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Complexity, appeal and challenges of combinatorial games☆

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Abstract

Studying the precise nature of the complexity of games enables *gamesters* to attain a deeper understanding of the difficulties involved in certain new and old open game problems, which is a key to their solution. For *algorithmicians*, such studies provide new interesting algorithmic challenges. Substantiations of these assertions are illustrated on hand of many sample games, leading to a definition of the tractability, polynomiality and efficiency of subsets of games. In particular, there are tractable games that need not be polynomial, polynomial games that need not be efficient. We also define and explore the nature of the subclasses *PlayGames* and *MathGames*.

Keywords: Complexity of combinatorial games; PlayGames; MathGames

1. Introduction

In this talk, I would like to sell you the idea that the complexity and algorithmic nature of combinatorial games (simply *games* in the sequel) is quite unlike that of existential decision and optimization problems. A study of the precise nature of the complexity of games enables *gamesters* to attain a deeper understanding of the difficulties involved in certain new and old open game problems, which is a key to their solution. An illustration of this will be given in Section 5. *Algorithmicians*, on the other hand, will find new, interesting algorithmic challenges in the analysis of game complexities, in addition to the fun of playing games.

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For one thing, the very notion of tractability and intractability need to be redefined for games. In particular, it will turn out that tractability and polynomiality are *not* synonymous notions as for decision problems. Secondly, the notion of efficient and inefficient strategies is largely dichotomous for decision or optimization problems; in contrast, there is a wide panorama of games, spanning the gap between the very inefficient and the very efficient. Thirdly, whereas in the existential decision problems area there are only few (older) problems whose complexity have not yet been determined, such as graph isomorphism, the complexity of the majority of combinatorial games is still unknown. For decision problems, high complexity is normally a liability; for games it may be an asset.

Another idiosyncrasy of games is that they have only a very meager representation in the set of NP-complete problems, but a rich presence in the Pspace-complete and Exptime-complete sets, due to the alternating quantifiers expressing the win in a two-player games. Thus, the study of games offers insights into higher complexity classes.

Certain questions about games needing only a single existential quantifier may be NP-complete. Fraenkel et al. [33] showed, inter alia, that the question whether White can jump all of Black's kings in a checkers position involving only kings is polynomial on the $n \times n$ checkerboard, but NP-complete on a planar graph. Demaine *et al.* [20] showed that the question whether a player can win in a single move in an $n \times n$ phutball game is NP-complete. Of course puzzles involve only a single existential quantifier, so they are, if not in P, natural candidates for being NP-complete. A recent NP-completeness result for the puzzle *Clickomania* was proved by Biedl et al. [10]. An older one is *generalized instant insanity* [74]. Surprisingly, there are some puzzles that are Pspace-complete, such as a certain pebbling game of Gilbert et al. [55] and *Sokoban* [19], where blocks have to be pushed into target squares. The Pspace-completeness of such puzzles stems from the fact that vertices may be repebbled or squares may be revisited.

There are some important practical and theoretical approximability results for decision problems. Few are known for games. End positions of Berlekamp's Nimstring (see WW [7, Chapter 16]) become "reasonably tractable" and often strategies which win at Nimstring also win at dots-and-boxes, another game analyzed by Berlekamp [6], WW [7], so the former can be said to approximate the latter. Also Amazons has been analyzed by Berlekamp and associates, and in [5] he asserts that the simple "orthodox" values of all $2 \times N$ starting positions, in which a pair of Amazons of opposite color begin anywhere on an initially empty board, which may have jagged edges at either or both ends, are very good approximations. Another form of possible approximability result: guarantee a win in $(n/2) + \varepsilon$ plays of a game, out of n games. Another: find good approximations to the values of general hot partizan games.

We shall illustrate these challenges by means of sample games. All of them will be *two-player* games.

In Section 2, we show that the standard notion of tractability adopted for existential decision and optimization problems is unsuitable for games. The discussion is illustrated with two games. The important notion of *game sums* is explored in Section 3, together with its ramifications on game complexities. Seven games accompany the discourse. The essence of the previous sections is then used in Section 4 to formulate a definition of the tractability, polynomiality and efficiency of games. Annotations and illustrations with two further games are also included. This definition can be applied to shed light on the true nature of some unsolved games. An application is given in Section 5. In Section 6, we explore the nature and lure of games, considering aspects of the so-called *PlayGames* and *MathGames*. Section 7 presents a wrap-up of the various complexity issues covered in this paper, and some of their ramifications.

Numbered "Homework" problems are exercises that a reader may solve easily or after a moderate effort. Numbered "Problems" are research problems that I do not know a solution for.

2. Games and the common tractability notion

In algorithmics, we have learned that if any part of a process is exponential, then the process is defined to be *intractable*. Let us examine this common wisdom with respect to games.

Nim. This is one of the simplest games: given a finite number of tokens, arranged in piles. A move consists of selecting a pile and removing from it a positive number of tokens, possibly the entire pile. The player making the last move wins, the opponent loses.

The game has a very easy winning strategy: the XOR of the binary representation of the pile sizes is computed. If the XOR is nonzero, the Next player can win, i.e., the player who moves from the current position u (an N-position), by moving to a position with XOR zero. Otherwise the previous player can win, i.e., the player who moved to u (a P-position). In particular, for the case of two piles, the P-positions are precisely those where the pile sizes are the same.

Input size: $\Theta(\Sigma_{i=1}^{k} \log n_i)$, where n_i is the size of the ith pile.

Strategy Computation: Linear in input size.

So this is a perfect case of a tractable problem.

What about length of play?

Well, it turns out that the loser can force the winner to spend exponential time before consummating a win! Consider two piles of the same size n, which is a P-position. The loser can keep taking a single token from a pile, which has to be matched by the winner who takes a single token from the other pile, equalizing the pile sizes. The play thus lasts $\Omega(n)$ steps, which is exponential in the input size. A less trivial manifestation of exponential delay can be effected by playing Nim with more that two piles.

The problem with exponential length of play is but the tip of the iceberg of game-complexity idiosyncrasies!

