

THE EUCLIDEAN TRAVELING SALESMAN PROBLEM IS NP-COMPLETE*

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Abstract. The Traveling Salesman Problem is shown to be NP-Complete even if its instances are restricted to be realizable by sets of points on the Euclidean plane.

The Traveling Salesman Problem (TSP) can be stated as follows: Given $(n-1)n/2$ integers denoting the distances between all pairs of n cities, find a tour, that is, a simple path visiting all cities, so that the total traversed distance is the least possible. Recently there have been increasingly many reasons for one to believe that the TSP is extremely hard. There is evidence that there are no polynomial time algorithms for obtaining an exact solution (even if the distances are restricted to be 0-1 [5]) or a solution of some guaranteed accuracy [8]. Moreover, the problem of simply verifying whether a given solution is (exactly or approximately) optimal also seems to be intractable [7].

Although the motivation for the TSP can probably be traced back to the Euclidean case (the cities are points on the map and the distances are the usual Euclidean metric) there is little known about the complexity of the Euclidean TSP. There is a general feeling in the literature (e.g. [6]) that the Euclidean TSP is considerably easier than the general case, either because the heuristics seem to perform better, or because special methods of attack are applicable. For example, it is almost always easy in the Euclidean TSP to exhibit edges that are not contained in any optimal tour (namely the chords of the convex hull of the cities), whereas the same task seems to be considerably harder in the general case (see Theorem 4 in [7]). Nevertheless, in this paper it is shown essentially that the Euclidean TSP cannot be "much" easier than the general problem, at least as far as exact solutions are concerned.

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In fact, we are dealing with two problems. The first, the *tour-TSP*, is the ordinary TSP. The other, the *path-TSP*, is the problem facing traveling salesmen who can start from any city, and are not particularly interested in returning to the starting city of their tour. The path-TSP can be especially useful as a more precise model for some problems arising in applications, like the hole drilling problems [6].

The following suggests that the computational requirements of these problems are closely related to each other:

Theorem 1. *The problems tour-TSP and path-TSP reduce to each other in linear time, by reductions increasing the number of cities by only an additive constant.*

Proof. Starting with the path-TSP, create a new city with equal distances from all other cities. An optimal tour in the resulting TSP corresponds in a natural way to an optimal path in the original.

For the opposite direction suppose that we have n cities c_1, \dots, c_n and that k equals n times the largest distance between any two cities. Let d be the distance function (generalized to denote the length of paths and tours). We create a new copy c'_1 of c_1 , and modify d as follows:

$$d'(c_i, c_j) = d(c_i, c_j) \quad \text{if } i, j \neq 1,$$

$$d'(c_1, c_j) = d(c_1, c_j) + 2k \quad \text{for all } j,$$

$$d'(c'_1, c_j) = d(c_1, c_j) + 2k \quad \text{for all } j,$$

$$d'(c_1, c'_1) = 3k.$$

We claim that any optimal path in this TSP has c_1 and c'_1 as endpoints. To show this, we first note that c_1 and c'_1 cannot be adjacent in any optimal path, since an optimal path cannot contain the longest link. Moreover any path with c_1, c'_1 as endpoints has length at most $5k$.

Now suppose that an optimal path has endpoints $\{c_i, c_j\} \neq \{c_1, c'_1\}$. If only one of c_i, c_j is c_1 or c'_1 , then this path has length at least $6k$. If none of c_i, c_j is c_1 or c'_1 , then this path has length at least $8k$ (since c_1 and c'_1 cannot be adjacent). This establishes that any optimal path in the modified TSP has c_1 and c'_1 as endpoints.

Now, every such path p corresponds to a tour t in the original TSP with $d'(p) = d(t) + 4k$. Also any tour t in the original TSP can be obtained from such a path. Consequently, minimizing paths in the resulting TSP corresponds to minimizing tours in the original. \square

Since there are algorithms solving the TSP with n cities in $O(n^2 2^n)$ time [3], the above result suggests that the complexities of these two problems are within a constant factor from each other.

It is not clear though how the Euclidean cases of these problems relate to each

other. Obviously the above reductions do not work. Of course, one way to reduce the Euclidean cases to each other is to show that both are NP-Complete (Theorems 2 and 3).

