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A revisit of the scheme for computing treewidth and minimum fill-in

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ABSTRACT

In this paper, we reformulate the scheme introduced by Bouchitté and Todinca in [1], which computes treewidth and minimum fill-in of a graph using a dynamic programming approach. We will call the scheme *BT scheme*. Although BT scheme was originally designed for computing treewidth and minimum fill-in, it can be used for computing other graph parameters defined in terms of minimal triangulation. In this paper, we reformulate BT scheme so that it works for computing other graph parameters defined in terms of minimal triangulation, and give examples of other graph parameters.

1. Introduction

In [1,2], Bouchitté and Todinca introduced a dynamic programming approach for computing treewidth and minimum fill-in of a graph, and they showed that, using the dynamic programming approach, treewidth and minimum fill-in can be computed in polynomial time in the number of minimal separators. We will call the dynamic programming approach the *BT scheme*. Although BT scheme was originally designed for computing treewidth and minimum fill-in, it can be used for computing other graph parameters defined in terms of minimal triangulation. (Note that computing treewidth and minimum fill-in both can be translated into problems on minimal triangulation.) Indeed, several variants of BT scheme have been developed to compute other graph parameters/problems parameter by parameter: *tree-length* [3] via *chordal sandwich problem*, *treecost* [4], and the perfect phylogeny problem [5]. To unify those variants, we reformulate BT scheme so that it works for computing other graph parameters defined in terms of minimal triangulation.

The importance of establishment of BT scheme is that it unifies the polynomial computability of treewidth and minimum fill-in for the several graph classes: circle graphs [6,7], circular-arc graphs [8,7], cographs [9], chordal bipartite graphs [10,11], weakly chordal graphs [12], and d-trapezoid graphs [13]. Those graph classes have a polynomial number of minimal separators. In fact, it was conjectured that treewidth and minimum fill-in are computable in polynomial time for the classes of graphs with a polynomial number of minimal separators [14,15], and Bouchitté and Todinca [1,2] proved that the conjecture holds.

BT scheme is based on two types of recursive formulas: one is on minimal separators in [16]:

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$$tw(G) = \min_{S \in \Delta_G} \max_{C \in \mathcal{C}(S)} tw\big(R(S,C)\big),$$

$$mfi(G) = \min_{S \in \Delta_G} \Big(fill(S) + \sum mfi\big(R(S,C)\big)\Big),$$

and the other is on *potential maximal cliques* in [1]:

$$tw\big(R(S,C)\big) = \min_{S \subset \Omega \subseteq (S,C)} \max\big(|\Omega| - 1, tw\big(R(S_i, C_i)\big)\big),$$

$$mfi\big(R(S,C)\big) = \min_{S \subset \Omega \subseteq (S,C)} \Big(fill(\Omega) - fill(S) + \sum mfi\big(R(S_i, C_i)\big)\Big),$$

where S and Ω mean a minimal separator and a potential maximal clique, respectively. (See Section 3 for details.) To modify BT scheme so as to be able to compute not only treewidth and minimum fill-in but also other graph parameters defined in terms of minimal triangulation, we reformulate the recursive formulas in Section 3.

It is known that treewidth (tw), minimum fill-in (mfi), and chordal sandwich problem between G_1 and G_2 ($csp(G_1, G_2)$) can be expressed as follows (see Section 2 for the notation):

- $tw(G) = \min_{H \in MT(G)} \max_{M \in MC(H)} |M| - 1$,
- $mfi(G) = \min_{H \in MT(G)} \sum_{e \in FE_G(H)} 1$,
- $csp(G_1, G_2) = \min_{H \in MT(G_1)} \sum_{e \in FE_G(H)} g(e)$, where $g(e) = \begin{cases} 0 & \text{if } e \in E(G_2) \\ 1 & \text{otherwise.} \end{cases}$

As we will show in Section 6, tree-length (tl) can be represented as

- $tl(G) = \min_{H \in MT(G)} \max_{M \in MC(H)} dist_G(M)$.

To unify those expressions, we consider two types of graph parameters, one is *clique type*: graph parameters expressed as $\min_{H \in MT(G)} \max_{M \in MC(H)} f(M)$, and the other is *fill-in type*: graph parameters expressed as $\min_{H \in MT(G)} \sum_{e \in FE_G(H)} f(e)$. The former corresponds to treewidth and the latter to minimum fill-in. Then, we show that BT scheme works for the graph parameters of both clique and fill-in types.

2. Definitions and fundamental results

Let G be a graph and U be a subset of $V(G)$.

notation For a vertex v in G, $N(v)$ denotes the neighbor set of v, and $N(U)$ denotes the set $\bigcup_{u \in U} N(u) - U$. $G[U]$ denotes the subgraph of G induced by U. We denote by $\mathcal{C}_G(U)$ the set of connected components of $G[V \setminus U]$, and by G_U the graph obtained from G by completing U, i.e., by adding an edge between every pair of non-adjacent vertices of U. For convenience, for a connected component $C \in \mathcal{C}_G(U)$, we often make no distinction between the component C and its vertex set $V(C)$, so C be used in the sense of $V(C)$. We will drop the subscript G when it is clear from the context. For example, we will write simply $\mathcal{C}(U)$ instead of $\mathcal{C}_G(U)$. $MC(G)$ denotes the set of maximal cliques of G. For $x, y \in V(G)$, $dist_G(x, y)$ denotes the distance between u and v in G. We denote by $fill_G(U)$ the number of non-edges of U in G.

component A component $C \in \mathcal{C}_G(U)$ is a *full component associated with* U if for each vertex $u \in U$ there is a vertex in $v \in C$ such that $\{u, v\} \in E(G)$.

separator ([2]) A subset $S \subseteq V(G)$ is an *a, b-separator* of G for two non-adjacent vertices $a, b \in V(G)$ if the removal of S from G separates a and b in different connected components. An a, b-separator S is *minimal* if no proper subset of S separates a and b. S is a *minimal separator* of G if there are two vertices a and b for which S is a minimal a, b-separator. We denote by Δ_G the set of all minimal separators of G.

triangulation ([2]) A graph is chordal if every cycle of length at least four has a *chord* (i.e. an edge joining two vertices that are not adjacent in the cycle). A *triangulation* of $G = (V, E)$ is a chordal graph $H = (V, E \cup F)$ such that $E \cap F = \emptyset$, and F is called the *fill-in edges* of H. We denote F by $FE_G(H)$. H is a *minimal* triangulation of G if no proper subgraph of H is a triangulation of G. $MT(G)$ denotes the set of minimal triangulations of G. It is known that $\Delta_H \subseteq \Delta_G$ (see e.g. Theorem 2.9 in [1]).