This fact about Nim is rather embarrassing, since Nim, as one of the simplest games, is supposed to be *the* prototype of a polynomial game. We shall, however, overcome our embarrassment quickly, and retain Nim in the class of tractable games. The reason is simply that whereas we dislike computing in more than polynomial time, the human race relishes to see some of its members being tortured for an exponential length of time, from before the era of the Spanish inquisition and matadors, through soccer and tennis, to chess and Go!

The convention of accepting exponential length of play into the class of tractability and respectability, does not seem to have a parallel in the realm of existential complexity (polynomial and NP-complete optimization problems), where the lack of an exponential time component is a prerequisite to tractability, by definition. This little dent in our accepted view of tractability already suggests that this notion, tailored for optimization problems, has to be modified for games. Later, we will meet further reasons.

Homework 1. Find a game with a polynomial strategy that lasts exponentially long irrespective of the choice of moves of the loser or winner.

Problem 2. Nim is a *succinct* game in the sense that its input size is logarithmic. Give an example of a nonsuccinct game which has a polynomial winning strategy, but its length of play is exponential.

Problem 3. Is there an NP-hard succinct game?

Note. The succinct versions of many NP-complete decision and optimization problems are polynomial in their succinct input size. Just one example of many is “independent set” which is NP-complete for a general graph, but polynomial for the case when all degrees are $\leqslant 2$. The succinct versions tend to be easy, in general.

In contrast, there are many succinct versions of games that appear to be complex, but whose true complexity is unknown. Of course there are some exceptions for both classes of problems. Galperin and Wigderson [51] showed that certain very simple game questions are Pspace-complete in a certain model of a very specialized succinct representation of graphs. But in that model, the nonsuccinct versions are likely to be even harder. There are also games where the general form is complex and the succinct one easy. For example, Kayles is Pspace-complete on a general graph [75], but polynomial on a simple path [58]; see also ([7, Chapter 4]; [7] which will be cited as WW in the sequel). Its input size is $\log |V|$, the length of the simple path on which succinct Kayles is played. (The polynomiality stems from the periodicity of the Sprague–Grundy function for Kayles. (Kayles is played on a graph. A move is to place a counter on an unoccupied vertex which is not adjacent to any occupied vertex. Equivalently, to delete a node and all its neighbors.)

Tractability for succinct games can sometimes be established by demonstrating ultimate periodicity or additive periodicity of the g-function. A potentially mutually beneficial interaction with the “theory of combinatorics of words”, where questions of periodicity are of major concern, might be explored.

If a game has a polynomial winning strategy, do we consider it tractable no matter how long it lasts? Having begun with Nim, we shall illustrate this question with a game whose name has Nim as a prefix, namely *Nimania*, a mild case of *Dancing Mania*, sometimes observed in post-*pneumonia* patients [37–39].

Nimania. Given a positive integer n. Play begins by subtracting 1 from n. If $n=1$, the result is the empty set, and the game ends with player I winning. If $n>1$, one additional copy of the resulting number $n-1$ is adjoined, so at the end of the first

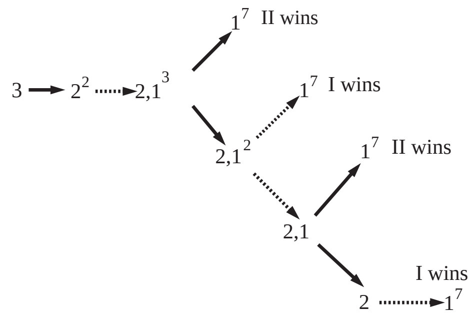

Fig. 1. Player I can win Nimania for $n=3$ in 13 moves. Solid arrows indicate player I's moves, and dashed arrows those of player II.

move there are two (indistinguishable) copies of $n-1$ (denoted $(n-1)^2$). At the kth stage, where $k \geqslant 1$, a move consists of selecting a copy of a positive integer m of the present position, and subtracting 1 from it. If $m=1$, the copy is deleted. If $m>1$, then k additional copies of the resulting number $m-1$ are adjoined to the existing numbers. The player making the last move wins; the opponent loses.

It can be shown that since the numbers in successive positions decrease (though the number of them increases), the game terminates. Who wins? For $n=1$ we saw above that player I wins. For $n=2$, player I moves to 1^2, player II to 1, hence player I again wins. For $n=3$, Fig. 1 shows that by following the lower path, player I can win in 13 moves. Unlike the cases $n=1$ and 2, however, not all moves of player I are winning for $n=3$.

An attempt to resolve the case $n=4$ by constructing a diagram similar to Fig. 1 is rather frustrating. It turns out that for $n=4$ the loser can delay the winner so that play lasts over 2^{44} moves! (There are $60^2 \times 24 \times 365 = 31,536,000$ s/year. If one move is made every second, player I will thus have to spend 557,845 years of his life to consummate his win.) We have proved, however, the following surprising facts:

(i) Player I can win for every $n \geqslant 1$.
(ii) For $n \geqslant 4$, player I cannot hope to see a win being consummated in any reasonable amount of time: the smallest number of moves is $\geqslant 2^{2^{n-2}}$, and the largest is an Ackermann function.
(iii) For $n \geqslant 4$, player I has a *robust* winning strategy: most of the time player I can make random moves; only near the end of play does player I have to pay attention (as we saw for the case $n=3$).

Since the length of play is at least doubly exponential, it seems reasonable to say that Nimania, in contrast to Nim, is intractable, though the winning strategy is robust. The complexity of computing the next move is constant; the high complexity is due to the sheer length of play.

So we have established that if play lasts for more than exponential time, it is intractable. Of course, there are numerous additional reasons for intractability in other games, some of which we will meet later.

There are, on the other hand, games that last only a constant number of moves, but the computation of the next winning move is hard. Rabin [73] gave a game of length 3 (two moves for player I, with one move of player II in-between) where player II can win, but it is undecidable to compute a winning move. Other short but intractable games are given in Jones [60], Jones and Fraenkel [61].

3. Game sums

Definition 4. (i) A game is *impartial* if the *options* (moves) of all positions are the same for both players. Otherwise the game is *partizan*.

(ii) The *game graph* of a game Γ is a digraph $G=(V,U)$, in which every vertex $u\in V$ represents a game position, and there is a directed edge $(u,v)\in E$ if and only if there is a move from u to v in Γ.

