ELSEVIER

Theoretical Computer Science 168 (1996) 303–320

Theoretical Computer Science

Universality of a reversible two-counter machine

Kenichi Morita *

Department of Industrial and Systems Engineering, Faculty of Engineering, Hiroshima University, Higashi-Hiroshima-shi 739, Japan

Abstract

A k-counter machine (CM(k)) is an automaton having k counters as an auxiliary memory. It has been shown by Minsky that a CM(2) can simulate any Turing machine and thus it is universal. In this paper, we investigate the computing ability of reversible (i.e., backward deterministic) CMs. We first show that any irreversible CM(k) can be simulated by a reversible CM($k+2$). In this simulation, however, the reversible CM($k+2$) leaves a large number as a garbage in some counter when it halts. We then show that, if k more counters are added, this garbage information is erased reversibly. Finally, we prove that any reversible CM(k) ($k = 1, 2, 3, \ldots$) can be simulated by a reversible CM(2). From these results computation-universality of a reversible CM(2) is established.

1. Introduction

A k-counter machine (CM(k)) is an automaton with k counters, each of which can store an arbitrary nonnegative integer. In one time step, a finite-state control of a CM(k) can increment or decrement the contents of a counter by one, or can test whether it is 0 or not. Minsky [8] showed that a CM(2) can simulate any Turing machine and thus it is universal.

In this paper, we study a "reversible" version of CM. A reversible computing system is a backward deterministic system, i.e., roughly speaking, each computational configuration of it has at most one predecessor. Until now, various reversible systems, such as reversible Turing machines, reversible cellular automata, reversible logic gates, have been studied (see, e.g., [4, 12, 13, 15] for a general survey). One interesting point of a reversible system is that it is closely related to physical reversibility and the problem of energy dissipation in a computing process. It is known to be possible to construct a reversible computer that works without dissipating energy in an ideal situation [2, 3, 5]. It is also interesting from a computational viewpoint that several systems have universal computing ability even if reversibility constraint is added. Bennett [1] showed that

* Tel.: +81 (824) 24 77 09; fax: +81 (824) 22 71 95; e-mail: morita@ke.sys.hiroshima-u.ac.jp.

PII S0304-3975(96)00081-3

any (irreversible) Turing machine can be simulated by a reversible one without leaving garbage symbols on the tape. Reversible cellular automata have also been shown to be computation-universal for both one-dimensional [10] and two-dimensional cases [7, 11, 14].

Here, we investigate the computing ability of reversible CMs. In Section 2, we give definitions of a CM and its reversibility. In Section 3, we first show that any irreversible CM(k) M can be simulated by a reversible CM($k+2$) M'. But M' leaves a large number as a garbage, in which a "history" of computation is encoded, when it halts. We then show a garbage-less construction of a CM($2k+2$) M'' that simulates M by applying the method of Bennett [1] to CM. In Section 4, we prove that any reversible CM(k) ($k = 1,2,3,\ldots$) can be simulated by a reversible CM(2). From these results computation-universality of a reversible CM(2) is obtained.

2. Definitions

We define a counter machine (CM) as a kind of multi-tape Turing machine whose heads are read-only ones and whose tapes are all blank except the leftmost squares as shown in Fig. 1 (a similar formulation is used e.g. in [6]). This definition is convenient for giving the notion of reversibility.

Definition 2.1. A *k-counter machine* (CM(k)) is a system

$$M = (k, Q, \delta, q_0, q_f),$$

where k is the number of tapes (or counters), Q is a nonempty finite set of internal states, $q_0 \in Q$ is an initial state, and $q_f \in Q$ is a final (halting) state. M uses

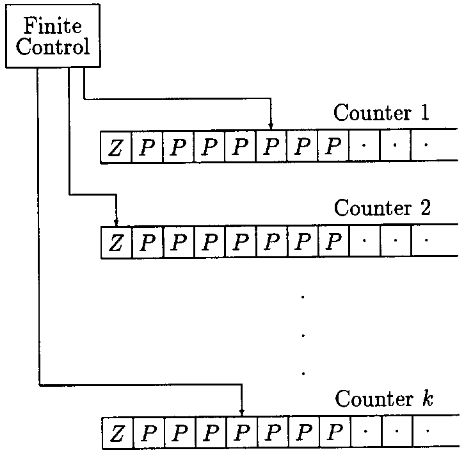

Fig. 1. A k-counter machine (CM(k)).

$\{Z,P\}$ as a tape alphabet (P is a blank symbol). δ is a move relation which is a subset of $(Q \times \{1,\ldots,k\} \times \{Z,P\} \times Q) \cup (Q \times \{1,\ldots,k\} \times \{-,0,+\} \times Q)$ (where "−", "0", and "+" denote left-shift, no-shift, and right-shift of a head, respectively). Tapes are one-way (rightward) infinite. The leftmost squares of the tapes contain the symbol "Z"s, and all the other squares contain "P"s (Z and P stand for "zero" and "positive").

Each element of δ is called a *quadruple*, and is either of the form

$$[q,i,s,q'] \quad \text{or} \quad [q,i,d,q'],$$

where $q,q' \in Q$, $i \in \{1,\ldots,k\}$, $s \in \{Z,P\}$, $d \in \{-,0,+\}$. The quadruple $[q,i,s,q']$ means that if M is in the state q and the ith head is reading the symbol s then change the state into q'. It is used to test whether the contents of a counter are zero or positive. On the other hand, $[q,i,d,q']$ means that if M is in the state q then shift the ith head to the direction d and change the state into q'. It is used to increment or decrement a counter by one (or make no change if $d = 0$).

Definition 2.2. An *instantaneous description* (ID) of a CM(k) $M = (k,Q,\delta,q_0,q_f)$ is a $(k+1)$-tuple

$$(q,n_1,n_2,\ldots,n_k) \in Q \times \mathbf{N}^k,$$

where $\mathbf{N} = \{0,1,\ldots\}$. It represents that M is in the state q and the counter i keeps n_i (we assume the position of the leftmost square of a tape is 0). The transition relation $\vdash_{\overline{M}}$ over IDs of M is defined as follows:

$$\begin{aligned} &(q,n_1,\ldots,n_{i-1},n_i,n_{i+1},\ldots,n_k) \\ \vdash_{\overline{M}}\ &(q',n_1,\ldots,n_{i-1},n_i',n_{i+1},\ldots,n_k) \end{aligned}$$

holds iff one of the following conditions (1)–(5) is satisfied.