potential maximal clique ([2]) A vertex set Ω of G is called a *potential maximal clique* if there is a minimal triangulation H of G such that Ω is a maximal clique of H. We denote by Π_G the set of all potential maximal cliques of G. For convenience, we stretch $MT(\cdot)$ slightly as follows: for a potential maximal clique Ω in G, $MT(G, \Omega)$ denotes the set $\{H \mid H \in MT(G) \text{ and } \Omega \in MC(H)\}$.

block Let S be a minimal separator of G. For $C \in \mathcal{C}(S)$, we say that $(S, C) = S \cup C$ is a *block associated with* S (or simply *block* of S). A block (S, C) is a *full* if C is a full component associated with S. The graph $R(S, C)$ obtained from

$G[S \cup C]$ by completing S, i.e. $R(S,C) = G_S[S \cup C]$, is called the *realization* of the block (S,C). We also use the term *block* for a potential maximal clique: Let Ω be a potential maximal clique of G. For $C \in \mathcal{C}(\Omega)$ and $S := N(C) \cap \Omega$, we say that $(S,C) = S \cup C$ is a *block associated with* Ω (or simply *block* of Ω).

Remark 1. It is known that, for a potential maximal clique Ω of a graph G, $\{N(C) \cap \Omega \mid C \in \mathcal{C}(\Omega)\}$ is exactly the minimal separators of G contained in Ω (see Theorem 3.14 in [1]). It is also known that Ω is a potential maximal clique if and only if

1. $G - \Omega$ has no full components associated to Ω,
2. $G_{\mathcal{S}}[\Omega]$ is a clique, where $\mathcal{S} = \{N(C) \cap \Omega \mid C \in \mathcal{C}(\Omega)\}$

(see Theorem 3.15 in [1]).

graph parameters The *treewidth* of G, denoted by $tw(G)$, is the minimum, over all triangulations H of G, of $\omega(H) - 1$, where $\omega(H)$ is the maximum clique size of H. The *minimum fill-in* of G, denoted by $mfi(G)$, is the smallest value of $|E(H) \setminus E(G)|$, where the minimum is taken over all triangulations H of G. The *tree-length* of G, denoted by $tl(G)$, is the smallest integer k for which G has a *tree decomposition* (see e.g. [16] for the definition) such that the distance in G between any pair of vertices that appear in the same bag of the tree decomposition is at most k. The *tree-length-sum* of G, denoted by $tls(G)$, is defined as the minimum number k for which there exists a triangulation H of G such that $k = \sum_{e \in E(H)} dist_G(e)$.

3. Reformulation of the recursive formulas

We first define two types of graph parameters.

Definition 1 *(Clique type and fill-in type).* Let G be a graph. A graph parameter f_c with a function g_c is of *clique type* if f_c can be described as

$$f_c(G) = \min_{H \in MT(G)} \max_{M \in MC(H)} g_c(M),$$

where g_c is a function from $2^{V(G)}$ to positive reals and can be computed in polynomial time in the size of G.

A graph parameter f_f with a function g_f is of *fill-in type* if f_f can be described as

$$f_f(G) = \min_{H \in MT(G)} \sum_{e \in FE_G(H)} g_f(e),$$

where g_f is a function from $V(G) \times V(G)$ to positive reals and can be computed in polynomial time in the size of G. Notice that $g_f(e)$ means $g_f((u,v))$ for $e = (u,v)$.

For convenience, for a graph G and a set $U \subseteq V(G)$, we denote the value of $\sum_{\{xy:\ x,y \in U\} \setminus E(G)} g_f(xy)$ by $fill_G(U, g_f)$. Clearly, treewidth and minimum fill-in can be expressed by the two types of graph parameters: setting $g_c(M)$ as $|M| - 1$ for each clique M and $g_f(e)$ as 1 for each edge e, we have $f_c(G) = tw(G)$ and $f_f(G) = mfi(G)$, respectively. Furthermore, tree-length and tree-length-sum can be also expressed by the two types of graph parameters: setting $g_c(M)$ as $\max_{x,y \in M} dist_G(x,y)$ for each clique M and $g_f(xy)$ as $dist(x,y)$ for each edge xy, we have $f_c(G) = tl(G)$ and $f_f(G) = tls(G)$, respectively.

The scheme for computing treewidth and minimum fill-in is based on the following two theorems.

Theorem 1. *(See [16].) Let G be a non-complete graph. Then,*

$$tw(G) = \min_{S \in \Delta_G} \max_{C \in \mathcal{C}(S)} tw\big(R(S,C)\big),$$

$$mfi(G) = \min_{S \in \Delta_G} \left(fill(S) + \sum_{C \in \mathcal{C}(S)} mfi\big(R(S,C)\big) \right).$$

Theorem 2. *(See [1].) Let (S,C) be a full block of a graph G. Then,*

$$tw\big(R(S,C)\big) = \min_{S \subset \Omega \subseteq (S,C)} \max\big(|\Omega| - 1, tw\big(R(S_i,C_i)\big)\big),$$

$$mfi\big(R(S,C)\big) = \min_{S \subset \Omega \subseteq (S,C)} \Big(fill(\Omega) - fill(S) + \sum mfi\big(R(S_i,C_i)\big) \Big),$$

where (S_i, C_i) are the blocks associated with Ω in $R(S,C)$.

In this paper, we prove that the above two theorems can be extended into a more flexible setting, that is, they can be reformulated as the next two theorems.

The following theorem is corresponding to Theorem 1, and we call it *Minimal Separator Recursion (MSR) theorem*.

Theorem *(MSR theorem). Let G be a non-complete graph, f_c a graph parameter of clique type, and f_f with g_f a graph parameter of fill-in type. Then,*

$$f_c(G) = \min_{S \in \Delta_G} \max_{C \in \mathcal{C}(S)} f_c\bigl(R(S,C)\bigr),$$

$$f_f(G) = \min_{S \in \Delta_G} \Bigl(\mathit{fill}_G(S, g_f) + \sum_{C \in \mathcal{C}(S)} f_f\bigl(R(S,C)\bigr) \Bigr).$$

The following theorem is corresponding to Theorem 2, and we call it *Potential Maximal Clique Recursion (PMCR) theorem*.