The game graph G has normally exponential size in the input size of Γ. This holds for both the seemingly complex game of chess, as for the easy Nim, since in both cases, every combination of any finite number of tokens in the game, translates into a *single* vertex of G.

Thus games have an a priori exponential complexity, quite unlike optimization and decision problems, which do not seem to exhibit an a priori bias towards polynomiality or nonpolynomiality.

For both impartial and partizan games, the potential of tractability is enhanced if the game breaks up into a sum. As Elwyn Berlekamp remarked, the situation is similar to that in other scientific endeavors, where we often attempt to decompose a given system into its functional components. This approach may yield improved insights into hardware, software or biological systems, human organizations, and abstract mathematical objects such as groups. In most cases, there are interesting issues concerning the interactions between subsystems and their neighbors.

The game of Nim is the *disjoint sum* of its component piles. Some other games decompose into sums. If a game decomposes into a disjoint sum of its components, a tractable strategy can sometimes be recovered, such as for Nim. In particular, the exponentially large game graph does not need to be constructed in these cases.

Consider the following examples.

Welter's game is an example of a game which is not a *disjoint* sum of its components. It is played on a semiinfinite strip ruled into squares, numbered consecutively from left to right, beginning with 0. Initially, a finite number of tokens is placed on distinct squares. A move consists of selecting a token and moving it to any *unoccupied* lower numbered square [81,82], see Fig. 2, where, say, the token on 5 can be placed only onto any one of the squares 3, 2, or 0. The player first unable to move loses;

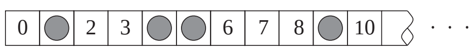

Fig. 2. Welter's game.

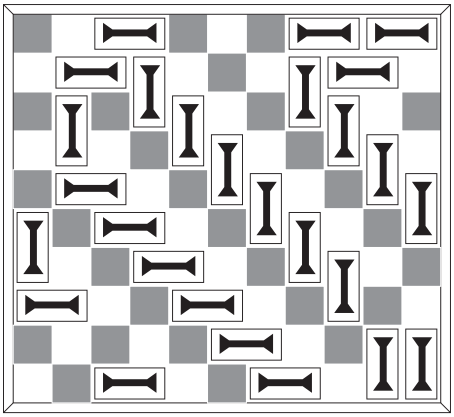

Fig. 3. Domineering position after the 14th move of L.

the opponent wins. Note that the game is equivalent to playing Nim with the proviso that the piles have *distinct* sizes at all times. This proviso makes the sum nondisjoint. A polynomial strategy can be recovered (Conway [ONAG in the sequel] [15, Chapter 13]), see also WW [7, Chapter 15]. Its validity proof is rather intricate. It also appears to be very difficult to generalize this game. For this and other properties of Welter's game, see [2,62,22,72,63].

Domineering. A chessboard or other doubly ruled board is tiled with dominoes. Every dominoe covers two adjacent squares. Left tiles vertically, Right horizontally. The player first unable to move loses, the opponent wins. See ONAG [15, Chapter 10], WW [7, Chapter 5], [3,13,65,83,85]. After the initial moves, the board may break up into a sum of partial boards. See Fig. 3 for a 10×11 board.

Domineering is partizan, unlike Nim, which is impartial. But for both cases, the game decomposes naturally into a disjoint sum of games, though this holds for domineering only *ultimately*. Chess does not appear to break up into sums in a natural way, but certain endgames of Go do.

Grundy's game. Given a finite number of piles of finitely many tokens, select a pile and split it into two nonempty piles of *different* sizes. The player first unable to move loses; the opponent wins. The game is a sum of its piles. But it is succinct. Though the Sprague–Grundy function has been computed for pile size at least up to 10^7, and a strong tendency to period 3 has been observed, no periodicity has been established. Ultimate periodicity for Grundy's game has been conjectured in WW [7, Chapter 4].

Homework 5. Find a strategy for the following game: given a finite number of piles of finitely many tokens, select a pile and split it into two nonempty piles. The player first unable to move loses; the opponent wins.

Geography. Geography games simulate on a graph the familiar word game in which two players alternately name a country (or town) subject to the restriction that the first

$$\begin{array}{cc} & \begin{matrix} & & & & j & & \end{matrix} \\ & \\ i & \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ & \vdots & & & & & \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 1 \end{pmatrix} \end{array}$$

Fig. 4. An initial position of chomp.

letter of every country matches the last letter of the previously named country, and that no country is named twice. The most common variations depend on whether the graph is undirected (U) or directed (D), and on whether no vertex (V) or no edge (E) can be repeated.

Play begins at some initially marked vertex. For vertex geography, a move consists of marking an as yet unmarked follower vertex of the last marked vertex. The player first unable to move loses; the opponent wins.

The game is nonsuccinct, but does not decompose into a sum. Directed edge geography (DEG) was proved to be Pspace complete by Schaefer [75]. The same holds for DVG. In fact, both remain Pspace-complete even for bipartite planar graphs with in/out degrees at most 2 and degree at most 3 [67,34]. Fraenkel et al. [43] showed that UEG is Pspace-complete, but polynomial for the bipartite case. It was also pointed out there that UVG is polynomial; other variations are mentioned there in the introduction.

Poset games. These are games played on partially ordered sets. The next three games are instances of poset games. The first is chomp [49,52], in which two players alternately move on a given $m \times n$ matrix of 1's (see Fig. 4). For a technical reason there is a single 0 at the origin. A move consists of pointing to some 1, say at location (i, j), and removing the entire north-east sector (i.e., replacing all the 1's by 0's inside the sector). The player removing the last 1 wins. The input size is $\log(mn)$, which is succinct. In addition, this game is not the sum of totally ordered sets, as Nim; rather it is the product of two Nim-piles. Also, it does not seem to decompose into sums.

Neither tractability nor intractability are known for general m, n. However, there is a neat proof that player I can win: If taking the element (m, n) (the "largest" element) is an opening winning move, then player I can make it and win. If it is a losing move, then there is a winning answer, say taking element (i, j). Player I's first move is then to take $(i, j)\ldots.$ This argument holds in general for poset games with a largest element, but it is nonconstructive. Incidentally, computer simulations of chomp suggest that, more often than not, an opening winning move is to take an element *other* than the largest.