(1) $[q,i,Z,q'] \in \delta$ and $n_i = n_i' = 0$.
(2) $[q,i,P,q'] \in \delta$ and $n_i = n_i' > 0$.
(3) $[q,i,-,q'] \in \delta$ and $n_i - 1 = n_i'$.
(4) $[q,i,0,q'] \in \delta$ and $n_i = n_i'$.
(5) $[q,i,+,q'] \in \delta$ and $n_i + 1 = n_i'$.

We denote reflexive and transitive closure of $\vdash_{\overline{M}}$ by $\vdash_{\overline{M}}^{*}$, and n-step transition by $\vdash_{\overline{M}}^{n}$ $(n = 0,1,\ldots)$.

Definition 2.3. Let $M = (k,Q,\delta,q_0,q_f)$ be a CM(k), and

$$\alpha_1 = [p_1,i_1,x_1,p_1'] \quad \text{and} \quad \alpha_2 = [p_2,i_2,x_2,p_2']$$

be two distinct quadruples in δ. We say α_1 and α_2 *overlap in domain* iff the following holds, where $D = \{-,0,+\}$.

$$p_1 = p_2 \ \wedge\ [i_1 \neq i_2 \ \vee\ x_1 = x_2 \ \vee\ x_1 \in D \ \vee\ x_2 \in D]$$

We say α_1 and α_2 *overlap in range* iff the following holds.

$$p_1' = p_2' \ \wedge \ [i_1 \neq i_2 \ \vee \ x_1 = x_2 \ \vee \ x_1 \in D \ \vee \ x_2 \in D]$$

A quadruple α is called *deterministic* (*reversible*, respectively) iff there is no other quadruple in δ which overlaps in domain (range) with α. M is called *deterministic* (*reversible*, respectively) iff every quadruple in δ is deterministic (reversible).

For example, the following pair

$$[q_1, 2, P, q_3] \quad \text{and} \quad [q_4, 2, +, q_3]$$

overlaps in range, while the pair

$$[q_1, 2, Z, q_3] \quad \text{and} \quad [q_4, 2, P, q_3]$$

does not. As seen from this definition, every ID of a deterministic (reversible, respectively) CM(k) has at most one ID that immediately follows (precedes) it. Hereafter, we consider only deterministic reversible and deterministic irreversible CM(k)s.

3. Simulating an irreversible counter machine by a reversible one

In this section we show that any (irreversible) CM can be simulated by a reversible one by adding some extra counters. As a preparation, we define a notion of "state-degeneration degree" for CMs, and show a lemma on it (a similar notion for Turing machines is in [9]).

Definition 3.1. Let $M = (k, Q, \delta, q_0, q_f)$ be a deterministic CM(k). A state $q \in Q$ is called *state-degenerative* iff there are at least two distinct quadruples $[q_1, i_1, x_1, q]$ and $[q_2, i_2, x_2, q]$ in δ. If there are exactly k such quadruples in δ, we say that the *state-degeneration degree* of q is k, and denote it as sdeg(q)=k. That is,

$$\mathrm{sdeg}(q) = \ |\{\alpha \in \delta \mid \exists q', i, x \ (\alpha = [q', i, x, q])\}|.$$

State-degeneration degree of M is defined as

$$\mathrm{sdeg}(M) = \ \max\{\mathrm{sdeg}(q) \mid q \in Q\}.$$

Lemma 3.1. *For any deterministic* CM(k) $M = (k, Q, \delta, q_0, q_f)$, *there is a deterministic* CM(k) $M' = (k, Q', \delta', q_0, q_f)$ *with* sdeg(M')$\leqslant 2$ *such that*

$$(q_0, m_1, \ldots, m_k) \mathrel{\vdash^{*}_{M}} (q_f, n_1, \ldots, n_k)$$
$$\text{iff}$$
$$(q_0, m_1, \ldots, m_k) \mathrel{\vdash^{*}_{M'}} (q_f, n_1, \ldots, n_k)$$

holds for all $m_1, \ldots, m_k, n_1, \ldots, n_k \in \mathbf{N}$.

Proof. Choose a state $q \in Q$ such that $\mathrm{sdeg}(q) > 2$ (if no such q exists, we have done). If $\mathrm{sdeg}(q) = k$ there are k quadruples

$$[q_{r_1}, i_1, x_1, q],\ [q_{r_2}, i_2, x_2, q],\ \ldots,\ [q_{r_k}, i_k, x_k, q].$$

In M', these k quadruples are replaced by the $2k-2$ quadruples shown below, where $q_{s_1}, \ldots, q_{s_{k-2}}$ are new states (see Fig. 2).

Repeating this procedure for all $q \in Q$ such that $\mathrm{sdeg}(q) > 2$, a CM(k) M' with $\mathrm{sdeg}(M') \leqslant 2$ is obtained. It is clear that M' simulates M. $\square$

We now show that any irreversible CM can be simulated by a reversible one by adding two extra counters to keep a "history" of a computation (but the history is left as a garbage when it halts).

$$\begin{array}{lllll}
(1) & [q_{r_1}, & i_1, & x_1, & q_{s_1}] \\
 & [q_{r_2}, & i_2, & x_2, & q_{s_1}] \\
(2) & [q_{s_1}, & 1, & 0, & q_{s_2}] \\
 & [q_{r_3}, & i_3, & x_3, & q_{s_2}] \\
 & & \vdots & & \\
(j) & [q_{s_{j-1}}, & 1, & 0, & q_{s_j}] \\
 & [q_{r_{j+1}}, & i_{j+1}, & x_{j+1}, & q_{s_j}] \\
 & & \vdots & & \\
(k-2) & [q_{s_{k-3}}, & 1, & 0, & q_{s_{k-2}}] \\
 & [q_{r_{k-1}}, & i_{k-1}, & x_{k-1}, & q_{s_{k-2}}] \\
(k-1) & [q_{s_{k-2}}, & 1, & 0, & q] \\
 & [q_{r_k}, & i_k, & x_k, & q]
\end{array}$$

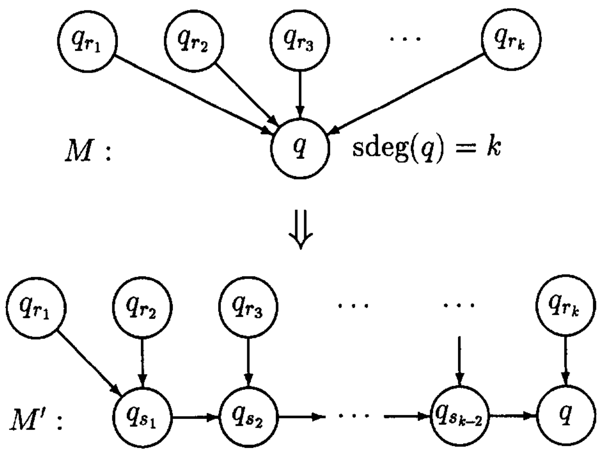

Fig. 2. Reducing the state-degeneration degree to 2 by adding new states $q_{s_1}, \ldots, q_{s_{k-2}}$.