Theorem *(PMCR theorem). Let G be a graph, (S, C) be a full block of G, Ω be a potential maximal clique in (S, C) of G, and $(S, C : \Omega)$ be the blocks associated with Ω in (S, C). Let f_c with g_c and f_f with g_f be graph parameters of clique type and fill-in type, respectively. Then,*

$$f_c\bigl(R(S,C)\bigr) = \min_{S \subset \Omega \subseteq (S,C)} \max\Bigl(g_c(\Omega), \max_{(S_i,C_i) \in (S,C:\Omega)} f_c\bigl(R(S_i,C_i)\bigr) \Bigr),$$

$$f_f\bigl(R(S,C)\bigr) = \min_{S \subset \Omega \subseteq (S,C)} \Bigl(\mathit{fill}_G(\Omega, g_f) - \mathit{fill}_G(S, g_f) + \sum_{(S_i,C_i) \in (S,C:\Omega)} f_f\bigl(R(S_i,C_i)\bigr) \Bigr).$$

For clique type, a very similar study can be found in [17]. Our proofs for MSR and PMCR theorems on clique type are essentially the same as the corresponding proofs in [17]. The principal difference between them is as follows. In [17], the optimal value based on a function f over all tree decompositions of a graph G is considered, where f is a *monotone* function (from $2^{V(G)}$ to non-negative reals) which corresponds to our g_c in Definition 1. Since the optimal value is taken over all tree decompositions of G, it should be calculated over all triangulations (not necessarily minimal) of G. As a result, however, for determining the optimal value, it is sufficient to check over all minimal triangulations, thanks to the monotonicity of f. On the other hand, our g_c is not necessarily monotone, but the optimal value should be calculated over all *minimal* triangulations.

4. Proof of MSR theorem

In this section, we show MSR theorem holds: We first prove for clique type in MSR theorem, then prove for fill-in type. To this end, we use the following useful lemma which plays an important role.

Lemma 3. *(See [16, Lemma 3.1].) Let $S \in \Delta_G$ and let $C_1, C_2, \dots, C_r$ be the components of $G[V \setminus S]$. Suppose H_j is a minimal triangulation of $R(S, C_j)$ for any $j \in \{1, 2, \dots, r\}$. Then the graph $H = (V(H), E(H))$ with $V(H) = V(G)$ and $E(H) = \bigcup_{j=1}^{r} E(H_j)$ is a minimal triangulation of G.*

Conversely, let H be a minimal triangulation of G with $S \in \Delta_H$. Then, $H[S \cup C]$ is a minimal triangulation of the realization $R(S, C)$ for each component C of $G[V \setminus S]$.

Remark 2. Let H be a minimal triangulation of a graph G, S a minimal separator of H, and M a maximal clique of H. Then, M is a subset of $S \cup C$ for some component C of $H[V \setminus S]$ (otherwise, M intersects at least two components C_1, C_2 of $H[V \setminus S]$, but this is impossible because M is a clique).

4.1. Proof of clique type

Theorem *(Clique type recursion on minimal separator). Let G be a non-complete graph, and f with g be a graph parameter of clique type (i.e. $f(G) = \min_{H \in MT(G)} \max_{M \in MC(H)} g(M)$). Then,*

$$f(G) = \min_{S \in \Delta_G} \max_{C \in \mathcal{C}(S)} f\bigl(R(S,C)\bigr).$$

Proof. First, we show that $f(G) \geqslant \min_{S \in \Delta_G} \max_{C \in \mathcal{C}(S)} f(R(S,C))$. To show this, we will prove that $f(G) = \min_{H \in MT(G)} \max_{M \in MC(H)} g(M) \geqslant \max_{C \in \mathcal{C}(S)} f(R(S,C))$ for *some* minimal separator S of G. Let H_{opt} be an optimal triangulation of G with respect to f, i.e., H_{opt} is a minimal triangulation of G such that $f(G) = \max_{M \in MC(H_{opt})} g(M)$. Let S be a minimal separator of H_{opt}, i.e., $S \in \Delta_{H_{opt}} \subseteq \Delta_G$. Then,

$$\begin{aligned}
f(G) &= \max_{M \in MC(H_{opt})} g(M) \quad \text{(by the choice of } H_{opt}\text{)} \\
&= \max_{C \in \mathcal{C}(S)} \max_{M \in MC(H_{opt}[S \cup C])} g(M) \quad \text{(by Remark 2)} \\
&\geqslant \max_{C \in \mathcal{C}(S)} \min_{H \in MT(R(S,C))} \max_{M \in MC(H)} g(M) \quad \bigl(\text{as } H_{opt}[S \cup C] \in MT\bigl(R(S,C)\bigr) \text{ by Lemma 3}\bigr) \\
&= \max_{C \in \mathcal{C}(S)} f\bigl(R(S,C)\bigr).
\end{aligned}$$

Next, we show that $f(G) \leqslant \min_{S \in \Delta_G} \max_{C \in \mathcal{C}(S)} f(R(S,C))$. To show this, we will prove that $\min_{H \in MT(G)} \max_{M \in MC(H)} g(M) \leqslant \max_{C \in \mathcal{C}(S)} f(R(S,C))$ for *any* minimal separator S of G. Let S be a minimal separator of G, $C_1, \dots, C_r$ the components of $G[V \setminus S]$, H_i $(1 \leqslant i \leqslant r)$ a minimal triangulation of $R(S, C_i)$ such that $f(R(S,C_i)) = f(H_i)$ $(= \max_{M \in MC(H_i)} g(M))$, and H_0 the graph such that $V(H_0) = V(G)$ and $E(H_0) = \bigcup_{1 \leqslant j \leqslant r} E(H_j)$. Then,

$$\begin{aligned}
f(G) &= \min_{H \in MT(G)} \max_{M \in MC(H)} g(M) \\
&\leqslant \max_{M \in MC(H_0)} g(M) \quad \bigl(\text{since } H_0 \in MT(G) \text{ by Lemma 3}\bigr) \\
&= \max_{M \in \bigcup_{1 \leqslant i \leqslant r} MC(H_0[S \cup C_i])} g(M) \quad \text{(by Remark 2)} \\
&= \max_{1 \leqslant i \leqslant r} \max_{M \in MC(H_i)} g(M) \\
&= \max_{1 \leqslant i \leqslant r} f\bigl(R(S,C_i)\bigr) \quad \Bigl(\text{as } f\bigl(R(S,C_i)\bigr) = \max_{M \in MC(H_i)} g(M)\Bigr).
\end{aligned}$$

Thus, we have $f(G) = \min_{S \in \Delta_G} \max_{C \in \mathcal{C}(S)} f(R(S,C))$.

