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Counting the number of games

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Abstract

We give upper and lower bounds on $g(n)$ equal to the number of games born by day n. In particular, we give an upper bound of $g(n+1) \leqslant g(n) + 2^{g(n)} + 2$. For the lower bound, for all $\alpha < 1$, for sufficiently large n, $g(n+1) \geqslant 2^{g(n)^{\alpha}}$.

1. Introduction

For a complete introduction to combinatorial game theory, see [1] or [3]. For a terse introduction to combinatorial game theory axioms sufficient for reading this paper, see [4].

Define $\mathscr{G}_n$, the *games born by day n*, recursively as follows:

$$\mathscr{G}_0 \stackrel{\text{def}}{=} \{0\},$$
$$\mathscr{G}_n \stackrel{\text{def}}{=} \{\{\mathscr{G}^{\mathrm{L}} | \mathscr{G}^{\mathrm{R}}\} : \mathscr{G}^{\mathrm{L}}, \mathscr{G}^{\mathrm{R}} \subseteq \mathscr{G}_{n-1}\}.$$

Previously known upper and lower bounds on the number of games, $g(n)$, born by day n are, to the best of our knowledge, unpublished. Clearly, $g(n) \leqslant 4^{g(n-1)}$ since there are $2^{g(n-1)}$ choices for subset G^{L} and for G^{R}. Lower bounds can be obtained by counting only those games with names. For instance, it is not hard to see that there are $2^{n+1}-1$ *numbers* born by day n.

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2. Upper bounds

Let $\mathcal{N}_n$ be the set of new games born on day $n+1$, i.e.,

$$\mathcal{N}_n = \mathcal{G}_{n+1} \backslash \mathcal{G}_n.$$

For any game $G \in \mathcal{N}_n$, define the *top cover*, $\lceil G \rceil$, the set of minimal games in $\mathcal{G}_n$ greater than G. Similarly the *bottom cover*, $\lfloor G \rfloor$, contains the maximal games in $\mathcal{G}_n$ less than G. i.e.,

$$\lceil G \rceil = \{H \in \mathcal{G}_n : H > G \text{ and for no } H' \text{ in } \mathcal{G}_n \text{ is } H > H' > G\},$$
$$\lfloor G \rfloor = \{H \in \mathcal{G}_n : H < G \text{ and for no } H' \text{ in } \mathcal{G}_n \text{ is } H < H' < G\}.$$

In this paper, when a relation is applied to a game and a set it is assumed to hold for all elements of the set. We compare two sets of games similarly. For example, if $\mathcal{S}_1$ and $\mathcal{S}_2$ are sets, $\mathcal{S}_1 \leqslant \mathcal{S}_2$ if and only if for all $G_1 \in \mathcal{S}_1$ and $G_2 \in \mathcal{S}_2$, $G_1 \leqslant G_2$. An *anti-chain* (in the partial order $\mathcal{G}_n$) is a subset of $\mathcal{G}_n$ containing no two comparable elements. Call a pair of anti-chains, $(\mathcal{T}, \mathcal{B})$, *admissible* if $\mathcal{T} > \mathcal{B}$. In this paper, we use the symbol $G_1 \lhd\!\!\mid G_2$ to mean G_1 is less than or incomparable with G_2, i.e., $G_1 \not\geqslant G_2$. Similarly $G_1 \mid\!\!\rhd G_2$ if and only if $G_1 \not\leqslant G_2$.

Theorem 1. *There is a* 1–1 *correspondence between* $G \in \mathcal{N}_n$ *and admissible pairs* $(\mathcal{T}, \mathcal{B})$. *In particular*, $\lceil G \rceil = \mathcal{T}$ *and* $\lfloor G \rfloor = \mathcal{B}$ *if and only if*

$$G = \{\mathcal{L} | \mathcal{R}\}, \tag{1}$$

where

$$\mathcal{L} = \{H^{\mathrm{L}} \in \mathcal{G}_n : H^{\mathrm{L}} \lhd\!\!\mid \mathcal{T}\} \tag{2}$$

and

$$\mathcal{R} = \{H^{\mathrm{R}} \in \mathcal{G}_n : H^{\mathrm{R}} \mid\!\!\rhd \mathcal{B}\}. \tag{3}$$

Proof. Note that for any G, $(\lceil G \rceil, \lfloor G \rfloor)$ is an admissible pair. The following two lemmas complete the proof. □

Lemma 2. *For any admissible pair* $(\mathcal{T}, \mathcal{B})$, *there is at most one game* $G \in \mathcal{N}_n$ *such that* $\mathcal{T} = \lceil G \rceil$ *and* $\mathcal{B} = \lfloor G \rfloor$.