We will now give a more precise definition of the Euclidean TSP. The cities can be given in terms of a list of pairs of integers denoting the coordinates with respect to some coordinate system. It is not clear what the distance matrix should be. If we take it to be the (infinite precision) real-valued Euclidean metric, it is a nontrivial task to show that the resulting problem is in NP, since there is no obvious upper bound for the precision required in order to compare the length of a tour or path with a given integer. In what follows we will assume that the elements of the distance matrix are the integral parts of this metric. Any desired precision can be thus obtained by increasing the scale accordingly. Moreover in the constructions that will follow we will also allow rational coordinates, with the understanding that the scale will be eventually multiplied by an adequately large integer, so that all coordinates become integral and any necessary precision is obtained.

We will refer to lists of pairs of integers or, equivalently, "maps" of sets of cities with integer-valued coordinates, as *configurations*. The configuration of Fig. 1a will be referred to as a *2-chain*. The vertical distance between the two parallel rows is 1; horizontal distances within the chain are 2. A 2-chain can be traversed by optimal paths either in *mode 1* (Fig. 1b) or in *mode 2* (Fig. 1c). We will abbreviate a 2-chain schematically by Fig. 1d.

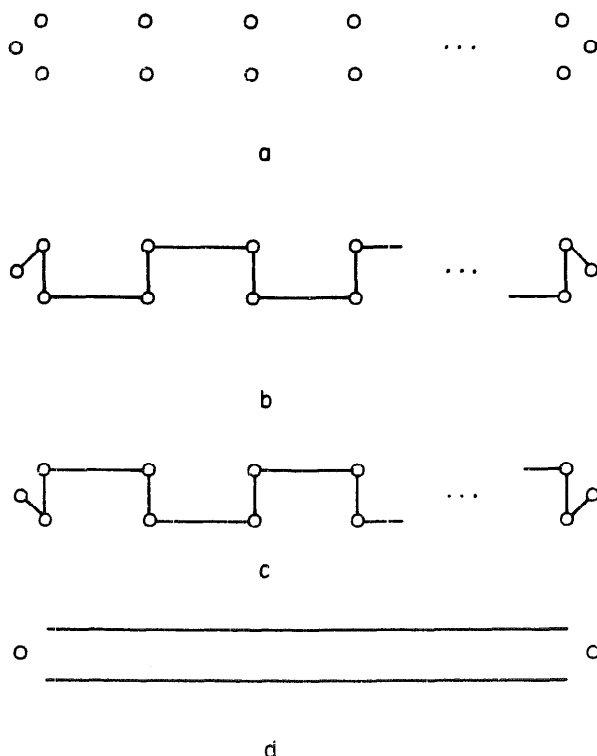


Fig. 1. The 2-chain.

A *1-chain* is a configuration like the one in Fig. 2a. (The distance between any closest pair of cities within the chain is 1). We will use for 1-chains the schematic abbreviation of Fig. 2b.

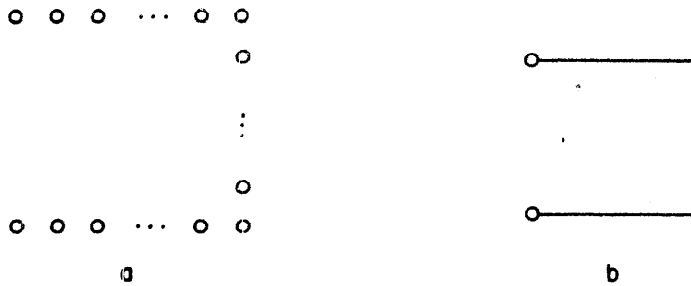


Fig. 2. A 1-chain.

The configuration *H* is shown in Fig. 3a (abbrev. as in Fig. 3b). The distance between *A* and *B* is 1, the distance between *D* and *D'* is 6, and that between *A* and *C* is 7. For this configuration the following statement is true:

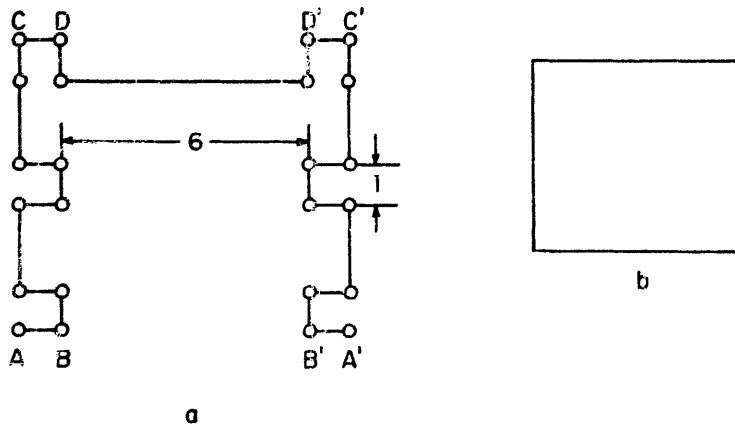


Fig. 3. The configuration *H*.