4.2. Proof of fill-in type

Theorem *(Fill-in type recursion on minimal separator). Let $G = (V, E)$ be a non-complete graph and f with g be a graph parameter of fill-in type. Then*

$$f(G) = \min_{S \in \Delta_G} \Bigl(\mathit{fill}_G(S, g) + \sum_{C \in \mathcal{C}(S)} f\bigl(R(S,C)\bigr) \Bigr),$$

Proof. First, we show that $f(G) \geqslant \min_{S \in \Delta_G} (\mathit{fill}_G(S,g) + \sum_{C \in \mathcal{C}(S)} f(R(S,C)))$. To show this, we will prove that $f(G) \geqslant \mathit{fill}_G(S,g) + \sum_{C \in \mathcal{C}(S)} f(R(S,C))$ for *some* minimal separator S of G. Let H_{opt} be an optimal triangulation of G with respect to f, i.e., H_{opt} is a minimal triangulation of G and $f(G) = \sum_{e \in FE_G(H_{opt})} g(e)$. Let S be a minimal separator of H_{opt} (hence, $S \in \Delta_G$). From Lemma 3, for each component C of $G[V \setminus S]$, $H_{opt}[S \cup C]$ is a minimal triangulation of $R(S,C)$. Hence, letting $C_1, C_2, \dots, C_r$ be the connected components of $G[V \setminus S]$,

$$\begin{aligned}
f(G) &= \sum_{e \in E(H_{opt})} g(e) - \sum_{e \in E(G)} g(e) \\
&= \Bigl(\sum_{1 \leqslant i \leqslant r} \sum_{e \in E(H_{opt}[S \cup C_i])} g(e) - (r-1) \sum_{e \in E(H_{opt}[S])} g(e) \Bigr) \\
&\quad - \Bigl(\sum_{1 \leqslant i \leqslant r} \sum_{e \in E(R(S,C_i))} g(e) - \mathit{fill}_G(S,g) - (r-1) \sum_{e \in E(H_{opt}[S])} g(e) \Bigr) \\
&= \mathit{fill}_G(S,g) + \sum_{1 \leqslant i \leqslant r} \Bigl(\sum_{e \in E(H_{opt}[S \cup C_i])} g(e) - \sum_{e \in E(R(S,C_i))} g(e) \Bigr) \\
&\geqslant \mathit{fill}_G(S,g) + \sum_{1 \leqslant i \leqslant r} f\bigl(R(S,C_i)\bigr) \quad \bigl(\text{as } H_{opt}[S \cup C_i] \in MT\bigl(R(S,C_i)\bigr) \text{ by Lemma 3}\bigr).
\end{aligned}$$

Next, we show that for *any* $S \in \Delta_G$, $f(G) \leqslant \mathit{fill}_G(S,g) + \sum_{C \in \mathcal{C}(S)} f(R(S,C))$. Let $S \in \Delta_G$ and $C_1, C_2, \dots, C_r$ be the connected components of $G[V \setminus S]$. Furthermore, for each $1 \leqslant i \leqslant r$, let H_i be a minimal triangulation of $R(S,C_i)$ such that $f(R(S,C_i)) = \sum_{e \in FE_{R(S,C_i)}(H_i)} g(e) = \sum_{e \in E(H_i)} g(e) - \sum_{e \in E(R(S,C_i))} g(e)$. Now, let consider the graph H_0 whose vertex set is $V(H_0) = V(G)$ and whose edge set is $E(H_0) = \bigcup_{i=1}^{r} E(H_i)$. From Lemma 3, the graph H_0 is a minimal triangulation of G. Thus,

$$
\begin{aligned}
f(G) &= \min_{H\in MT(G)} \sum_{e\in E(H)} g(e) - \sum_{e\in E(G)} g(e) \\
&\leqslant \sum_{e\in E(H_0)} g(e) - \sum_{e\in E(G)} g(e) \quad (\text{since } H_0 \text{ is a minimal triangulation of } G) \\
&= \Bigl(\sum_{1\leqslant i\leqslant r}\sum_{e\in E(H_0[H_i])} g(e) - (r-1)\sum_{e\in E(H_0[S])} g(e)\Bigr) \\
&\quad - \Bigl(\sum_{1\leqslant i\leqslant r}\sum_{e\in E(R(S,C_i))} g(e) - \mathit{fill}_G(S,g) - (r-1)\sum_{e\in E(H_0[S])} g(e)\Bigr) \\
&= \mathit{fill}_G(S,g) + \sum_{1\leqslant i\leqslant r}\Bigl(\sum_{e\in E(H_i)} g(e) - \sum_{e\in E(R(S,C_i))} g(e)\Bigr) \\
&= \mathit{fill}_G(S,g) + \sum_{1\leqslant i\leqslant r} f\bigl(R(S,C_i)\bigr).
\end{aligned}
$$

5. Proof of PMCR theorem

Before proving the theorem, let us recall some results in [1] which we will use frequently in our proofs.

Lemma 4. *(See Corollary 3.12 in [1].) Let Ω be a potential maximal clique of G and let $S \in \Delta_G(\Omega)$. Then S is strictly contained in Ω and $\Omega - S$ is in a full connected component associated with S.*

Theorem 5. *(See Theorem 4.3 in [1].) Let H be a minimal triangulation of G and let Ω be a maximal clique of H. Then for each block (S_i, C_i) associated with Ω in G, the graph $H_i = H[S_i \cup C_i]$ is a minimal triangulation of the realization $R(S_i, C_i)$.*

Conversely, let Ω be a potential maximal clique of G. For each block (S_i, C_i) associated with Ω in G, let H_i be a minimal triangulation of $R(S_i, C_i)$. Then $H = (V(G), E(H))$ with $E(H) = \bigcup_{i=1}^{p} E(H_i) \cup \{\{x, y\} \mid x, y \in \Omega\}$ is a minimal triangulation of G.

Remark 3. Let Ω be a potential maximal clique of a graph G, $(S_1, C_1), \ldots, (S_r, C_r)$ the blocks associated with Ω in G, H_i a minimal triangulation of $R(S_i, C_i)$ for $1 \leqslant i \leqslant r$, and H_0 the graph with $V(H_0) = V(G)$ and $E(H_0) = \bigcup_{i=1}^{p} E(H_i) \cup E(G)$. Then, for a maximal clique M in H_0 of G, one of the following two cases holds:

1. M is Ω (by Remark 1) or
2. M is in $H_0[S_i \cup C_i]$ for some block (S_i, C_i) associated with Ω. Because, any maximal clique cannot intersect more than one component. Also note that if there is a vertex $v \in (\Omega \cap M)\backslash S_i$ then $M \subseteq \Omega$ (in which case M is Ω).