Theorem 3.1. *For any deterministic* CM(k) $M = (k, Q, \delta, q_0, q_f)$, *there is a deterministic reversible* CM($k+2$) $M' = (k+2, Q', \delta', q_0, q_f)$, *such that*

$$(q_0, m_1, \ldots, m_k) \mid\!\frac{*}{M}\; (q_f, n_1, \ldots, n_k)$$

$$\text{iff}$$

$$\exists h \in \mathbf{N}\,[(q_0, m_1, \ldots, m_k, 0, 0) \mid\!\frac{*}{M'}\; (q_f, n_1, \ldots, n_k, h, 0)]$$

holds for all $m_1, \ldots, m_k, n_1, \ldots, n_k \in \mathbf{N}$.

Proof. By Lemma 3.1 we assume sdeg(M) = 2 (if sdeg(M) = 1 then M is already reversible, so we need not consider this case). Further assume M has no quadruple in which q_0 appears as the fourth element (i.e., q_0 appears only at time 0).

We now construct CM($k+2$) M' that simulates M. M' uses k counters to simulate those of M, and keeps the history of its computation by the counter $k+1$ in order to make M' reversible. The counter $k+2$ is for working.

The state set Q' and the quadruple set δ' of M' are constructed as follows.

1. For each reversible quadruple $[q_s, i_s, x_s, q_t]$ in δ, include the states q_s and q_t in Q', and include the following quadruple in δ'.

$$[q_s,\ i_s,\ x_s,\ q_t] \tag{1.1}$$

2. For each pair of quadruples $[q_r, i_r, x_r, q_t]$ and $[q_s, i_s, x_s, q_t]$ in δ which overlap in range, include the states $q_r, q_s, q_t, q(r,t,j), q(s,t,j), q(t,\ell)$ $(j = 1, \cdots, 5,\ \ell = 1, \cdots, 6)$ in Q', and the following quadruples in δ'.

$$
\begin{array}{llll}
[q_r, & i_r & x_r, & q(r,t,1)] \\
[q(r,t,1), & k+2, & Z, & q(r,t,2)] \\
[q(r,t,2), & k+1, & Z, & q(t,1)] \\
[q(r,t,2), & k+1, & P, & q(r,t,3)] \\
[q(r,t,3), & k+1, & -, & q(r,t,4)] \\
[q(r,t,4), & k+2, & +, & q(r,t,5)] \\
[q(r,t,5), & k+2, & P, & q(r,t,2)]
\end{array}
\qquad
\begin{array}{r}
(2.1)\\(2.2)\\(2.3)\\(2.4)\\(2.5)\\(2.6)\\(2.7)
\end{array}
$$

$$
\begin{array}{llll}
[q_s, & i_s, & x_s, & q(s,t,1)] \\
[q(s,t,1), & k+2, & Z, & q(s,t,2)] \\
[q(s,t,2), & k+1, & Z, & q(t,5)] \\
[q(s,t,2), & k+1, & P, & q(s,t,3)] \\
[q(s,t,3), & k+1, & -, & q(s,t,4)] \\
[q(s,t,4), & k+2, & +, & q(s,t,5)] \\
[q(s,t,5), & k+2, & P, & q(s,t,2)] \\
[q(t,1), & k+2, & Z, & q_t] \\
[q(t,1), & k+2, & P, & q(t,2)] \\
[q(t,2), & k+2, & -, & q(t,3)]
\end{array}
\qquad
\begin{array}{r}
(2.8)\\(2.9)\\(2.10)\\(2.11)\\(2.12)\\(2.13)\\(2.14)\\(2.15)\\(2.16)\\(2.17)
\end{array}
$$

$$[q(t,3),\ k+1,\ +,\ q(t,4)] \tag{2.18}$$
$$[q(t,4),\ k+1,\ P,\ q(t,5)] \tag{2.19}$$
$$[q(t,5),\ k+1,\ +,\ q(t,6)] \tag{2.20}$$
$$[q(t,6),\ k+1,\ P,\ q(t,1)] \tag{2.21}$$

When M executes a reversible quadruple, M' simply does so by (1.1). On the other hand, when M executes an irreversible quadruple, M' writes the information which quadruple is used into the counter $k+1$. This is done by (2.1)–(2.21). Since $\mathrm{sdeg}(M) = 2$, there are always two possibilities of executed quadruple, say $[q_r, i_r, x_r, q_t]$ and $[q_s, i_s, x_s, q_t]$. Thus the choice sequence of quadruples (i.e., history) can be expressed in a binary number, and M' holds it in the counter $k+1$.

We first consider the case $[q_r, i_r, x_r, q_t]$ is used by M. Assume the counter $k+1$ keeps n, which represents the history up to this moment, and the counter $k+2$ keeps 0. After simulating the operation of M by (2.1), M' transfers the number n from the counter $k+1$ to the counter $k+2$ by (2.2)–(2.7). Then, using (2.15)–(2.21), M' multiplies the contents of the counter $k+2$ by 2. Thus the result $2n$ is obtained in the counter $k+1$. Next, consider the case $[q_s, i_s, x_s, q_t]$ is used by M. Quadruples (2.8)–(2.14) act essentially the same as (2.1)–(2.7). However, in this case, the quadruple (2.20) (rather than (2.15)) is executed first among (2.15)–(2.21). By this, the result $2n+1$ is obtained in the counter $k+1$.

Consequently, the information which quadruple was executed is kept as the least significant bit of the number in the counter $k+1$. Due to this operation M' becomes reversible. Indeed, it is easily verified that M' is deterministic and reversible (for example, the pairs of quadruples [(2.2), (2.7)], [(2.9), (2.14)], [(2.3), (2.21)], and [(2.10), (2.19)] do not overlap in range). □

Example 3.1. Consider a deterministic irreversible CM(2) $M_a = (2, Q, \{Z, P\}, \delta, q_0, q_f, P)$ having the following quadruples as δ.

$[q_0,\ 1,\ 0,\ q_1]$	$(M_a\text{-}1)$
$[q_1,\ 2,\ Z,\ q_f]$	$(M_a\text{-}2)$
$[q_1,\ 2,\ P,\ q_2]$	$(M_a\text{-}3)$
$[q_2,\ 2,\ -,\ q_3]$	$(M_a\text{-}4)$
$[q_3,\ 1,\ +,\ q_1]$	$(M_a\text{-}5)$

M_a adds two numbers given in the counters 1 and 2, and stores the result in the counter 1. For example, $(q_0, 2, 2) \mid\!\frac{8}{M_a}\ (q_f, 4, 0)$. Note that $\mathrm{sdeg}(M_a) = 2$. The state transition of M_a is shown in Fig. 3.

