11)^{m_1} \binom{1}{p} (00)^{n_1} v_1 \cdots v_{k-1} (00)^{n_k} 01, \\ & 01(11)^{m_k} u_{k-1} \cdots u_1 (11)^{m_1} \binom{0}{p} 0(00)^{n_0} 1(00)^{n_1} v_1 \cdots v_{k-1} (00)^{n_k} 1, \\ & 01(11)^{m_k} u_{k-1} \cdots u_1 (11)^{m_1} \binom{0}{p} (00)^{n_1} v_1 \cdots v_{k-1} (00)^{n_k} 1 \mid \\ & k \geq 0, m_i \geq 0, n_i \geq 0, (u_i = 0, v_i = 1) \text{ or } (u_i = v_i = 01) \end{aligned} \right\}.$$

It is easy to see that this set is not regular but a linear deterministic context-free language being similar to that one from *DTM*(3,1).

Summarizing, one obtains

Theorem 4. *All sets accepted by Turing machines from $DTM(2,2)$ are regular except one case only (up to symmetries) giving a linear deterministic context-free language. The halting problem for $DTM(2,2)$ is decidable.*

It should be noted that the machine last mentioned is not the busy beaver machine for $DTM(2,2)$ since that one is the following which can be shown easily:

	<i>p</i>	<i>q</i>
0	1 <i>Lq</i>	1 <i>Rp</i>
1	<i>Rq</i>	<i>H</i>

It should also be mentioned that in most cases the sets $L_s(T,p)$ and $L_c(T,p)$ are identical, but that there are some exceptions.

7. $DTM(4,4)$

Here only a machine accepting a language being not context-free is presented. This machine is

	<i>p</i>	<i>q</i>	<i>r</i>	<i>s</i>
0	<i>R</i>	<i>R</i>	<i>L</i>	<i>L</i>
1	<i>Rq</i>	0 <i>Rp</i>	<i>L</i>	<i>Lq</i>
2	<i>R</i>	<i>H</i>	<i>Rp</i>	...
3	<i>Lr</i>	2 <i>Ls</i>

where the open positions are irrelevant and may be filled by M .

For the accepted language the following fact holds:

$$L_s(T,p) \cap \binom{2}{p} 1^* 33 = \left\{ \binom{2}{p} 1^{2^n} 33 \mid n \geq 0 \right\}.$$

showing that $L_s(T,p)$ is not context-free.

Recently, it has been shown [3] that there exist Turing machines in $DTM(2,5)$ and $DTM(3,6)$ closely related to Collatz problems with unknown solutions.

Finally, in 1992 the existence of a universal Turing machine in $DTM(3,10)$ was shown [8], and in 1995 the existence of another one in $DTM(18,2)$ [9].

There exists also an unpublished proof by Pavlockaya on the decidability of the halting problem for $DTM(3,2)$.

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References

- [1] V. Diekert and M. Kudlek, Small deterministic Turing machines, Papers on Automata and Languages, Dept. of Math., Karl Marx Univ. of Economics, Budapest, 1988–4, (1989) 77–87.
- [2] G.T. Herman, The Halting problem of one state Turing machines with n -dimensional tape, *Z. Math. Logik Grundl. Math.* **14** (1968) 185–191.
- [3] P. Michel, Busy beaver competition and Collatz-like problems, *Arch. Math. Logic* **32** (1993) 351–367.
- [4] M.I. Minsky, Size and structure of universal Turing machines using tag systems, in: *Recursive Function Theory, Symposia in Pure Mathematics* Vol. 5 (AMS, Providence, RI, 1962) 229–238.
- [5] M.I. Minsky, *Computation. Finite and Infinite Machines* (Prentice-Hall, Englewood Cliffs, NJ, 1967–1972).
- [6] L.M. Pavlockaya, Dostatočnye usloviya razrešivosti problemy ostanovki dlya mašin Tyuringa, *Problemy Kibernetiki* **33** (1978) 91–118.
- [7] Yu.V. Rogozhin, Sem' universal'nyh mašin Tyuringa, *Matematičeskie Issledovaniya* **69** (1982) 76–90.
- [8] Yu.V. Rogozhin, Universal'naya mašina Tyuringa s 10 sostayaniyah i 3 simbolami, *Izvestiya Akademii Nauk Respubliki Moldova Matematika* (10) (1992) 80–82.
- [9] Yu.V. Rogozhin, On the notion of universality and Shannon's problem for Turing machines, Presented at MCU/UMC'95, Paris.
- [10] S. Watanabe, 5-symbol 8-state and 5-symbol 6-state universal Turing machines, *J. ACM* **8** (1961) 476–483.