Proof. Suppose one such G exists. It suffices to show $G = \{\mathcal{L} | \mathcal{R}\}$, where $\mathcal{L}$ and $\mathcal{R}$ are given by Eqs. (2) and (3). Every left option G^{L} is in $\mathcal{L}$ since otherwise $G^{\mathrm{L}} \geqslant T$ for some $T \in \mathcal{T}$, and $G^{\mathrm{L}} \geqslant G$ which is never true. Similarly each $G^{\mathrm{R}} \in \mathcal{R}$. It remains to show the additional left options in $\mathcal{L}$ (and, by a parallel argument, in $\mathcal{R}$) are of no consequence. The *Gift Horse Principle* [1] states that the value of game G is unaffected by introducing new left options less than or incomparable with G. But if some $H^{\mathrm{L}} \in \mathcal{L}$ exceeded G then H^{L} (or some element between H^{L} and G) must be in $\mathcal{T}$. (No H^{L} equals G since G is a *new* day $n+1$ game.) □

Lemma 3. *For any admissible pair* $(\mathscr{T},\mathscr{B})$, *let* G *be given by* (1). *Then* $\lceil G\rceil=\mathscr{T}$ *and* $\lfloor G\rfloor=\mathscr{B}$.

Proof. We will show $\lceil G\rceil=\mathscr{T}$. (The case $\lfloor G\rfloor=\mathscr{B}$ is symmetric.)

We will first show that if $T\in\mathscr{T}$ then $T>G$ by exhibiting a winning strategy for Left (moving first or second) on $T-G$. Since $T>\mathscr{B}$, $T\in\mathscr{R}$ and Left can win moving first to $T-T$. If Right moves first to some $T-H^{\mathrm{L}}$ for $H^{\mathrm{L}}\in\mathscr{L}$, Left has a winning response since $T \mathrel{|\!\!\triangleright} H^{\mathrm{L}}$. Lastly, if Right moves on T to some $T^{\mathrm{R}}-\{\mathscr{L}|\mathscr{R}\}$, observe that $T^{\mathrm{R}} \mathrel{|\!\!\triangleright} T>\mathscr{B}$, and hence $T^{\mathrm{R}}\in\mathscr{R}$ and Left plays to $T^{\mathrm{R}}-T^{\mathrm{R}}$.

Next, we will prove that if $T'\in\mathscr{G}_n$ and $T'\geqslant G$ then $T'\geqslant T$ for some $T\in\mathscr{T}$, establishing the lemma. Suppose, to the contrary, that $T' \mathrel{\triangleleft\!\!|} \mathscr{T}$. Then $T'\in\mathscr{L}$ and Right wins moving first from $T'-G$ to $T'-T'$ and so $T' \mathrel{\triangleleft\!\!|} G$. $\square$

Corollary 4 (to Theorem 1). *For any subset* $\mathscr{S}$ *of* $\mathscr{G}_n$, *define*

$$f(\mathscr{S})=|\{G\in\mathscr{N}_n:\mathscr{S}=\lceil G\rceil\cup\lfloor G\rfloor\}|.$$

Then $f(\mathscr{S})\leqslant 2$. *In particular*,

(1) $f(\mathscr{S})=1$ *if and only if* $\mathscr{S}$ *is the union of non-empty anti-chains* $\mathscr{T}\cup\mathscr{B}$ *with* $\mathscr{T}>\mathscr{B}$, *and*
(2) $f(\mathscr{S})=2$ *if and only if* $\mathscr{S}$ *is an anti-chain.*
(3) *In all other cases*, $f(\mathscr{S})=0$.

We need only use $f(\mathscr{S})\leqslant 2$ to show $|\mathscr{N}_n|$ is bounded by twice the number of subsets of day n games, proving the following theorem due to Hickerson [5]:

Theorem 5. $|\mathscr{N}_n|\leqslant 2^{g(n)+1}$.

Dan Hoey [6] tightened this upper bound by using Corollary 4 more strongly.