A deterministic reversible CM(4) $M_a' = (4, Q', \{Z, P\}, \delta', q_0, q_f, P)$ constructed by the method of Theorem 3.1 has the following 24 quadruples.

1. Quadruples corresponding to the reversible ones $(M_a\text{-}2)$–$(M_a\text{-}4)$ of M_a:

$[q_1,\ 2,\ Z,\ q_f]$
$[q_1,\ 2,\ P,\ q_2]$
$[q_2,\ 2,\ -,\ q_3]$

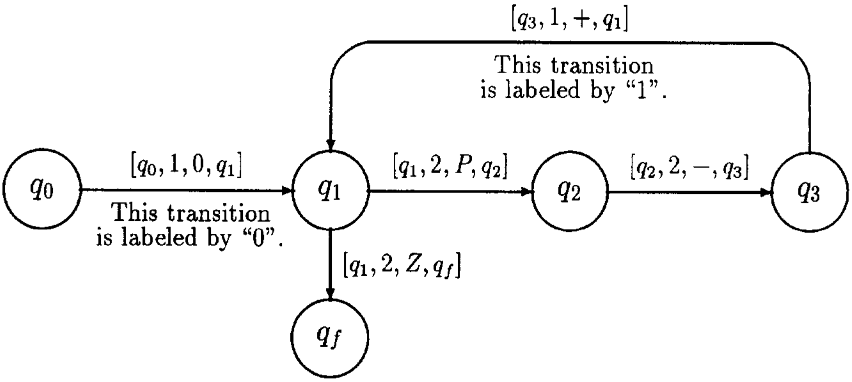

Fig. 3. A CM(2) M_a that performs addition.

2. Quadruples corresponding to the irreversible pair $[q_0, 1, 0, q_1]$ (M_a-1) and $[q_3, 1, +, q_1]$ (M_a-5):

$$
\begin{array}{llll}
[q_0, & 1, & 0, & q(0,1,1)] \\
[q(0,1,1), & 4, & Z, & q(0,1,2)] \\
[q(0,1,2), & 3, & Z, & q(1,1)] \\
[q(0,1,2), & 3, & P, & q(0,1,3)] \\
[q(0,1,3), & 3, & -, & q(0,1,4)] \\
[q(0,1,4), & 4, & +, & q(0,1,5)] \\
[q(0,1,5), & 4, & P, & q(0,1,2)]
\end{array}
$$

$$
\begin{array}{llll}
[q_3, & 1, & +, & q(3,1,1)] \\
[q(3,1,1), & 4, & Z, & q(3,1,2)] \\
[q(3,1,2), & 3, & Z, & q(1,5)] \\
[q(3,1,2), & 3, & P, & q(3,1,3)] \\
[q(3,1,3), & 3, & -, & q(3,1,4)] \\
[q(3,1,4), & 4, & +, & q(3,1,5)] \\
[q(3,1,5), & 4, & P, & q(3,1,2)]
\end{array}
$$

$$
\begin{array}{llll}
[q(1,1), & 4, & Z, & q_1] \\
[q(1,1), & 4, & P, & q(1,2)] \\
[q(1,2), & 4, & -, & q(1,3)] \\
[q(1,3), & 3, & +, & q(1,4)] \\
[q(1,4), & 3, & P, & q(1,5)] \\
[q(1,5), & 3, & +, & q(1,6)] \\
[q(1,6), & 3, & P, & q(1,1)]
\end{array}
$$

When M_a executes the irreversible quadruple $[q_0, 1, 0, q_1]$, bit "0" is attached to the binary number kept in the counter 3 as the least significant bit. On the other hand, when M_a executes $[q_3, 1, +, q_1]$, bit "1" is attached. For example, the addition 2+2 is carried out by M'_a in the following way.

$$
\begin{aligned}
& (q_0,2,2,0,0) \vdash^{4}_{M'_a} (q_1,2,2,0,0) \\
\vdash^{1}_{M'_a} & (q_2,2,2,0,0) \vdash^{1}_{M'_a} (q_3,2,1,0,0) \\
\vdash^{6}_{M'_a} & (q_1,3,1,1,0) \vdash^{1}_{M'_a} (q_2,3,1,1,0) \\
\vdash^{1}_{M'_a} & (q_3,3,0,1,0) \vdash^{16}_{M'_a} (q_1,4,0,3,0) \\
\vdash^{1}_{M'_a} & (q_f,4,0,3,0)
\end{aligned}
$$

The reversible CM$(k+2)$ M' constructed in Theorem 3.1, however, leaves in general a very large number in the counter $k+1$ when it halts. This number is in fact a garbage information, but it cannot be simply erased by a reversible CM. If we want to erase it reversibly, we must add a backward computing process that "undoes" the forward computing process as in the case of a reversible Turing machine [1] (of course, copying process of results should be inserted between the forward and backward computing processes). This method for CM is shown in the following theorem.

Theorem 3.2. *For any deterministic CM(k) $M = (k, Q, \delta, q_0, q_f)$, there is a deterministic reversible CM$(2k+2)$ $M'' = (2k+2, Q'', \delta'', q_0,\ p_0)$, such that*

$$(q_0, m_1, \ldots, m_k) \vdash^{*}_{M} (q_f, n_1, \ldots, n_k)$$

iff

$$(q_0, m_1, \ldots, m_k,\ 0,0,0,\ldots,0) \vdash^{*}_{M''} (p_0, m_1, \ldots, m_k, 0, 0, n_1, \ldots, n_k)$$

holds for all $m_1, \ldots, m_k, n_1, \ldots, n_k \in \mathbf{N}$.

Proof. Assume M has no quadruple in which q_0 appears as the fourth element. By using the method in Theorem 3.1, we first convert M to an equivalent reversible CM$(k+2)$ $M' = (k+2, Q', \delta', q_0, q_f)$. We then construct M'' from M'. Like M', M'' uses the counters 1 through k to simulate those of M, and the counters $k+1$ and $k+2$ for keeping history and for working. The remaining k counters are used for recording the result of the computation. The entire computation process of M'' is divided into three stages. They are forward computation stage, copy stage, and backward computation stage. The state set Q'' and the quadruple set δ'' of M'' are as follows.

I. *Forward computation stage*: Internal states and quadruples needed for this stage are exactly the same as M' in Theorem 3.1.

II. *Copy stage*: In this stage, the contents of the counters 1 through k are copied to the counters $k+3$ through $2k+2$ using the counter $k+2$ for working.