Problem 6. Give a constructive, preferably polynomial, strategy for chomp.

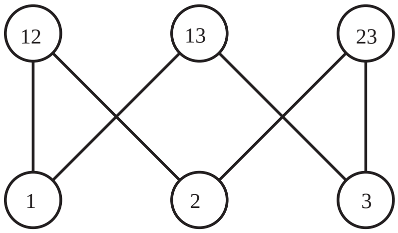

Fig. 5. A superset game on A_3^2.

We say, informally, that a *P-position* in a game is any position u from which the *P*revious player can force a win, that is, the opponent of the player moving from u. An *N-position* is any position v from which the *N*ext player can force a win, that is, the player who moves from v. A *tie position* is an end position which is a win for neither player, and a *D-position* is a draw position, i.e., a "dynamic tie" position: a player cannot force a win but has a next nonlosing move. Denote by $\mathscr{P}$ the set of all P-positions of a game, by $\mathscr{N}$ the set of all its N-positions, by $\mathscr{D}$ the set of all its D-positions, and by $F(u)$ the set of all (immediate) followers of position u. Then we have, $u \in \mathscr{P}$ if and only if $F(u) \subseteq \mathscr{N}$, $u \in \mathscr{N}$ if and only if $F(u) \cap \mathscr{P} \neq \emptyset$, and $u \in \mathscr{D}$ if and only if $F(u) \cap \mathscr{P} = \emptyset$ and $F(u) \cap \mathscr{D} \neq \emptyset$.

Normal play of a game is when the player making the last move in a game wins; *misère* play, when the player making the last move loses.

Superset game. Put $A_n^k = \{B \subseteq \{1,\ldots,n\} : 0 < |B| \leqslant k\}$. A move in this two-player game consists of pointing at an as yet unremoved subset and removing it, together with all sets *containing* it. For normal play, we then clearly have $A_n^1 \in \mathscr{P}$ if and only if $n \equiv 0 \pmod 2$. Gale and Neyman [50] showed that $A_n^2 \in \mathscr{P}$ if and only if $n \equiv 0 \pmod 3$. We may add to this the trivial statement, $A_n^0 \in \mathscr{P}$ if and only if $n \equiv 0 \pmod 1$. It is therefore conjectured there that $A_n^k \in \mathscr{P}$ if and only if $n \equiv 0 \pmod{k+1}$. A superset game on A_3^2 is shown in Fig. 5. It can be verified easily that it is a P-position, consistent with the conjecture $A_n^k \in \mathscr{P}$ if and only if $n \equiv 0 \pmod{k+1}$.

Incidentally, note that $A_n^n \in \mathscr{N}$ by the above nonconstructive argument, and so if the conjecture is true, then the unique winning move *is* to remove the largest element $\{1,\ldots,n\}$, much unlike the observed behavior of chomp. At the end of [42], the g-values of the first few positions of A_n^k have been computed.

The superset game is also succinct, and its doubly exponential game graph does not decompose into a sum.

Problem 7. Settle the Gale–Neyman conjecture.

von Neumann's Hackendot is played on a forest. A player points to an as yet unremoved vertex, and removes the unique path from that vertex to the root of the tree the vertex belongs to. This removal breaks up the tree into a forest, in general. The game is an N-position when begun on a tree, by the above nonconstructive argument. An interesting tractable strategy for normal play of the game was given by Úlehla [80].

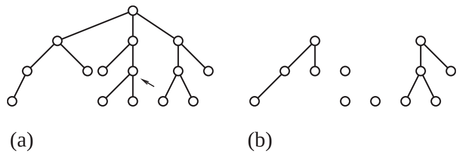

Fig. 6. A game of Hackendot.

See also WW [7, Chapter 17]. A typical game position of Hackendot is shown given in Fig. 6(a). The result after one move is seen in Fig. 6(b).

This game is nonsuccinct and it decomposes into a disjoint sum of its trees. These properties seem to contribute to its demonstrated tractability.

We point out that recently a high-school student, Byrnes [14], has proved a theorem about the periodicity of the g-function of certain restricted poset games, with an application to chomp.

4. What are tractable, polynomial and efficient games?

The above sample games and many others led us to suggest the following complexity definition for subsets of games.

Definition 8. A subset T of combinatorial games with a *polynomial* strategy has the following properties. For normal play of every $G \in T$, and every position u of G:

(a) The P-, N-, D- or tie-label of u can be computed in polynomial time.

(b) The next optimal move (from an N- to a P-position; from a D- to a D-position, from a tie- to a tie-position) can be computed in polynomial time.

(c) The winner can consummate a win in at most an exponential number of moves.

(d) The subset T is closed under summation, i.e., $G_1, G_2 \in T$ implies $G_1 + G_2 \in T$ (so (a), (b), (c) hold for $G_1 + G_2$ for every independently chosen position of G_1 and for every independently chosen position of G_2).

A subset $T_1 \subseteq T$ for which (a)–(d) hold also for misère play is a subset of games with an *efficient* strategy.

A superset $T^1 \supseteq T$ for which (a)–(c) hold is a superset of games with a *tractable* strategy.

A game in some such T or T_1 or T^1 is called *polynomial* or *efficient* or *tractable*, respectively.

A decidable game which has no tractable strategy is called *intractable*.

Ten comments about Definition 8 and its ramifications are as follows.

(1) Every efficient game is polynomial, every polynomial game is tractable. But a tractable game need not be polynomial, a polynomial game need not be efficient, quite unlike optimization and decision problems, where polynomiality and tractability were defined to be synonymous. Examples are given below. See also Section 7, (1).

(2) Instead of "polynomial time" in (a) and (b) we could have specified some low polynomial bound, so that some games complete in P (see, for example [1]), and possibly two-player games on cellular automata [29–31], would be excluded. But the decision about how low that polynomial bound should be would be largely arbitrary, and we would lose the closure under composition of polynomials. Hence we preferred not to do this.

(3) In (b), we could have included also a P-position, i.e., the requirement that the loser can compute in polynomial time a next move that makes play last as long as possible. In a way, this is included in (c). A more explicit enunciation on the speed of losing does not seem to be part of the requirements for a tractable strategy.