Lemma 6. *(See Lemma 4.6 in [1].) Let $R(S, C)$ be the realization of some full block (S, C) and let $H(S, C)$ be a minimal triangulation of $R(S, C)$. Then there is a maximal clique Ω of $H(S, C)$ such that $S \subset \Omega$ and Ω is a potential maximal clique of G.*

5.1. Proof of clique type

Theorem *(Clique type recursion on potential maximal clique). Let G be a graph, S a minimal separator in G, (S, C) a full block of G, $H(S, C)$ a minimal triangulation of the realization $R(S, C)$, f with g a graph parameter of clique type. Then,*

$$f\bigl(R(S,C)\bigr) = \min_{S\subset\Omega\subseteq(S,C)} \max\Bigl(g(\Omega), \max_{(S_i,C_i)\in(S,C:\Omega)} f\bigl(R(S_i,C_i)\bigr)\Bigr),$$

where $(S, C : \Omega)$ denotes the blocks associated with a potential maximal clique Ω with $S \subset \Omega \subseteq S \cup C$.

Proof. First, we show that, for *some* potential maximal clique Ω in G such that $S \subset \Omega \subseteq (S, C)$, $f(R(S, C)) \geqslant \max(g(\Omega), \max_{(S_i,C_i)\in(S,C:\Omega)} f(R(S_i, C_i)))$ holds. Let $H_{opt}^{(S,C)}$ be a minimal triangulation of $R(S, C)$ such that $f(R(S, C)) = \max_{M\in MC(H_{opt}^{(S,C)})} g(M)$. Then, by Lemma 6, there is a maximal clique Ω in $H_{opt}^{(S,C)}$ such that S is strictly contained in Ω and Ω is a potential maximal clique of G. Now, for the Ω, let $(S_1, C_1), \ldots, (S_r, C_r)$ denote the blocks associated with Ω in $R(S, C)$. Then, by Theorem 5, for each (S_i, C_i), $H_{opt}^{(S,C)}[S_i \cup C_i]$ is a minimal triangulation of $R(S_i, C_i)$. Thus,

$$
\begin{aligned}
f\bigl(R(S,C)\bigr) &= \min_{H\in MT(R(S,C))} \max_{M\in MC(H)} g(M) \\
&= \max_{M\in MC(H_{opt}^{(S,C)})} g(M) \quad \bigl(\text{from the setting of } H_{opt}^{(S,C)}\bigr)
\end{aligned}
$$

$$= \max\Big(g(\Omega), \max_{1\leqslant i\leqslant r} \max_{M\in MC(H_{opt}^{(S,C)}[S_i\cup C_i])} g(M)\Big)$$

$$\geqslant \max\Big(g(\Omega), \max_{1\leqslant i\leqslant r} \min_{H\in MT(R(S_i,C_i))} \max_{M\in MC(H)} g(M)\Big)$$

$$= \max\Big(g(\Omega), \max_{1\leqslant i\leqslant r} f\big(R(S_i,C_i)\big)\Big).$$

The second to the third line follows from the facts that Ω is a maximal clique in $H_{opt}^{(S,C)}$ and that any maximal clique cannot intersect C_i and C_j for some $1\leqslant i<j\leqslant r$. As $H_{opt}^{(S,C)}[S_i\cup C_i]\in MT(R(S_i,C_i))$ by Theorem 5, the fourth line can be derived.

Next, we show that, for *any* potential maximal clique Ω in G such that $S\subset\Omega\subseteq(S,C)$, $f(R(S,C))\leqslant\max(g(\Omega), \max_{(S_i,C_i)\in(S,C:\Omega)} f(R(S_i,C_i)))$ holds. Let Ω be a potential maximal clique in G such that $S\subset\Omega\subseteq(S,C)$. Note that from Lemma 6 there is such an Ω. For the Ω, let $(S_1,C_1),\dots,(S_r,C_r)$ denote the blocks associated with Ω in $R(S,C)$. Recall that $MT(R(S,C),\Omega)$ denotes the set of minimal triangulations of $R(S,C)$ in which Ω is a maximal clique. Then,

$$f\big(R(S,C)\big) = \min_{H\in MT(R(S,C))} \max_{M\in MC(H)} g(M)$$

$$\leqslant \min_{H\in MT(R(S,C),\Omega)} \max_{M\in MC(H)} g(M) \quad \big(\text{since } MT\big(R(S,C),\Omega\big)\subseteq MT\big(R(S,C)\big)\big)$$

$$= \min_{H\in MT(R(S,C),\Omega)} \max_{M\in\{\Omega\}\cup\bigcup_{1\leqslant i\leqslant r} MC(H[S_i\cup C_i])} g(M)$$

$$= \max\Big(g(\Omega), \min_{H\in MT(R(S,C),\Omega)} \max_{1\leqslant i\leqslant r} \max_{M\in MC(H[S_i\cup C_i])} g(M)\Big)$$

$$\leqslant \max\Big(g(\Omega), \max_{1\leqslant i\leqslant r} \min_{H\in MT(R(S_i,C_i))} \max_{M\in MC(H)} g(M)\Big) \quad (\text{by the fact below})$$

$$= \max\Big(g(\Omega), \max_{(S_i,C_i)\in(S,C:\Omega)} f\big(R(S_i,C_i)\big)\Big).$$

Since Ω is a maximal clique in $H\in MT(R(S,C),\Omega)$ and any maximal clique cannot intersect C_i and C_j for some $1\leqslant i<j\leqslant r$, the third line can be derived from the second line. The fifth line follows from the fact that

$$\min_{H\in MT(R(S,C),\Omega)} \max_{1\leqslant i\leqslant r} \max_{M\in MC(H[S_i\cup C_i])} g(M) \leqslant \max_{1\leqslant i\leqslant r} \min_{H\in MT(R(S_i,C_i))} \max_{M\in MC(H)} g(M).$$

To show the fact, it is sufficient to prove that there is a minimal triangulation H_0 in $MT(R(S,C),\Omega)$ such that

$$\max_{1\leqslant i\leqslant r} \max_{M\in MC(H_0[S_i\cup C_i])} g(M) = \max_{1\leqslant i\leqslant r} \min_{H\in MT(R(S_i,C_i))} \max_{M\in MC(H)} g(M).$$