1. Include $c(1,1)$ in Q'', and include the following quadruple in δ''.

$$[q_f,\ k+2,\ Z,\ c(1,1)] \tag{3.1}$$

2. For each $i \in \{1,\dots,k\}$, include $c(i,1),\dots,c(i,5)$, $c(i+1,1)$, and $d(i,1),\dots,d(i,6)$ in Q'', and include the following quadruples in δ''.

$$[c(i,1),\quad i,\quad Z,\quad d(i,1)] \tag{4.1}$$
$$[c(i,1),\quad i,\quad P,\quad c(i,2)] \tag{4.2}$$
$$[c(i,2),\quad i,\quad -,\quad c(i,3)] \tag{4.3}$$
$$[c(i,3),\quad k+2,\quad +,\quad c(i,4)] \tag{4.4}$$
$$[c(i,4),\quad k+2,\quad P,\quad c(i,1)] \tag{4.5}$$
$$[d(i,1),\quad k+2,\quad Z,\quad c(i+1,1)] \tag{4.6}$$
$$[d(i,1),\quad k+2,\quad P,\quad d(i,2)] \tag{4.7}$$
$$[d(i,2),\quad k+2,\quad -,\quad d(i,3)] \tag{4.8}$$
$$[d(i,3),\quad i,\quad +,\quad d(i,4)] \tag{4.9}$$
$$[d(i,4),\quad i+k+2,\quad +,\quad d(i,5)] \tag{4.10}$$
$$[d(i,5),\quad i,\quad P,\quad d(i,1)] \tag{4.11}$$

The contents of the counter i is first transferred to the counter $k+2$ by (4.1)–(4.5), and then to the counters i and $i+k+2$ by (4.6)–(4.11).

3. Include p_f in Q'', and the following quadruple in δ''.

$$[c(k+1,1),\ k+2,\ Z,\ p_f] \tag{5.1}$$

III. *Backward computation stage*: In this stage, the computation performed in the forward computation stage is undone, and thus the contents of counter $k+1$ is erased reversibly. We define x^{-1} for $x \in \{+,0,-,Z,P\}$ as follows.

$$x^{-1} = \begin{cases} - & \text{if } x = + \\ 0 & \text{if } x = 0 \\ + & \text{if } x = - \\ Z & \text{if } x = Z \\ P & \text{if } x = P \end{cases}$$

For each quadruple $[q_s, i_s, x_s, q_t]$ in δ', include the states p_s and p_t in Q'', and the following quadruple in δ''.

$$[p_t,\ i_s,\ x_s^{-1},\ p_s] \tag{6.1}$$

Note that (6.1) is the reverse quadruple of $[q_s, i_s, x_s, q_t]$ in the sense that it undoes the operation of the latter quadruple. Since quadruples in δ' are all deterministic and reversible, (6.1) is also so.

By the above quadruples (1.1)–(6.1), M'' acts as follows for all $m_1,\dots,m_k,n_1,\dots,n_k \in \mathbf{N}$ such that $(q_0,m_1,\dots,m_k) \mathrel{|\!\!\frac{*}{M}} (q_f,n_1,\dots,n_k)$:

$$(q_0,m_1,\dots,m_k,0,0,0,\dots,0)$$

$$\mathrel{|\!\!\frac{*}{M''}} (q_f,n_1,\ \dots,n_k,\ h,0,0,\dots,0)$$

$$\vdash_{M''}^{*} (p_f, n_1, \ldots, n_k, h, 0, n_1, \ldots, n_k)$$

$$\vdash_{M''}^{*} (p_0, m_1, \ldots, m_k, 0, 0, n_1, \ldots, n_k)$$

for some $h \in \mathbf{N}$. $\square$

Remark. In the copy stage of the above construction, all the counters 1 through k are copied. But it is of course not necessary. We can copy only needed results and reduce the number of counters.

4. Universality of a reversible two-counter machine

The next proposition has been shown by Minsky [8].

Proposition 4.1 (Minsky [8]). *For any Turing machine T there is a* CM(5) *M that simulates T.*

His formulation of CM is slightly different from ours. But, it is easily seen (from the proof of Proposition 4.1) that five counters are enough to simulate a Turing machine for our CM.

Minsky further showed that any k-counter machine can be simulated by a two-counter machine by using a Gödel number.

Proposition 4.2 (Minsky [8]). *For any* CM(k) *M ($k = 1, 2, \ldots$) there is a* CM(2) *M' that simulates M.*

We now show a reversible version of Proposition 4.2.

Theorem 4.1. *For any deterministic reversible* CM(k) *$M = (k, Q, \delta, q_0, q_f)$, there is a deterministic reversible* CM(2) *$M' = (2, Q', \delta', q_0', q_f')$, such that*

$$(q_0, m_1, \ldots, m_k) \vdash_{M}^{*} (q_f, n_1, \ldots, n_k)$$
$$\text{iff}$$
$$(q_0, p_1^{m_1} \cdots p_k^{m_k}, 0) \vdash_{M'}^{*} (q_f, p_1^{n_1} \cdots p_k^{n_k}, 0)$$

holds for all $m_1, \ldots, m_k, n_1, \ldots, n_k \in \mathbf{N}$, where p_i denotes the ith prime number (i.e., $p_1 = 2$, $p_2 = 3$, $p_3 = 5, \ldots$).

Proof. From CM(k) M, the sets Q' and δ' of M' are constructed as follows.

1. For each quadruple $[q_r, i, 0, q_s]$ in δ, include the states q_r and q_s in Q', and include the same quadruple in δ'.

$$[q_r,\ 1,\ 0,\ q_s] \tag{7.1}$$

2. For each quadruple $[q_r, i, +, q_s]$ in δ, include the states $q_r, q_s, q(r,j), q(r,6,\ell)$ ($j = 1,\ldots,7$, $\ell = 1,\ldots,p_i$) in Q', and the following $p_i + 10$ quadruples in δ'.

$$\begin{array}{llll}
[q_r, & 2, Z, & q(r,1)] & (8.1) \\
[q(r,1), & 1, Z, & q(r,5)] & (8.2) \\
[q(r,1), & 1, P, & q(r,2)] & (8.3) \\
[q(r,2), & 1, -, & q(r,3)] & (8.4) \\
[q(r,3), & 2, +, & q(r,4)] & (8.5) \\
[q(r,4), & 2, P, & q(r,1)] & (8.6) \\
[q(r,5), & 2, Z, & q_s] & (8.7) \\
[q(r,5), & 2, P, & q(r,6)] & (8.8) \\
[q(r,6), & 2, -, & q(r,6,1)] & (8.9) \\
[q(r,6,1), & 1, +, & q(r,6,2)] & (8.10.1) \\
[q(r,6,2), & 1, +, & q(r,6,3)] & (8.10.2) \\
 & \vdots & & \\
[q(r,6,p_i-1), & 1, +, & q(r,6,p_i)] & (8.10.p_i-1) \\
[q(r,6,p_i), & 1, +, & q(r,7)] & (8.10.p_i) \\
[q(r,7), & 1, P, & q(r,5)] & (8.11)
\end{array}$$

By quadruples (8.1)–(8.6) the contents of the counter 1 is transferred to the counter 2. Then by (8.7)–(8.11) it is multiplied by p_i and stored in the counter 1. In this way, $[q_r, i, +, q_s]$ of M is simulated by M'. It is easy to verify that the above quadruples are all deterministic and reversible in δ', since $[q_r, i, +, q_s]$ is deterministic and reversible in δ.