Theorem 6. $g(n+1)\leqslant g(n)+2^{g(n)}+2$.

Proof. On day 0, the theorem holds. On subsequent days, the partial order of $\mathscr{G}_n$ has a top and bottom (n and $-n$) each comparable to all other elements in $\mathscr{G}_n$. Hence, no subset $\mathscr{S}$ of $\mathscr{G}_n$ containing n or $-n$ will have an isolated element (incomparable with all other games in $\mathscr{S}$) unless $\mathscr{S}$ is the singleton set $\{n\}$ or $\{-n\}$, and any subset $\mathscr{S}$ of $\mathscr{G}_n$ containing both n and $-n$ will have a 3-chain unless $\mathscr{S}=\{n,-n\}$. So,

$$\begin{aligned}|\mathscr{N}_n| &\leqslant |\{\mathscr{S}\subseteq\mathscr{G}_n:\mathscr{S}\text{ has no 3-chain and at least one isolated element}\}|\\ &\quad+|\{\mathscr{S}\subseteq\mathscr{G}_n:\mathscr{S}\text{ has no 3-chain}\}|\\ &\leqslant(2+2^{g(n)-2}-1)+(4+3(2^{g(n)-2}-1))\\ &=2+2^{g(n)}.\qquad\square\end{aligned}$$

This bound can be tightened still further by making stronger use of the fact that $\mathscr{S}$ cannot have a 3-chain. For example,

Theorem 7. $g(n+1) \leqslant g(n) + [g(n-1)^2 + \frac{5}{2}\, g(n-1) + 2]2^{g(n)-2g(n-1)}$.
(*The right-hand side is upper bounded by* $[2g(n-1)^2/4^{g(n-1)}] \cdot 2^{g(n)}$ *for* $n \geqslant 2$.)

Proof. The length of the longest chain of games born by day n is exactly $2g(n-1)+1$ [4]; call this value k. Then the number of possibilities for the elements of $\mathscr{S}$ in such a chain is at most $\binom{k}{2} + k + 1$. When two elements are taken from the chain, $\mathscr{S}$ determines at most one game in $\mathscr{G}_{n+1}$. The number of possibilities for elements of $\mathscr{S}$ outside the chain is at most $2^{g(n)-k}$. Hence,

$$g(n+1) \leqslant g(n) + \left(\binom{k}{2} + 2(k+1) \right) 2^{g(n)-k}$$
$$\leqslant g(n) + \left[g(n-1)^2 + \frac{5}{2}\, g(n-1) + 2 \right] 2^{g(n)-2g(n-1)}. \qquad \square$$

3. Lower bounds

In this section, we give a lower bound of $g(n) \geqslant 2^{g(n-1)^{\alpha}}$ where $\alpha > 0.51$ and $\alpha \to 1$ as $n \to \infty$. In addition, if $a(n)$ is the longest day n anti-chain, we show $a(n+1) \geqslant \binom{a(n)}{\lfloor a(n)/2 \rfloor}$.

We will first bound $g(n+1)$ in two ways: the first expression is simpler, and the second is tighter.

Theorem 8.

$$g(n+1) \geqslant 2^{g(n)/2g(n-1)} \tag{4}$$

and

$$g(n+1) \geqslant (8g(n-1) - 4)\,(2^{(g(n)-2)/(2g(n-1)-1)} - 1). \tag{5}$$

Proof. The games born on day n form a distributive lattice [2], and the length of every maximal chain in the lattice is exactly $l = 2g(n-1) + 1$ [4]. To obtain the first inequality, observe that one anti-chain must be of length $\geqslant g(n)/l$. By Theorem 1, each non-empty anti-chain $\mathscr{S}$ determines 4 day $n+1$ games, those with admissible pairs $(\mathscr{S}, \{-n\})$, $(\mathscr{S}, \{\})$, $(\{n\}, \mathscr{S})$, and $(\{\}, \mathscr{S})$. So,

$$g(n+1) \geqslant 4 \cdot 2^{g(n)/(2g(n-1)+1)} - 1$$

which we bound to give (4).