For each $1\leqslant i\leqslant r$, let H_i be a minimal triangulation of $R(S_i,C_i)$ such that H_i minimizes $\max_{M\in MC(H)} g(M)$ among all minimal triangulations H of $R(S_i,C_i)$. Then, take the graph $(S\cup C, \bigcup_{1\leqslant i\leqslant r} E(H_i)\cup E(R(S,C)))$ as H_0. Note that, by Remark 1, Ω is a maximal clique in H_0, and by Theorem 5 H_0 is a minimal triangulation of $R(S,C)$. Hence, H_0 is in $MT(R(S,C),\Omega)$. Thus, we have

$$\max_{1\leqslant i\leqslant r} \max_{M\in MC(H_0[S_i\cup C_i])} g(M) = \max_{1\leqslant i\leqslant r} \max_{M\in MC(H_i)} g(M)$$

$$= \max_{1\leqslant i\leqslant r} \min_{H\in MT(R(S_i,C_i))} \max_{M\in MC(H)} g(M) \quad (\text{from the setting of } H_i).$$

As a result, we have

$$f(G) = \min_{S\subset\Omega\subseteq(S,C)} \max\Big(g(\Omega), \max_{(S_i,C_i)\in(S,C:\Omega)} f\big(R(S_i,C_i)\big)\Big).$$

5.2. Proof of fill-in type

Theorem *(Fill-in type recursion on potential maximal clique). Let G be a graph, S a minimal separator, (S,C) a full block of G, Ω a potential maximal clique in G such that $S\subset\Omega\subseteq(S,C)$, $(S,C:\Omega)$ the blocks associated with Ω in (S,C). Let f with g be a graph parameter of fill-in type. Then,*

$$f\big(R(S,C)\big) = \min_{S\subset\Omega\subseteq(S,C)} \Big(\mathit{fill}_G(\Omega,g) - \mathit{fill}_G(S,g) + \sum_{(S_i,C_i)\in(S,C:\Omega)} f\big(R(S_i,C_i)\big)\Big).$$

Proof. First, we show that, for *some* potential maximal clique Ω with $S \subset \Omega \subseteq (S, C)$,

$$f\bigl(R(S,C)\bigr) \geqslant fill_G(\Omega, g) - fill_G(S, g) + \sum_{(S_i,C_i)\in(S,C:\Omega)} f\bigl(R(S_i,C_i)\bigr)$$

holds. First let H_{opt} be a minimal triangulation of $R(S,C)$ such that $f(R(S,C)) = \sum_{e\in FE_{R(S,C)}(H_{opt})} g(e)$. Then, take a maximal clique Ω in H_{opt} such that $S \subset \Omega \subseteq (S,C)$ and Ω is a potential maximal clique in G. (Note that there is such a maximal clique Ω by Lemma 6.)

From Theorem 5, for each $(S_i, C_i) \in (S, C : \Omega)$, $H_{opt}[S_i \cup C_i]$ is a minimal triangulation of $R(S_i, C_i)$. Thus,

$$\begin{aligned}
f\bigl(R(S,C)\bigr) &= \sum_{e\in FE_{R(S,C)}(H_{opt})} g(e) \quad \text{(from the setting of } H_{opt}\text{)}\\
&= \sum_{e\in E(H_{opt})} g(e) - \sum_{e\in E(R(S,C))} g(e)\\
&= \Biggl(\sum_{e\in E(H_{opt}[\Omega])} g(e) + \sum_{(S_i,C_i)\in(S,C:\Omega)} \Biggl(\sum_{e\in E(H_{opt}[S_i\cup C_i])} g(e) - \sum_{e\in E(H_{opt}[S_i])} g(e) \Biggr)\Biggr)\\
&\quad - \Biggl(\sum_{e\in E(G[\Omega])} g(e) + fill_G(S,g) + \sum_{(S_i,C_i)\in(S,C:\Omega)} \Biggl(\sum_{e\in R(S_i,C_i)} g(e) - \sum_{e\in E(R(S_i,C_i)[S_i])} g(e) \Biggr)\Biggr)\\
&= \sum_{e\in E(H_{opt}[\Omega])} g(e) - \sum_{e\in E(G[\Omega])} g(e) - fill_G(S,g) + \sum_{(S_i,C_i)\in(S,C:\Omega)} \Biggl(\sum_{e\in E(H_{opt}[S_i\cup C_i])} g(e) - \sum_{e\in R(S_i,C_i)} g(e) \Biggr)\\
&= fill_{R(S,C)}(\Omega,g) - fill_{R(S,C)}(S,g) + \sum_{(S_i,C_i)\in(S,C:\Omega)} \Biggl(\sum_{e\in E(H_{opt}[S_i\cup C_i])} g(e) - \sum_{e\in R(S_i,C_i)} g(e) \Biggr)\\
&\geqslant fill_{R(S,C)}(\Omega,g) - fill_{R(S,C)}(S,g) + \sum_{(S_i,C_i)\in(S,C:\Omega)} \Biggl(\min_{H\in MT(R(S_i,C_i))} \sum_{e\in E(H)} g(e) - \sum_{e\in R(S_i,C_i)} g(e) \Biggr)\\
&= fill_{R(S,C)}(\Omega,g) - fill_{R(S,C)}(S,g) + \sum_{(S_i,C_i)\in(S,C:\Omega)} f\bigl(R(S_i,C_i)\bigr).
\end{aligned}$$

Next, we show that, for *any* Ω such that $S \subset \Omega \subseteq (S, C)$,

$$f\bigl(R(S,C)\bigr) \leqslant fill_{R(S,C)}(\Omega,g) - fill_{R(S,C)}(S,g) + \sum_{(S_i,C_i)\in(S,C:\Omega)} f\bigl(R(S_i,C_i)\bigr).$$

So, let Ω be a potential maximal clique in G such that $S \subset \Omega \subseteq (S, C)$, and let $(S_1, C_1), \dots, (S_r, C_r)$ denote the blocks associated with Ω in $R(S, C)$. Again from Theorem 5,

$$\begin{aligned}
f\bigl(R(S,C)\bigr) &= \min_{H\in MT(R(S,C))} \sum_{e\in FE_{R(S,C)}(H)} g(e)\\
&\leqslant \min_{H\in MT(R(S,C),\Omega)} \sum_{e\in FE_{R(S,C)}(H)} g(e)\\
&= fill_{R(S,C)}(\Omega,g) - fill_{R(S,C)}(S,g) + \min_{H\in MT(R(S,C),\Omega)} \sum_{1\leqslant i\leqslant r} \Biggl(\sum_{e\in E(H[S_i\cup C_i])} g(e) - \sum_{e\in R(S_i,C_i)} g(e) \Biggr)\\
&\leqslant fill_{R(S,C)}(\Omega,g) - fill_{R(S,C)}(S,g) + \sum_{1\leqslant i\leqslant r} \Biggl(\min_{H\in MT(R(S_i,C_i))} \Biggl(\sum_{e\in E(H)} g(e) - \sum_{e\in R(S_i,C_i)} g(e) \Biggr)\Biggr)\\
&= fill_{R(S,C)}(\Omega,g) - fill_{R(S,C)}(S,g) + \sum_{(S_i,C_i)\in(S,C:\Omega)} f\bigl(R(S_i,C_i)\bigr).
\end{aligned}$$

The third to the fourth line follows from the fact that

$$\min_{H\in MT(R(S,C),\Omega)} \sum_{1\leqslant i\leqslant r} \sum_{e\in E(H[S_i\cup C_i])} g(e) \leqslant \sum_{1\leqslant i\leqslant r} \Biggl(\min_{H\in MT(R(S_i,C_i))} \sum_{e\in E(H)} g(e) \Biggr).$$

Algorithm 1: BT scheme.