3. For each quadruple $[q_r, i, -, q_s]$ in δ, include the states $q_r, q_s, q(r,j), q(r,5,\ell)$ ($j = 1,\ldots,7$, $\ell = 1,\ldots,p_i$) in Q', and the following $p_i + 10$ quadruples in δ'.

$$\begin{array}{llll}
[q_r, & 2, Z, & q(r,1)] & (9.1) \\
[q(r,1), & 1, Z, & q(r,5)] & (9.2) \\
[q(r,1), & 1, P, & q(r,2)] & (9.3) \\
[q(r,2), & 1, -, & q(r,3)] & (9.4) \\
[q(r,3), & 2, +, & q(r,4)] & (9.5) \\
[q(r,4), & 2, P, & q(r,1)] & (9.6) \\
[q(r,5), & 2, Z, & q_s] & (9.7) \\
[q(r,5), & 2, P, & q(r,5,1)] & (9.8) \\
[q(r,5,1), & 2, -, & q(r,5,2)] & (9.9.1) \\
[q(r,5,2), & 2, -, & q(r,5,3)] & (9.9.2) \\
 & \vdots & & \\
[q(r,5,p_i-1), & 2, -, & q(r,5,p_i)] & (9.9.p_i-1) \\
[q(r,5,p_i), & 2, -, & q(r,6)] & (9.9.p_i) \\
[q(r,6), & 1, +, & q(r,7)] & (9.10) \\
[q(r,7), & 1, P, & q(r,5)] & (9.11)
\end{array}$$

Quadruples (9.1)–(9.6) are just the same as (8.1)–(8.6). By (9.7)–(9.11) the contents of the counter 2 is divided by p_i and stored in the counter 1, and thus $[q_r, i, -, q_s]$ of M is simulated. It is also easy to verify that the above quadruples are all deterministic and reversible.

4. For each pair of quadruples $[q_r, i, Z, q_s]$ and $[q_r, i, P, q_t]$ in δ, include the states q_r, $q(r,j,1)$, $q(r,j,2)$ $(j = 0,\ldots,p_i)$ in Q', and the following $3p_i + 3$ quadruples in δ'.

$$
\begin{array}{llllr}
[q_r, & 2, & Z, & q(r,0,1)] & (10.0) \\
[q(r,0,1), & 1, & Z, & q'(t,0,1)] & (10.0.1) \\
[q(r,0,1), & 1, & P, & q(r,0,2)] & (10.0.2) \\
[q(r,0,2), & 1, & -, & q(r,1,1)] & (10.0.3) \\
[q(r,1,1), & 1, & Z, & q'(s,1,1)] & (10.1.1) \\
[q(r,1,1), & 1, & P, & q(r,1,2)] & (10.1.2) \\
[q(r,1,2), & 1, & -, & q(r,2,1)] & (10.1.3) \\
[q(r,2,1), & 1, & Z, & q'(s,2,1)] & (10.2.1) \\
[q(r,2,1), & 1, & P, & q(r,2,2)] & (10.2.2) \\
[q(r,2,2), & 1, & -, & q(r,3,1)] & (10.2.3) \\
 & \vdots & & & \\
[q(r,p_i-2,1), & 1, & Z, & q'(s,p_i-2,1)] & (10.p_i-2.1) \\
[q(r,p_i-2,1), & 1, & P, & q(r,p_i-2,2)] & (10.p_i-2.2) \\
[q(r,p_i-2,2), & 1, & -, & q(r,p_i-1,1)] & (10.p_i-2.3) \\
[q(r,p_i-1,1), & 1, & Z, & q'(s,p_i-1,1)] & (10.p_i-1.1) \\
[q(r,p_i-1,1), & 1, & P, & q(r,p_i-1,2)] & (10.p_i-1.2) \\
[q(r,p_i-1,2), & 1, & -, & q(r,p_i,1)] & (10.p_i-1.3) \\
[q(r,p_i,1), & 2, & +, & q(r,p_i,2)] & (10.p_i.1) \\
[q(r,p_i,2), & 2, & P, & q(r,0,1)] & (10.p_i.3)
\end{array}
$$

Note that, if only $[q_r, i, Z, q_s]$ exists in δ (and $[q_r, i, P, q_t] \notin \delta$), then $3p_i + 2$ quadruples except (10.0.1) are added to δ'. On the other hand, if only $[q_r, i, P, q_t]$ exists in δ, then $2p_i + 4$ quadruples except $(10.j.1)$ $(j = 1,\ldots,p_i-1)$ are added to δ'.

In order to test whether the contents of the counter i of M is positive or zero, M' must check whether the contents of the counter 1 is divisible by p_i or not. This is performed by the above quadruples. When division is completed, the contents of the counter 1 becomes 0, and the quotient is in the counter 2. Then M' transits to the state $q'(t,0,1)$ if the remainder is 0, or $q'(s,j,1)$ if the remainder is $j(= 1,\ldots,p_i-1)$. Restoration of the original contents of the counter 1, and the transition to the state q_t or q_s are performed by the quadruples (11.0.1)–(11.p_i.2) below.