(4) Regarding (b), we have already observed in Section 1 that there are intractable games where the computation of the next move from an N-position is undecidable [73], and others, such as Nimania, where this computation is linear. A variety of intermediate complexities between these extremes are exhibited by other games.

(5) As was pointed out in Section 2, our convention of accepting exponential length of play into the class of tractable games does not seem to have a parallel in the realm of existential complexity (polynomial and NP-complete optimization problems). Note that (c) tends to relax the common notion of tractability, by permitting an element of exponential length, whereas the other items of Definition 8 are rather in the direction of tightening it.

(6) In Section 2, we saw that for Nim, play may last for an exponential number of moves. In general, for succinct games, the loser can delay the win for an exponential number of moves. Is there a "more natural" succinct game for which the loser cannot force an exponential delay? There are some succinct games for which the loser cannot force an exponential delay, such as Kotzig's Nim (WW [7, Chapter 15]) of length $4n$ and move set $M=\{n,2n\}$. This example is rather contrived, in that M is not fixed, and the game is not primitive in the sense of Fraenkel et al. [35, Section 3], i.e., the gcd of the move set is not 1. Is there a "natural" nonsuccinct game for which the loser can force precisely an exponential delay? Perhaps an epidemiography game with a sufficiently slowly growing function f (where at move k we adjoin $f(k)$ new copies; see [37,38]), played on a general digraph, can provide an example.

(7) There are several ways of compounding a given finite set of games—moving rules and ending rules. See, for example, [76], ONAG [15, Chapter 14]. Since the sum of games is the most natural, fundamental and important among the various compounds, we only required in (d) closure under game sums.

(8) One might consider a game efficient only if both its succinct and nonsuccinct versions fulfill conditions (a)–(d). But given a succinct game, there are often many different ways of defining a nonsuccinct variation; and given a nonsuccinct game, it is often not so clear what its succinct version is, if any. Hence, this requirement was not included in the definition.

(9) It would seem that instead of beginning the definition about a *subset* T, we could have begun right away with a *game* that satisfies the desired requirement. However, there may be different sets T, such as subsets of impartial games and subsets of partizan games, each of which satisfies (d), but their union does not. In fact, are there partizan games G_1, G_2, G_3 such that: (i) $G_1, G_2, G_3, G_1+G_2, G_2+G_3$ and all their options have

Table 1
A winning move in Moore's Nim_2

Decimal	Binary		Binary	Decimal
6	0110		0110	6
7	0111		0111	7
8	1000	$\longrightarrow$	0011	3
9	1001		0101	5
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polynomial-time strategies, (ii) $G_1 + G_3$ is NP-hard? If so, then such sets T are not disjoint. In this case two of them would contain G_2.

(10) At the beginning of the definition, "... every position u of G" is mentioned. At the end of (d) a similar remark is made once more. The reason for this "repetition" is that in (d) two copies of the same game could be used for G_1 and G_2 with the same—arbitrary—position in both. A trivial parity argument permits winning (or maintaining a draw) in such a situation for every game. The extra repetition was done to exclude from T such trivial cases.

Collections of games with a panorama of complexities bridging the gap between efficient and intractable games as per Definition 8 can be produced. Just about any imaginable perversity manifests itself in some game, and perturbs some of (a)–(d). Succinctness may affect (a). Rabin's game violates (b), and Nimania upsets (c). Misère play and interaction between tokens affect (d). Also partizan games violate (d) conditionally, in the sense that sums are Pspace-complete [71]; even if the component games have the form $\{a\|\{b|c\}\}$ with $a,b,c \in \mathbb{Z}$: [90], Moews (as cited in [9, Chapter 5]). Moore's Nim [70], WW [7, Chapter 15], and Wythoff's game (see Section 6), are not known to satisfy (d), but both are tractable.

Moore's Nim_k is a variation of Nim in which up to k piles can be reduced. Thus Nim is Nim_1. A tractable strategy can be given by expressing the pile sizes in binary as in Nim, but XOR-ing them to the base $k+1$. If this "sum" (without carries) is 0, we have a P-position. Otherwise, it is an N-position, and a move to 0 wins. For example, Table 1 depicts a winning move in Moore_2. No polynomial strategy seems to be known for this game.

Another curious strategy property is exhibited by

Two-player cellular automata games. This designates a collection of games, a subcollection of which has a barely tractable strategy. The collection depends on an integer parameter s. On the digraph depicted in Fig. 7, place a number of tokens on distinct vertices. A move consists of selecting an occupied vertex u, and *firing* its token into $q = \min(s, d_{\text{out}}(u))$ followers of u, where $d_{\text{out}}(u)$ is the outdegree of u. That is, u and q of its followers are "complemented": a token is placed at unoccupied vertices, and tokens are removed from occupied ones on every vertex of the selected "q-neighborhood". No move can be made from a leaf. The two players alternate moving, but for $s = 1$, a loop at u permits a player to pass. A player unable to move loses. The outcome may be a draw.

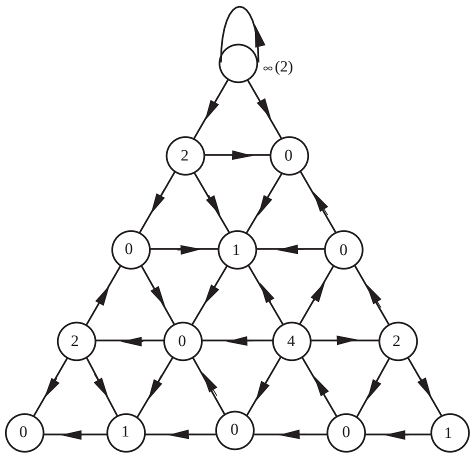

Fig. 7. Solving a cellular automata game with $s = 2$.

The labels in Fig. 7 are for the case $s = 2$. If the occupied vertices Nim-sum to a nonzero value, the player to move can win by moving to a position with Nim-sum 0, unless the sum is $\infty(K)$ with $0 \notin K$, in which case a draw can be maintained. See [28, Section 3], or [47] for the generalized Sprague–Grundy function and the generalized Nim-sum, which were first introduced by Smith [76].