Lemma 1. *Among all Traveling Salesman paths having as endpoints two of the cities *A, A', B, B', C, C', D* and *D'*, there are 4 optimal paths with length 32, namely those with endpoints (*A, A'*) (shown in Fig. 3a), (*B, B'*), (*C, C'*), (*D, D'*).*

The *Exact Cover Problem* is the following: Given a family *F* of subsets of the finite set *U*, is there a subfamily *F'* of *F*, consisting of disjoint sets, such that *F'* covers *U*. This problem is known to be NP-Complete [5].

Consider an instance (not necessarily Euclidean) of the TSP, that is, a set *E* of cities $\{c_1, \dots, c_n\}$ and a distance function *d*. A subset *G* of *E* is a *b-component* (*b* an integer) if for all $c \in G$ we have

$$\min \{d(c, c') : c' \notin G\} \geq b$$

and

$$\max \{d(c, c') : c' \in G\} < b,$$

and G is maximal with respect to this property. Note that the b -components, if any, are disjoint, but do not necessarily exhaust E .

A k -path (k an integer) in an instance E is a set of k node-disjoint, not closed paths covering E . Traveling Salesman paths are thus 1-paths.

An instance E is b -compact if, for all positive integers k , an optimal k -path has length less than b plus the length of an optimal $(k + 1)$ -path. Note that b -components are trivially b -compact.

The following technical Lemma is needed:

Lemma 2. *Suppose that in an instance E of the TSP we have N a -components G_1, \dots, G_N , such that the distance between any two components is at least $2a$, and G_0 , the remaining part of E , is a -compact. Suppose that any optimal Traveling Salesman path of E has its endpoints on G , and that they do not contain links between any two a -components of E . Let L_1, \dots, L_N be the lengths of the optimal 1-paths of G_1, \dots, G_N and L_0 the length of the optimal $(N + 1)$ -path of G . If there is a 1-path P of E consisting of the union of an optimal $(N + 1)$ -path of G_0 , N optimal 1-paths of G_1, \dots, G_N and $2N$ edges of length a connecting a -components to G_0 , then P is optimal. If no such 1-path exists, the optimal 1-path of E has length greater than $L = L_0 + L_1 + \dots + L_N + 2Na$.*

Proof. Suppose that the optimal 1-path of E traverses G_i by a k_i -path, $i = 1, \dots, N$; then it traverses G_0 by a k_0 -path, where $k_0 = \sum k_i + 1$. The length of this 1-path will be $L' \geq L'_0 + L'_1 + \dots + L'_N + 2(k_0 - 1)a$, where L'_i is the length of the optimal k_i -path of G_i , $i = 0, 1, \dots, N$. Since all G_i are a -compact, we have that $L'_i \geq L_i - (k_i - 1)a$, $i = 1, \dots, N$ and $L'_0 \geq L_0 - (k_0 - N)a$. $L' \geq \sum_{i=1}^N [L_i - (k_i - 1)a] + 2(k_0 - 1)a + L_0 - (k_0 - N)a = L$. Moreover, the equality is obtained only in the case where $k_i = 1$, $i = 1, \dots, N$ and all paths are optimal. \square

We can now prove the following theorem:

Theorem 2. *The Euclidean path-TSP is NP-Complete.*

Proof. We will reduce the Exact Cover Problem to it. The construction is essentially an elaboration on the proof (appearing in [2]) of the NP-Completeness of the planar directed Hamiltonian path problem.