Input: a connected graph G, all its potential maximal cliques and all its minimal separators S
Output: $f_c(G)$ and $f_f(G)$

1 compute all the full block (S,C) and sort them by the number of vertices;
2 **foreach** *full block* (S,C) *taken in increasing order* **do**
3   **if** (S,C) *is inclusion-minimal* **then**
4     $tw(R(S,C)) := |S \cup C| - 1$; // $\xrightarrow[\text{replace}]{}$ $f_c(R(S,C)) := g_c(S \cup C)$;
5     $mfi(R(S,C)) := fill(S \cup C)$; // $\xrightarrow[\text{replace}]{}$ $f_f(R(S,C)) := fill_G(S \cup C, g_f)$;
6   **else**
7     $tw(R(S,C)) := \infty$; // $\xrightarrow[\text{replace}]{}$ $f_c(R(S,C)) := \infty$;
8     $mfi(R(S,C)) := \infty$; // $\xrightarrow[\text{replace}]{}$ $f_f(R(S,C)) := \infty$;
9   **foreach** *potential maximal clique* Ω *with* $S \subset \Omega \subseteq (S,C)$ **do**
10   compute the blocks (S_i, C_i) associated with Ω s.t. $S_i \cup C_i \subset S \cup C$,
11     $tw(R(S,C)) = \min(tw(R(S,C)), \max_i(|\Omega| - 1, tw(R(S_i, C_i))))$;
12   // $\xrightarrow[\text{replace}]{}$ $f_c(R(S,C)) := \min(f_c(R(S,C)), \max(g(\Omega), \max_i f_c(R(S_i, C_i))))$,
13     $mfi(R(S,C)) = \min(mfi(R(S,C)), fill(\Omega) - fill(S) + \sum mfi(R(S_i, C_i)))$;
14   // $\xrightarrow[\text{replace}]{}$ $f_f(R(S,C)) := \min(f_f(R(S,C)), \text{fill}_G(\Omega, g_f) - \text{fill}_G(S, g_f) + \sum_i f_f(R(S_i, C_i)))$,
15 let Δ_G^* be the set of inclusion-minimal separators of G;
16 $tw(G) = \min_{S \in \Delta_G^*} \max_{C \in \mathcal{C}(S)} tw(R(S,C))$; // $\xrightarrow[\text{replace}]{}$ $f_c(G) = \min_{S \in \Delta_G^*} \max_{C \in \mathcal{C}(S)} f_c(R(S,C))$;
17 $mfi(G) = \min_{S \in \Delta_G} (fill(S) + \sum_{C \in \mathcal{C}(S)} mfi(R(S,C)))$; // $\xrightarrow[\text{replace}]{}$ $f_f(G) = \min_{S \in \Delta_G^*} (\text{fill}_G(S, g_f) + \sum_{C \in \mathcal{C}(S)} f_f(R(S,C)))$;

Fig. 1. Dynamic programming based on MSR theorem and PMCR theorem.

To show the fact, it is sufficient to prove that there is a minimal triangulation H_0 in $MT(R(S,C), \Omega)$ such that

$$\sum_{1 \leqslant i \leqslant r} \sum_{e \in E(H_0[S_i \cup C_i])} g(e) = \sum_{1 \leqslant i \leqslant r} \left(\min_{H \in MT(R(S_i, C_i))} \sum_{e \in E(H)} g(e) \right).$$

For each $1 \leqslant i \leqslant r$, let H_i be a minimal triangulation of $R(S_i, C_i)$ such that H_i minimizes $\sum_{e \in E(H)} g(e)$ among all minimal triangulations of $R(S_i, C_i)$. Then, let H_0 be the graph with $V(H_0) = S \cup C$ and $E(H_0) = \bigcup_{1 \leqslant i \leqslant r} E(H_i) \cup \{\{x,y\} \mid x, y \in \Omega\}$. Clearly, H_0 satisfies the above equality. And H_0 is in $MT(R(S,C), \Omega)$ by Theorem 5 and Remark 1.

As a result, we have

$$f\big(R(S,C)\big) = \min_{S \subset \Omega \subseteq (S,C)} \left(fill_{R(S,C)}(\Omega, g) - fill_{R(S,C)}(S, g) + \sum_{(S_i, C_i) \in (S, C:\Omega)} f\big(R(S_i, C_i)\big) \right).$$

5.3. Reformed BT scheme: Dynamic programming based on MSR and PMCR theorems

Fig. 1 shows BT scheme (computing treewidth and minimum fill-in) appeared in [18], and from which reformulated BT scheme can be obtained by replacing the recursive formulas in Theorem 1 and Theorem 2 with the recursive formulas in MSR theorem and PMCR theorem, respectively (i.e., the comment lines). In [18], Fomin et al. showed that the BT scheme can be implemented to run in $O(n^3 \cdot |\Pi_G|)$. (Recall that Π_G is the set of potential maximal cliques of G.)

As an immediate consequence of the running time analysis, we have the following theorem.