Homework 9. Play a cellular automata game on the digraph of Fig. 7 for $s = 3$, and compute the corresponding labels.

Suppose that a *cellata* (cellular automata) game is played on a digraph $G = (V, E)$, with $|V| = n$. It is natural to associate with it a game graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$, where $\mathbf{V}$ is the set of all n-dimensional binary vectors, and a 1 (0) designates that the corresponding vertex is occupied (unoccupied).

Despite the exponential size of this game graph, it turns out that the generalized Sprague–Grundy function γ can be restored by restricting attention to vectors of weight $\leqslant 2(s + 1)$. For the case $s = 1$, the so-called *annihilation games* [25,44,46], we can even formulate an $O(n^6)$ algorithm for the game. Misère play (last player losing) of annihilation games was analyzed by Ferguson [24]. The complexity for $s > 1$ is still open [29–31]. A special case of cellata games has applications to the efficient computation of optimal or nearly optimal linear error correcting codes. The "lexicode" method [17,16] produces a code of length n in $O(2^{2n})$ steps. The method of Fraenkel [27] and Fraenkel and Rahat [41], yields a code of length n and minimum distance d in $O(n^{d-1})$ steps.

The polynomiality of annihilation games has a curious property.

Kalmár [64] and Smith [76] defined a *strategy in the wide sense* to be a strategy that depends on the present position and on all its antecedents, from the beginning of play. Having defined this notion, both authors concluded that it seems logical that it

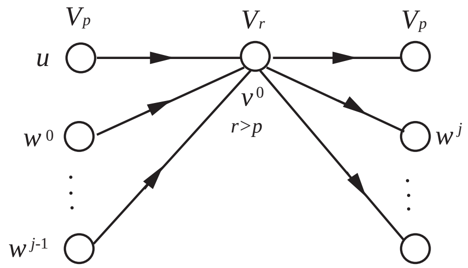

Fig. 8. Illustration of a strategy in the broad sense.

suffices to consider a *strategy in the narrow sense*, which is a strategy that depends only on the present position (the terminology *Markov strategy* suggests itself here). They then promptly restricted attention to strategies in the narrow sense.

Let us define a *strategy in the broad sense* to be a strategy that depends on the present position v and on *all* its (immediate) predecessors $u \in F^{-1}(v)$, whether or not such u is a position in the play of the game. This notion, if anything, seems to be even less needed than a strategy in the wide sense.

Yet, for annihilation games, the only strategy that we know which can produce a next winning move from an N-position in polynomial time, is a strategy in the broad sense. The reason is that γ is computed only for an induced subgraph $\mathbf{G}'$ of size $O(n^4)$, and so also the *counter function*, which points to the "correct" follower from an N-position is computed only for $\mathbf{G}'$. While $\mathbf{G}'$ suffices for restoring γ on all of $\mathbf{G}$, it restores a simulated counter c' which may lead to an ancestor rather than to a follower. This is illustrated schematically in Fig. 8: player II (the loser) moves from u with γ-value p to some v^0 with higher γ-value r. Then player I (the winner) wishes to move to some w^j with γ-value p and lower counter value c. The simulated counter c' may point to an ancestor w^0 rather than to the desired follower w^j. But $c'(w^0) < c'(v^0)$. Player I may then pretend that player II moved from w^0 to v^0, rather than from u. This procedure can continue only a finite number of times, so eventually player I will find a follower w^j of v^0 with γ-value p and simulated counter value $c'(w^j) < c'(u)$.

Annihilation games might have a polynomial strategy in the narrow sense, but we do not know of one. Perhaps the polynomial strategy in the broad sense suggested itself precisely because the game is "barely" polynomial, so to speak. Small perturbations of the annihilation games lead to Exptime- and Pspace-complete games [45,34,56].

Problem 10. Does a general cellata game have a polynomial strategy?

5. N-heap Wythoff game

In this section, we illustrate how the study of the complexity of a game in general and Definition 8 in particular, may lead to the solution of an old game problem.

Table 2
The first few P-positions of Wythoff's game

n	A_n	B_n
0	0	0
1	1	2
2	3	5
3	4	7
4	6	10
5	8	13
6	9	15
7	11	18
8	12	20
9	14	23
10	16	26

Wythoff's game. See [88,18,89,32,7,26,11,21,66]. Some of these references analyze various generalizations of the game.

The game is superficially similar to Nim, but played with two piles only. The moves are of two types: remove any positive number of tokens from a *single* pile, or take the *same* number of tokens from both piles. We denote game positions by (x, y) with $0 \leqslant x \leqslant y$, where x, y denote the two pile sizes, and proceed to examine normal play. Clearly $(0, 0)$ is a P-position. So is $(1, 2)$, as can be verified easily by considering all its followers. The P-positions (A_n, B_n) for $n \in \{0, \ldots, 10\}$ are listed in Table 2.

The table suggests the following interesting structure:

$$A_n = \text{mex}\{A_i, B_i : 0 \leqslant i < n\}, \quad B_n = A_n + n \quad \forall n \in \mathbb{Z}_{\geqslant 0},$$

where for any subset $S \subset \mathbb{Z}_{\geqslant 0}$, $S \neq \mathbb{Z}_{\geqslant 0}$, $\text{mex}\, S := \min(\mathbb{Z}_{\geqslant} 0 \setminus S) =$ least nonnegative integer not in S. We have indeed, $\mathscr{P} = \bigcup_{i=0}^{\infty} \{(A_i, B_i)\}$.

The strategy indicated by Table 2 is exponential, since it has to be computed up to $\mathrm{O}(\max(x, y))$ for the input (x, y) of size $\mathrm{O}(\log(xy))$. However, there exist two polynomial procedures for computing the P-positions [26]. One of them is based on the observation $(A_n, B_n) = (\lfloor n\phi \rfloor, \lfloor n\phi \rfloor + n)$ where $\phi = (1 + \sqrt{5})/2$ (the golden section). Thus the game is tractable, but no polynomial strategy for it is known. Why?