Suppose that we are given an instance of the Exact Cover Problem with $F = \{S_1, \dots, S_n\}$ and $U = \{u_1, \dots, u_m\}$. Let $p = \sum_{S \in F} |S|$. The basis for our construction is the "skeleton" shown schematically in Fig. 4. (An adequate value of a is 20 . $b = 4a + 2$). The skeleton consists of n 2-chains joined by 1-chains (in Fig. 6 n is even) with m copies of H between two consecutive 2-chains. Intuitively, the n rows of the skeleton correspond to the n sets of the exact cover problem, whereas the m

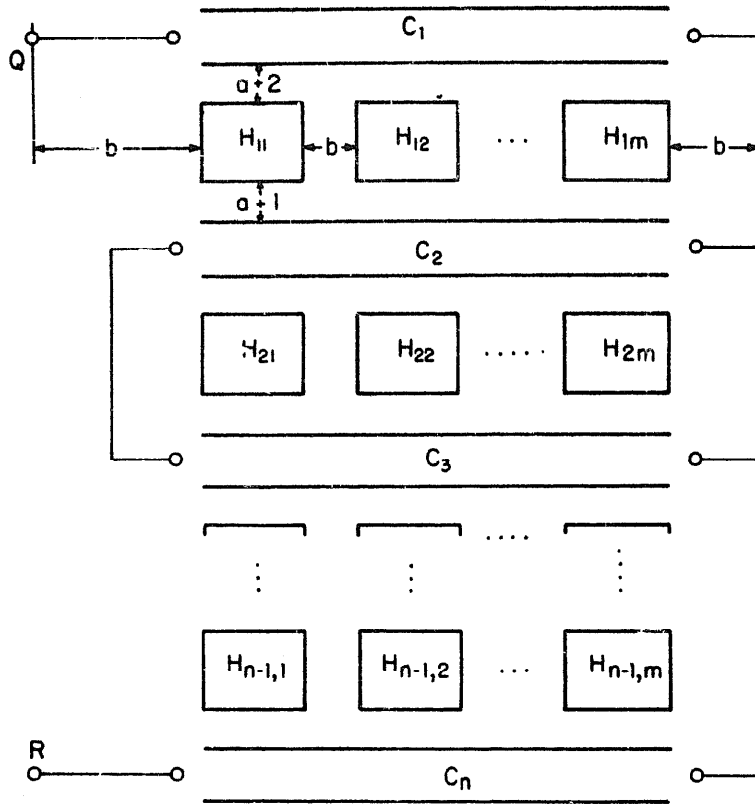


Fig. 4. The skeleton, schematically.

columns correspond to the m elements of U . Q and R are the endpoints of this configuration.

We note that, so far, only the parameters m and n are taken into account for the construction of the skeleton. The structure of the instance of the exact cover problem will affect the construction via the following modifications: In the chain C_j we replace the portion exactly above $H_{j,i}$ ($j \neq n$) (or equivalently, under $H_{j-1,i}$ if $j \neq 1$) by the configuration A of Fig. 5, if $u_i \notin S_j$, and by the configuration B of Fig. 6a if $u_i \in S_j$. (In Fig. 5 and 6a, 6b the configurations A and B are shown together with paths traversing them. The shortest distance shown is 1; the width of both configurations is 8, identical to that of H .) We observe that, no matter in which mode C_j is traversed by a Traveling Salesman path, from the configuration A we can also “visit”, in an optimal way, one of $H_{j,i}, H_{j-1,i}$ (in Fig. 5 C_j is traversed in

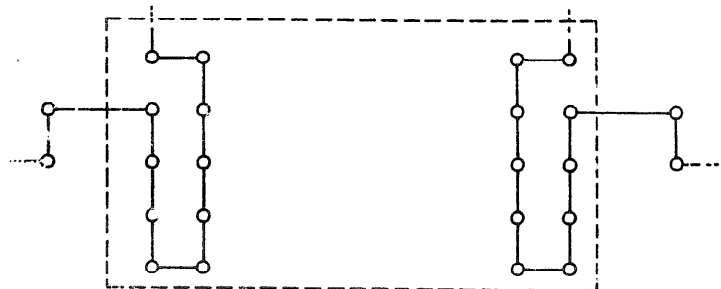


Fig. 5. The configuration A (within dotted lines).