5. For each state q_s such that $[q, i, x, q_s]$ exists in δ for some $q \in Q$, $i \in \{1,\ldots,k\}$, $x \in \{Z, P\}$, include $q_s, q'(s,j,1), q'(s,j,2)$ $(j = 0,\ldots,p_i)$ in Q', and the following

$2p_i+3$ quadruples in δ'. (Note that to such q_s there corresponds unique i because M is reversible.)

$$
\begin{array}{llllr}
[q'(s,0,1), & 2, & Z, & q_s] & (11.0.1)\\
[q'(s,0,1), & 2, & P, & q'(s,0,2)] & (11.0.2)\\
[q'(s,0,2), & 2, & -, & q'(s,p_i,1)] & (11.0.3)\\
[q'(s,p_i,1), & 1, & +, & q'(s,p_i,2)] & (11.p_i.1)\\
[q'(s,p_i,2), & 1, & P, & q'(s,p_i-1,1)] & (11.p_i.2)\\
[q'(s,p_i-1,1), & 1, & +, & q'(s,p_i-1,2)] & (11.p_i-1.1)\\
[q'(s,p_i-1,2), & 1, & P, & q'(s,p_i-2,1)] & (11.p_i-1.2)\\
 & \vdots & & & \\
[q'(s,2,1), & 1, & +, & q'(s,2,2)] & (11.2.1)\\
[q'(s,2,2), & 1, & P, & q'(s,1,1)] & (11.2.2)\\
[q'(s,1,1), & 1, & +, & q'(s,1,2)] & (11.1.1)\\
[q'(s,1,2), & 1, & P, & q'(s,0,1)] & (11.1.2)
\end{array}
$$

We can verify that the quadruples (10.0)–(10.p_i.2) and (11.0.1)–(11.p_i.2) are all deterministic and reversible from the fact that M is deterministic and reversible. □

Example 4.1. Consider a deterministic reversible CM(3) $M_t=(3,Q,\delta,q_0,q_f,P)$ having the following quadruples as δ.

$$
\begin{array}{ll}
[q_0,\ 2,\ Z,\ q_1] & (M_t\text{-}1)\\
[q_1,\ 1,\ Z,\ q_f] & (M_t\text{-}2)\\
[q_1,\ 1,\ P,\ q_2] & (M_t\text{-}3)\\
[q_2,\ 1,\ -,\ q_3] & (M_t\text{-}4)\\
[q_3,\ 2,\ +,\ q_4] & (M_t\text{-}5)\\
[q_4,\ 3,\ +,\ q_5] & (M_t\text{-}6)\\
[q_5,\ 2,\ P,\ q_1] & (M_t\text{-}7)
\end{array}
$$

M_t reversibly transfers the number given in the counter 1 to the counters 2 and 3. For example,

$$(q_0,3,0,0)\ \vdash^{*}_{M_t}\ (q_f,0,3,3).$$

A deterministic reversible CM(2) $M_t'=(2,Q',\ \delta',q_0,q_f,P)$ constructed by the method of Theorem 4.1 has the following 93 quadruples.

1. Quadruples corresponding to $[q_0,2,Z,q_1]$:

$$
\begin{array}{llll}
[q_0, & 2, & Z, & q(0,0,1)]\\
[q(0,0,1), & 1, & P, & q(0,0,2)]\\
[q(0,0,2), & 1, & -, & q(0,1,1)]
\end{array}
$$

$$
\begin{array}{l}
[q(0,1,1),\ 1,\ Z,\ q'(1,1,1)] \\
[q(0,1,1),\ 1,\ P,\ \ q(0,1,2)] \\
[q(0,1,2),\ 1,\ -,\ \ q(0,2,1)] \\
[q(0,2,1),\ 1,\ Z,\ q'(1,2,1)] \\
[q(0,2,1),\ 1,\ P,\ \ q(0,2,2)] \\
[q(0,2,2),\ 1,\ -,\ \ q(0,3,1)] \\
[q(0,3,1),\ 2,\ +,\ \ q(0,3,2)] \\
[q(0,3,2),\ 2,\ P,\ \ q(0,0,1)]
\end{array}
$$

2. Quadruples corresponding to the state q_1:

$$
\begin{array}{l}
[q'(1,0,1),\ 2,\ Z,\ \qquad\quad q_1] \\
[q'(1,0,1),\ 2,\ P,\ q'(1,0,2)] \\
[q'(1,0,2),\ 2,\ -,\ q'(1,3,1)] \\
[q'(1,3,1),\ 1,\ +,\ q'(1,3,2)] \\
[q'(1,3,2),\ 1,\ P,\ q'(1,2,1)] \\
[q'(1,2,1),\ 1,\ +,\ q'(1,2,2)] \\
[q'(1,2,2),\ 1,\ P,\ q'(1,1,1)] \\
[q'(1,1,1),\ 1,\ +,\ q'(1,1,2)] \\
[q'(1,1,2),\ 1,\ P,\ q'(1,0,1)]
\end{array}
$$

3. Quadruples corresponding to the pair $[q_1, 1, Z, q_f]$ and $[q_1, 1, P, q_2]$:

$$
\begin{array}{l}
[q_1,\ \qquad\quad 2,\ Z,\ \ q(1,0,1)] \\
[q(1,0,1),\ 1,\ Z,\ q'(2,0,1)] \\
[q(1,0,1),\ 1,\ P,\ \ q(1,0,2)] \\
[q(1,0,2),\ 1,\ -,\ \ q(1,1,1)] \\
[q(1,1,1),\ 1,\ Z,\ q'(f,1,1)] \\
[q(1,1,1),\ 1,\ P,\ \ q(1,1,2)] \\
[q(1,1,2),\ 1,\ -,\ \ q(1,2,1)] \\
[q(1,2,1),\ 2,\ +,\ \ q(1,2,2)] \\
[q(1,2,2),\ 2,\ P,\ \ q(1,0,1)]
\end{array}
$$

4. Quadruples corresponding to the state q_f:

$$
\begin{array}{l}
[q'(f,0,1),\ 2,\ Z,\ \qquad\quad q_f] \\
[q'(f,0,1),\ 2,\ P,\ q'(f,0,2)] \\
[q'(f,0,2),\ 2,\ -,\ q'(f,2,1)] \\
[q'(f,2,1),\ 1,\ +,\ q'(f,2,2)] \\
[q'(f,2,2),\ 1,\ P,\ q'(f,1,1)] \\
[q'(f,1,1),\ 1,\ +,\ q'(f,1,2)] \\
[q'(f,1,2),\ 1,\ P,\ q'(f,0,1)]
\end{array}
$$

5. Quadruples corresponding to the state q_2:

$$
\begin{array}{llll}
[q'(2,0,1), & 2, & Z, & q_2] \\
[q'(2,0,1), & 2, & P, & q'(2,0,2)] \\
[q'(2,0,2), & 2, & -, & q'(2,2,1)] \\
[q'(2,2,1), & 1, & +, & q'(2,2,2)] \\
[q'(2,2,2), & 1, & P, & q'(2,1,1)] \\
[q'(2,1,1), & 1, & +, & q'(2,1,2)] \\
[q'(2,1,2), & 1, & P, & q'(2,0,1)]
\end{array}
$$