It might be argued that the nondisjunctive move of taking from *both* piles is the source of the difficulty. Suppose, we play a take-away game on n piles of tokens. There are two types of moves. (I) Remove any positive number of tokens from a *single* pile, (II) Remove a nonnegative vector $(a_1, \ldots, a_n)$ from all the piles, with at least two of the $a_i > 0$. Blass et al. [12] gave necessary and sufficient conditions for this game to have the same strategy as Nim. In most cases, the strategy indeed remains that of Nim. In particular, taking $(k, k + 1)$ from two Nim piles, leaves it invariant, whatever $k \in \mathbb{Z}_{>0}$ is chosen at each move.

What is special about the removal of (k, k) is that it constitutes the set of P-positions of Nim. "Shortcircuiting" those by permitting to move from one to another must upset the Nim strategy, and it produces the interesting Wythoff game. See also [40,48]. This led us to the following conjecture, a special case of which is listed in [57, Problem 53].

Define an N-heap Wythoff game as follows: Given $N \geqslant 2$ heaps of finitely many tokens, whose sizes are $A^1, \ldots, A^N$, $A^1 \leqslant \cdots \leqslant A^N$. The moves are to take any positive number of tokens from a *single* heap or to take $(a_1, \ldots, a_N) \in \mathbb{Z}^N_{\geqslant 0}$ from *all* the heaps—a_i from the ith heap—subject to the conditions: (i) $a_i > 0$ for some i, (ii) $a_i \leqslant A^i$ for all i, (iii) $a_1 \oplus \cdots \oplus a_N = 0$, where $\oplus$ denotes Nim-addition. The player making the last move wins and the opponent loses. Note that the classical Wythoff game is the case $N = 2$. Let $N \geqslant 3$. For every fixed $(A^1, \ldots A^{N-2}) \in \mathbb{Z}^{N-2}_{>0}$ with $A^1 \leqslant \cdots \leqslant A^{N-2}$, denote the P-positions by $(A^1, \ldots, A^{N-2}, A^{N-1}_n, A^N_n)$, $A^{N-2} \leqslant A^{N-1}_n \leqslant A^N_n$ for all n. We conjecture:

There exists an integer $m = m(A^1, \ldots, A^{N-2})$ such that $A^{N-1}_n = \text{mex}(\{A^{N-1}_i, A^N_i : i < n\} \cup T)$, $A^N_n = A^{N-1}_n + n$ for all $n \geqslant m$, where T is a (small) set of integers which depends only on $A^1, \ldots, A^{N-2}$.

For example, for $N = 3$, $A_1 = 1$ we have $T = \{2, 17, 22\}$; and it seems that $m = 23$. A related conjecture is that:

For every fixed $(A^1, \ldots, A^{N-2}) \in \mathbb{Z}^{N-2}_{>0}$ there exist integers $a = a(A^1, \ldots, A^{N-2})$, $m = m(A^1, \ldots, A^{N-2}) \in \mathbb{Z}_{\geqslant 1}$, such that $A^{N-1}_n \in \{\lfloor n\phi \rfloor - (a+1), \lfloor n\phi \rfloor - a, \lfloor n\phi \rfloor - (a-1)\}$ for all $m \geqslant n$, where $\phi = (1 + \sqrt{5})/2$ (the golden section). Moreover, there is a certain fractal (Fibonacci-based) regularity to the relative appearance of each of the three values $\lfloor n\phi \rfloor - (a+1), \lfloor n\phi \rfloor - a, \lfloor n\phi \rfloor - (a-1)$, which may enable one to recover a polynomial strategy.

This appears to hold for $a = 4$, $m = 35$ when $N = 3$, $A^1 = 1$.

Problem 11. Settle the two conjectures.

6. The nature and lure of games

To explore the nature and the lure of games, we consider, informally, two subclasses.

(i) *Games people play* (*PlayGames*): games that are challenging to the point that people will purchase them and play them.

(ii) *Games mathematicians play* (*MathGames*): Games that are challenging to mathematicians or other scientists to play with and ponder about, but not necessarily to “the man in the street”.

Examples of PlayGames are chess, go, hex, reversi; of MathGames: Nim-type games, Wythoff games, annihilation games, octal games.

Some “rule of thumb” properties, which seem to hold for the majority of PlayGames and MathGames are listed below.

I. **Complexity.** Both PlayGames and MathGames tend to be computationally intractable. An assortment of intractability results, from NP-hardness to Exptime-completeness, can be found, e.g., in WW [7] (NP-hardness of redwood furniture and dots-and-boxes), [23,33,34,36,45,56,86]. For summaries of further complexity results see [54,59]. There are a few tractable MathGames, such as Nim, but most games still live in *Wonderland*: we are wondering about their as yet unknown complexity. Roughly speaking, however, NP-hardness is a necessary but not a sufficient condition for being a PlayGame! Some games on Boolean formulas are Exptime-complete, yet none of them seems to have the potential of commercial marketability.

II. **Boardfeel.** None of us may know an exact strategy from a midgame position of chess, but even a novice gets some feel who of the two players is in a stronger position, merely by looking at the board. This is what we loosely call *boardfeel*. Our informal definition of PlayGames and MathGames suggests that the former do have a boardfeel, whereas the latter do not. For many MathGames, such as Nim, a player without prior knowledge of the strategy has no inkling whether any given position is "strong" or "weak" for a player. Even two positions before ultimate defeat, the player sustaining it may be in the dark about the outcome, which will stump him. The player has no boardfeel. (Even many MathGames, including Nim-type games, can be played, equivalently, on a board.)

    Thus, in the boardfeel sense, simple games are complex and complex games are simple! This paradoxical property also does not seem to have an analog in the realm of decision problems. The boardfeel is the main ingredient which makes PlayGames interesting to play.

III. **Math appeal.** PlayGames, in addition to being interesting to play, also have considerable mathematical appeal. This has been exposed recently by the theory of partizan games established by Conway and applied to endgames of Go by Berlekamp, students and associates [4], Berlekamp and Kim [8], Berlekamp and Wolfe [9], Moews [68,69], Spight [77] and Takizawa [79]. On the other hand, MathGames have their own special combinatorial appeal, of a somewhat different flavor. They appeal to and are created by mathematicians of various disciplines, who find special intellectual challenges in analyzing them. As Winkler [84] called a subset of them: "games people don't play". We might also call them, in a more positive vein, "games mathematicians play". Both classes of games have applications to areas outside game theory. Examples: surreal numbers (PlayGames), error correcting codes (MathGames). Both provide enlightenment through bewilderment, as David Wolfe and Tom Rodgers put it at the beginning of the preface to [87].