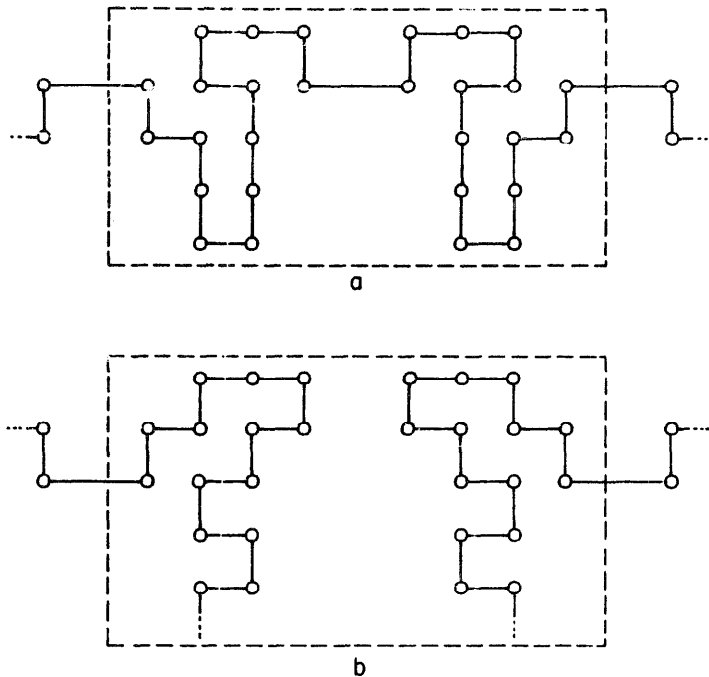


Fig. 6. The configuration B (within dotted lines).

mode 2 and $H_{j-1,i}$ is visited). From the configuration B we can visit one of $H_{j-1,i}, H_{j,i}$ in an optimal way only if C_j is traversed in mode 1 (Fig. 6b), whereas traversing C_j in mode 2 would force us to skip both $H_{j-1,i}, H_{j,i}$ (Fig. 6a) or deviate from optimality. Note that both configurations do not change the mode in which C_j is traversed.

The precision required for this configuration is the precision necessary in order to distinguish between a and $(a^2 + 1)^{1/2}$. Moreover, it should be clear that an optimal Traveling Salesman path for the resulting configuration must have Q and R as endpoints.

Now we observe that this configuration meets the hypotheses of Lemma 2 with $N = m(n - 1)$, $L_1 = \dots = L_N = 32$ and

$$L_0 = m(6a + 20) + n(4a + 13 + 2^{1/2}) + 8p + 2m - 2a - 11.$$

(By $2^{1/2}$ we denote the square root of 2 evaluated up to the precision required.)

We will examine when this configuration has an optimal path P , as described in the Lemma. Suppose that such a path P exists. P traverses all 1-chains in the obvious way, and all 2-chains in one of the two modes. It follows from the construction of A and B , that the mode of traversal of the 2-chains will not be affected by the presence of these configurations. Since its portion on G_0 has to be optimal, P must “visit” a component H from any configuration A encountered, and it must return (by Lemma 1) to the symmetric node of A , since its portion on H must be optimal, too. If P encounters a configuration B and the corresponding chain is traversed in mode 1, P will also visit a component H . However, if the corresponding chain is traversed in mode 2, P will traverse B without visiting any

configuration H , since all edges between G_0 and copies of H must have length a (and not $(a^2 + 1)^{1/2}$). Moreover this must happen exactly once for each "column" of the configuration, since there are $n - 1$ copies of H and n configurations A and B in each column. Hence, if we consider the fact that C_j is traversed in mode 2 (resp. mode 1) to mean that S_j is (resp. is not) contained in the cover, we see that the existence of path P , as described in Lemma 2, implies that F' covers U exactly.

Conversely, if U has an exact cover F' in F , we assign, as above, modes to the chains according to whether or not $S_j \in F'$. We are then in a position to exhibit a path P meeting the requirements of the Lemma. Hence the configuration at hand has a Traveling Salesman path of length no more than

$$L = mn(52 + 8a) + n(4a + 13 + 2^{1/2}) + 8p - m(30 + 2a) - 2a - 11$$

if and only if the given instance of the Exact Cover Problem is solvable. \square

The following result was independently shown in [1].

Theorem 3. *The Euclidean tour-TSP is NP-Complete.*

Proof. The construction is identical, except that we connect Q and a with a 1-chain and increase L accordingly. \square

Finally we note that, since all edges used in P are either vertical or horizontal (except for the edges at the ends of the 2-chains, for which a simple independent argument is applicable) the same construction can be used to prove that the rectilinear (or "Manhattan") TSP is NP-Complete¹.

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¹ This remark is due to Johnson [4].