Theorem 7. *Let G be a graph with n vertices. Let $f_c(G) = \min_{H \in MT(G)} \max_{M \in MC(H)} g_c(M)$ be a graph parameter with a function g_c, from 2^n to positive reals, computable in $p_c(|V(G)|)$ time. And let $f_f(G) = \min_{H \in MT(G)} \sum_{e \in FE_G(H)} g_f(e)$ be a graph parameter with a function g_f, from $V(G) \times V(G)$ to positive reals, computable in $p_f(n)$ time. Then, using reformulated BT scheme (that is, the dynamic programming based on MSR theorem and PMCR theorem), given a graph G, the list of all its potential maximal cliques, and the list of all its minimal separators, $f_c(G)$ can be computed in $O((p_c(n) + n^3) \cdot |\Pi_G|)$ and $f_f(G)$ can be computed in $O(n^3 \cdot p_f(n) \cdot |\Pi_G|)$.*

Proof. First, let us consider the cost of lines 4 and 5 in the reformulated BT scheme. Since the number of the blocks (S,C) is at most $O(n|\Delta_G|)$, the total cost of computing $g_c(S \cup C)$ (i.e., line 4) is at most $O(n \cdot p_c(n) \cdot |\Delta_G|)$. As $fill_G(U, g_f)$ can be computed in $O(n^2 \cdot p_f(n))$ for each $U \subseteq V(G) \times V(G)$, the total cost of computing $fill_G(S \cup C, g_f)$ (i.e., line 5) is $O(n^3 \cdot p_f(n) \cdot |\Delta_G|)$.

Next, let us see the total cost of computing $\text{fill}_G(\Omega, g_f)$ and $\text{fill}_G(S, g_f)$ in lines 14 and 17. The cost for computing $\text{fill}_G(\Omega, g_f)$ is $O(n^2 \cdot p_f(n) \cdot |\Pi_G|)$. The cost for computing $\text{fill}_G(S, g_f)$ is $O(n^2 \cdot p_f(n) \cdot |\Delta_G|)$.

Hence, $f_c(G)$ can be computed in $O(n \cdot p_c(n) \cdot |\Delta_G| + n^3|\Pi_G|)$ and $f_f(G)$ can be computed in $O(n^2 \cdot p_f(n) \cdot (|\Pi_G| + |\Delta_G|) + n^3|\Pi_G|)$. From $|\Delta_G| \leqslant n \cdot |\Pi_G|$, we have $O((p_c(n) + n^3) \cdot |\Pi_G|)$ for computing $f_c(G)$ and $O(n^3 \cdot p_f(n) \cdot |\Pi_G|)$ for computing $f_f(G)$.

The running time depends on the number of the minimal separators $|\Delta_G|$ and the potential maximal cliques $|\Pi_G|$. It is known that $|\Delta_G|$ is $O(1.6181^n)$ [19] and $|\Pi_G|$ is $O(1.7347^n)$ [20].

In [1], Bouchitté and Todinca showed that given a graph G, the list of all its potential maximal cliques, and the list of all its minimal separators, treewidth and minimum fill-in of G can be computed in polynomial time in the size of the inputs G and the lists, using BT scheme (i.e., standard dynamic programming based on Theorem 1 and Theorem 2). In [2], they proved that the number of the potential maximal cliques of a graph is polynomially bounded in the number of its minimal separators and in the size of the graph, and that the potential maximal cliques of a graph can be listed in polynomial time in its size and the number of its minimal separators. By combining those results, it can be concluded that, for classes of graphs with a polynomial number of minimal separators, the treewidth and the minimum fill-in can be computed in polynomial time by BT scheme. This gives the next corollary.

Corollary 8. *Graph parameters of clique type and of fill-in type both can be computed in polynomial time for classes of graphs with a polynomial number of minimal separators.*

6. Applications

We introduce a new graph parameter fill-in distance, and we show that the new graph parameter coincides with the tree-length.

Definition 2 *(Fill-in distance). Fill-in distance* of G, denoted by $fid(G)$, is defined as the minimum number k for which there exists a triangulation H of G such that $dist_G(u,v) \leqslant k$ for every edge $\{u,v\}$ in H.

Remark 4.

- $fid(G)$ can be formulated as $\min_{H \in MT(G)} \max_{e \in FE_G(H)} dist_G(e)$.
- G is a chordal graph iff $fid(G) = 1$.
- Because of the fact that for any triangulation H of G there is a minimal triangulation H' of G with $E(H') \subseteq E(H)$, fill-in distance can be restated as follows: $fid(G)$ is defined as the minimum number k for which there exists a *minimal* triangulation H of G such that $dist_G(u,v) \leqslant k$ for every edge $\{u,v\}$ in H.

Lemma 9. $tl(G) = fid(G)$.

Proof. Let $tl(G) \leqslant k$. Then, there is a tree decomposition $\mathcal{T} = (T, \chi)$ such that $dist_G(u,v) \leqslant k$ for any $X \in \chi$ and for any pair $\{u,v\}$ with $u, v \in X$. Consider a graph H constructed from $\mathcal{T}$ by completing each $X \in \chi$, i.e., by adding an edge between pair of non-adjacent vertices in X for each $X \in \chi$. Then, it is well known that H is a triangulation of G. Since $dist_G(u,v) \leqslant k$ for every edge $\{u,v\}$ in H, $fid(G)$ is at most k.

Let $fid(G) \leqslant k$. Then, there exists a triangulation H of G such that $dist_G(u,v) \leqslant k$ for every edge $\{u,v\}$ in H. It is well known that every chordal graph H has a tree decomposition $\mathcal{T}$ such that the set of bags in $\mathcal{T}$ equals the set of maximal cliques in H (cf. [21]). Clearly, $\mathcal{T}$ is also a tree decomposition of G. For any pair u, v in any clique of H, namely, for any pair u, v in any bag of $\mathcal{T}$, $dist_G(u,v) \leqslant k$. Hence, $tl(G)$ is at most k.

The graph parameter fill-in distance (i.e. tree-length) can be viewed as "bottleneck version". In this sense, tree-length-sum can be considered as "total-sum version".

Corollary 10. *The graph parameters tree-length and tree-length-sum can be computed by the new scheme.*

The following are some further examples.

Problem A. Given a graph G, find the minimum integer k such that there exist a minimal triangulation H of G where each maximal clique C of H contains at most k fill-in edges.

Problem B. Given a graph G and a subset Z of $V(G)$, find the minimum integer k such that there exist a minimal triangulation H of G where each maximal clique C of H contains at most k vertices of Z.

Problem C. Given a graph G and a proper coloring c of $V(G)$, does there exist a supergraph G' of G which is properly colored by c and which is triangulated? This problem has applications in perfect phylogeny [22].

Each problem above can be solved by using the following expressions:

Problem A. $\min_{H \in MT(G)} \max_{M \in MC(H)} |\{\{x, y\} \notin E(G) \mid x, y \in M\}|$.

Problem B. $\min_{H \in MT(G)} \max_{M \in MC(H)} |\{x \mid x \in M \cap Z\}|$.

Problem C. $\min_{H \in MT(G)} \max_{M \in MC(H)} |\{\{x, y\} \notin E(G) \mid c(x) = c(y)\}|$.

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