6. Quadruples corresponding to $[q_2, 1, -, q_3]$:

$$
\begin{array}{llll}
[q_2, & 2, & Z, & q(2,1)] \\
[q(2,1), & 1, & Z, & q(2,5)] \\
[q(2,1), & 1, & P, & q(2,2)] \\
[q(2,2), & 1, & -, & q(2,3)] \\
[q(2,3), & 2, & +, & q(2,4)] \\
[q(2,4), & 2, & P, & q(2,1)] \\
[q(2,5), & 2, & Z, & q_3] \\
[q(2,5), & 2, & P, & q(2,5,1)] \\
[q(2,5,1), & 2, & -, & q(2,5,2)] \\
[q(2,5,2), & 2, & -, & q(2,6)] \\
[q(2,6), & 1, & +, & q(2,7)] \\
[q(2,7), & 1, & P, & q(2,5)]
\end{array}
$$

7. Quadruples corresponding to $[q_3, 2, +, q_4]$:

$$
\begin{array}{llll}
[q_3, & 2, & Z, & q(3,1)] \\
[q(3,1), & 1, & Z, & q(3,5)] \\
[q(3,1), & 1, & P, & q(3,2)] \\
[q(3,2), & 1, & -, & q(3,3)] \\
[q(3,3), & 2, & +, & q(3,4)] \\
[q(3,4), & 2, & P, & q(3,1)] \\
[q(3,5), & 2, & Z, & q_4] \\
[q(3,5), & 2, & P, & q(3,6)] \\
[q(3,6), & 2, & -, & q(3,6,1)] \\
[q(3,6,1), & 1, & +, & q(3,6,2)] \\
[q(3,6,2), & 1, & +, & q(3,6,3)] \\
[q(3,6,3), & 1, & +, & q(3,7)] \\
[q(3,7), & 1, & P, & q(3,5)]
\end{array}
$$

8. Quadruples corresponding to $[q_4, 3, +, q_5]$:

$$
\begin{array}{llll}
[q_4, & 2, & Z, & q(4,1)] \\
[q(4,1), & 1, & Z, & q(4,5)] \\
[q(4,1), & 1, & P, & q(4,2)] \\
[q(4,2), & 1, & -, & q(4,3)] \\
[q(4,3), & 2, & +, & q(4,4)] \\
[q(4,4), & 2, & P, & q(4,1)] \\
[q(4,5), & 2, & Z, & q_5] \\
[q(4,5), & 2, & P, & q(4,6)] \\
[q(4,6), & 2, & -, & q(4,6,1)] \\
[q(4,6,1), & 1, & +, & q(4,6,2)] \\
[q(4,6,2), & 1, & +, & q(4,6,3)] \\
[q(4,6,3), & 1, & +, & q(4,6,4)] \\
[q(4,6,4), & 1, & +, & q(4,6,5)] \\
[q(4,6,5), & 1, & +, & q(4,7)] \\
[q(4,7), & 1, & P, & q(4,5)]
\end{array}
$$

9. Quadruples corresponding to $[q_5, 2, P, q_1]$:

$$
\begin{array}{llll}
[q_5, & 2, & Z, & q(5,0,1)] \\
[q(5,0,1), & 1, & Z, & q'(1,0,1)] \\
[q(5,0,1), & 1, & P, & q(5,0,2)] \\
[q(5,0,2), & 1, & -, & q(5,1,1)] \\
[q(5,1,1), & 1, & P, & q(5,1,2)] \\
[q(5,1,2), & 1, & -, & q(5,2,1)] \\
[q(5,2,1), & 1, & P, & q(5,2,2)] \\
[q(5,2,2), & 1, & -, & q(5,3,1)] \\
[q(5,3,1), & 2, & +, & q(5,3,2)] \\
[q(5,3,2), & 2, & P, & q(5,0,1)]
\end{array}
$$

For example, by the above quadruples, the computation of M_t

$$(q_0, 2, 0, 0) \mid\!\frac{12}{M_t}\; (q_f, 0, 2, 2)$$

is simulated by M_t' as follows:

$$(q_0, 2^2 3^0 5^0, 0) \mid\!\frac{3971}{M_t'}\; (q_f, 2^0 3^2 5^2, 0).$$

From Proposition 4.1, Theorems 3.1, 3.2, and 4.1, we can derive the following theorem.

Theorem 4.2. *For any deterministic Turing machine T there is a deterministic reversible* CM(2) *M that simulates T.*

5. Concluding remarks

In this paper we gave conversion methods from an irreversible CM to an equivalent reversible CM, and from a reversible CM(k) to an equivalent reversible CM(2) (these methods were tested by computer simulation). Thus, we can conclude that a reversible CM is computation-universal even if it has only two counters. Since reversible CM(2) is a very simple model of computation, its universality will be useful to show other reversible systems' universality.

Acknowledgements

The author is grateful to the referee for his valuable comments.

References

[1] C.H. Bennett, Logical reversibility of computation, *IBM J. Res. Dev.* **17** (1973) 525–532.
[2] C.H. Bennett, The thermodynamics of computation, *Internat. J. Theoret. Phys.* **21** (1982) 905–940.
[3] C.H. Bennett and R. Landauer, The fundamental physical limits of computation, *Sci. Amer.* **253** (1985) 38–46.
[4] C.H. Bennett, Notes on the history of reversible computation, *IBM J. Res. Dev.* **32** (1988) 16–23.
[5] E. Fredkin and T. Toffoli, Conservative logic, *Internat. J. Theoret. Phys.* **21** (1982) 219–253.
[6] J.E. Hopcroft and J.D. Ullman, *Introduction to Automata Theory, Languages, and Computation* (Addison-Wesley, Reading, MA, 1979).
[7] N. Margolus, Physics-like model of computation, *Physica* **10D** (1984) 81–95.
[8] M.L. Minsky, *Computation: Finite and Infinite Machines* (Prentice-Hall, Englewood Cliffs, NJ, 1967).
[9] K. Morita, A. Shirasaki and Y. Gono, A 1-tape 2-symbol reversible Turing machine, *Trans. IEICE Japan* **E72** (1989) 223–228.
[10] K. Morita and M. Harao, Computation universality of one-dimensional reversible (injective) cellular automata, *Trans. IEICE Japan* **E72** (1989) 758–762.
[11] K. Morita and S. Ueno, Computation-universal models of two-dimensional 16-state reversible cellular automata, *IEICE Trans. Inf. Syst.* **E75-D** (1992) 141–147.
[12] K. Morita, Reversibility in computation, *J. Inform. Process. Soc. Japan* **35** (1994) 306–314 (in Japanese).
[13] K. Morita, Reversible cellular automata, *J. Inform. Process. Soc. Japan* **35** (1994) 315–321 (in Japanese).
[14] T. Toffoli, Computation and construction universality of reversible cellular automata, *J. Comput. System Sci.* **15** (1977) 213–231.
[15] T. Toffoli and N. Margolus, Invertible cellular automata: a review, *Physica D* **45** (1990) 229–253.