IV. **Existence.** There are relatively few PlayGames around. It seems to be hard to invent a PlayGame that catches the masses. In contrast, MathGames abound. They appeal to a large subclass of mathematicians and other scientists, who cherish producing them and pondering about them. The large proportion of MathGames-papers in games bibliographies reflects this phenomenon.

We conclude, inter alia, that for PlayGames, high complexity is desirable. Whereas in all respectable walks of life we strive towards solutions or at least approximate solutions which are polynomial, there are two less respectable human activities in which high complexity is appreciated. These are cryptography (covert warfare) and games (overt warfare). The desirability of high complexity in cryptography—at least for the encryptor!—is clear. We claim that it is also desirable for PlayGames.

It is no accident that games and cryptography team up: in both there are adversaries, who pit their wits against each other! But games are, in general, considerably harder

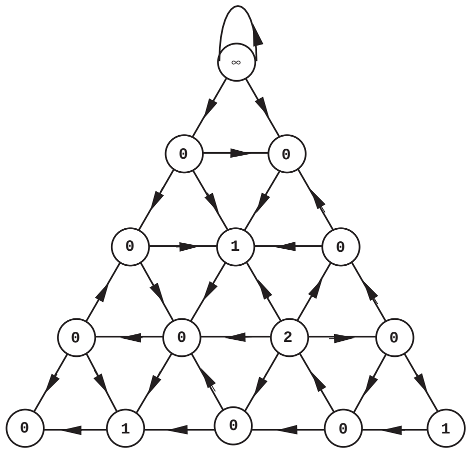

Fig. 9. Solving a cellular automata game with $s=3$.

than cryptography. For the latter, the problem whether the designer of a cryptosystem has a safe system can be expressed with two quantifiers only: $\exists$ a cryptosystem such that $\forall$ attacks on it, the cryptosystem remains unbroken? In contrast, the decision problem whether White can win if White moves first in a chess game, has the form: "$\exists\forall\exists\forall\cdots$ move: White wins?", expressing the question whether White has an opening winning move—with an unbounded number of alternating quantifiers.

Solution to Homework Problem 1. The game of "Scoring". See [28].

Solution to Homework Problem 5. It is easy to see that any position with k piles containing an even number of tokens is a P-position if and only if k is even. Indeed, every move reverses the parity of the number of piles containing an even number of tokens. For misère play the result is reversed, i.e., any position with k piles containing an even number of tokens is a P-position if and only if k is odd. In particular, all followers of every N-position are P-positions for both normal and misère play.

Solution to Homework Problem 9. The labels can be viewed in Fig. 9. We point out that it is "a lucky accident" that every *single* vertex of Figs. 7 and 9 could be labeled. In general, it is a *subset* of vertices that jointly get a label in the game-graph.

7. Epilog: a subset of 10 commandments for game complexities

The following summarizes some of the complexity issues that make games distinctive from existential decision and optimization problems.

(1) The notions of tractability and polynomiality are not synonymous for games. Nim is efficient, some Nim-type games are polynomial but not known to be efficient, Wythoff's game is tractable, but not known to be polynomial.

(2) Polynomiality of games is preserved even if length of play is a simple exponential. This is a relaxation of the requirement for decision problems, where no exponential element is permitted for tractability. The other polynomiality requirements for games are more stringent than for decision problems.
(3) The exponential size of the game graph renders games exponential a priori, unlike existential optimization problems, which do not exhibit such an a priori bias. There may be circumstances, such as decomposition into a disjoint sum, which can recover polynomiality.
(4) Most games lie in *Wonderland*; we are wondering about their as yet undetermined complexity status—quite unlike decision problems.
(5) Games have only a very meager representation in the set of NP-complete problems, but a rich one in the Pspace-complete and Exptime-complete sets of problems.
(6) The succinct forms of "most" NP-complete decision problems are polynomial; the complexity of "most" succinct games is unknown.
(7) Tractability for succinct games can sometimes be established by demonstrating ultimate periodicity or additive periodicity of the g-function. Perhaps the theory of "combinatorics of words" can contribute to establish such periodicity.
(8) For decision problems, high complexity is normally a liability; for games it is often an asset.
(9) The *boardfeel*, which makes simple games appear complex and complex games simple, does not seem to have an analog in the realm of decision problems. Neither do the notions of a strategy in the wide sense and in the broad sense.
(10) Unlike decision problems, only a few approximability results seem to be known as yet for games (some of which were mentioned in Section 1).

And the 11th commandment: It may be difficult to pull out a game from Wonderland and classify it into its precise complexity class. But it may be easier to check whether a game satisfies any of items (a)–(d) of Definition 8, and if so, to understand why the remaining items are hard to satisfy. This approach may lead to solutions for unsolved games or at least to reasonable conjectures.

In summing up, we remark that amusing oneself with games may sound like a frivolous occupation. But the fact is that the bulk of interesting and natural mathematical problems that are hardest in complexity classes beyond NP, such as Pspace, Exptime and Expspace, are two-player games; occasionally even one-player games (puzzles) or even zero-player games (Conway's "Life"). In addition to a natural appeal of the subject, there are applications or connections to various areas, including complexity, logic, graph and matroid theory, networks, error-correcting codes, surreal numbers, on-line algorithms and biology.

But when the chips are down, it is this "natural appeal" that compels both amateurs and professionals to become addicted to the subject. What is the essence of this appeal? Perhaps the urge to play games is rooted in our primal beastly instincts; the desire to corner, torture, or at least dominate our peers. An intellectually refined version of these dark desires, well hidden under the façade of local, national or international tournaments or scientific research, is the consuming strive "to beat them all", to be more clever than the most clever, in short—to create the tools to *Mathmas*ter them all in hot *comb*inatorial *comb*at! Reaching this goal is particularly satisfying and sweet in the context of combinatorial games, in view of their inherent high complexity.

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