We can tighten the bound by counting all single-level anti-chains. On day $n > 0$, the extreme (top and bottom) elements are $\pm n$. Using the remaining $g(n) - 2$ elements, we will bound the number of non-empty anti-chains occupying a single non-extreme level by $(g(n)-2)/(l-2)$. If these levels have $a_2, \ldots, a_{l-1}$ elements, then the number of non-empty anti-chains occupying a single level is $\sum_i (2^{a_i} - 1)$ which, by the convexity of 2^x, we can bound by summing the average length of an anti-chain

$$\sum_{2 \leqslant i \leqslant l-1} (2^{a_i} - 1) = \sum_i 2^{a_i} - (l-2) \geqslant (l-2)\,(2^{(g(n)-2)/(l-2)} - 1).$$

Again, each non-empty anti-chain yields 4 games, giving (5). □

Lemma 9. $g(n) \geqslant g(n-1)^2$.

Proof. The Lemma is true for $n<5$, for the number of games born by day n are 1, 4, 22, and 1474, for $n=0$, 1, 2, and 3. Applying (5) yields $g(4) \geqslant 3 \times 10^{12}$. Otherwise, applying induction to (4),

$$g(n) \geqslant 2^{g(n-1)/2g(n-2)} \geqslant 2^{\sqrt{g(n-1)}/2} \geqslant g(n-1)^2.$$

In the last step, note $2^{\sqrt{x}/2} \geqslant x^2$ when $x \geqslant 2000$, i.e., $g(n-1) \geqslant 2000$ or $n \geqslant 5$. □

Theorem 10. $g(n) = 2^{g(n-1)^{\alpha(n)}}$, *where* $\alpha(n) > 0.51$ *and* $\alpha(n) \to 1$ *as* $n \to \infty$.

Proof. Solving for $\alpha(n)$, and writing lg to mean $\log_2$,

$$\begin{aligned}\alpha(n) &= \frac{\lg \lg g(n)}{\lg g(n-1)} \\ &\geqslant \frac{\lg g(n-1) - \lg(2g(n-2))}{\lg g(n-1)} \qquad (6)\end{aligned}$$

$$\begin{aligned} &= 1 - \frac{1 + \lg g(n-2)}{\lg g(n-1)} \\ &\geqslant 1 - \frac{1 + \lg g(n-2)}{g(n-2)/2g(n-3)} \\ &\geqslant 1 - \frac{1 + \lg g(n-2)}{1/2\sqrt{g(n-2)}}. \qquad (7)\end{aligned}$$

This last quantity monotonically increases in n for $n \geqslant 3$ and limits to 1. For $n \leqslant 3$, $\alpha(n)$ can be calculated exactly from known values. Bounding $g(4)$ by (5) yields $\alpha(4) > 0.51$. Using (6), $\alpha(4) > 0.72$. Using (7) and monotonicity, $\alpha(n) > 0.99995$ for $n \geqslant 6$. □

Finally, define $a(n)$ to be the length of the longest anti-chain on day n. Since $g(n+1) \geqslant 2^{a(n)}$, the following lower bound on $a(n)$ suggests a faster order of growth for $\{g(n)\}$ than Theorems 10 and 8.

Theorem 11.

$$a(n+1) \geqslant \binom{a(n)+1}{\lceil a(n)/2 \rceil} \geqslant 2^{a(n)}/\sqrt{a(n)}.$$

Proof. An upper bound of $\binom{a(n)}{\lfloor a(n)/2 \rfloor}$ uses elementary techniques. Let the longest day n anti-chain be $\mathscr{A}(n)$. The set of games

$$\{\{n|\mathscr{S}\} : \mathscr{S} \subset \mathscr{A}(n) \text{ and } |\mathscr{S}| = \lfloor a(n)/2 \rfloor\}$$

is an anti-chain: Left can win moving first on the difference of any pair $\{n|\mathscr{S}_1\} - \{n|\mathscr{S}_2\}$ by moving to $\{n|\mathscr{S}_1\} - G$ where $G \in \mathscr{S}_2 \backslash \mathscr{S}_1$.

The proof of the theorem requires knowledge of results from [4]. Construct $A'(n)$ from $A(n)$ with the one additional game $\{n|-n\}$. All games in $A'(n)$ are incomparable and join-irreducible in the day $n+1$ distributive lattice. Let $J(\mathscr{S})$ be the day $n+1$ join of elements in $\mathscr{S}$. Birkhoff's construction of the day $n+1$ lattice from the join-irreducibles guarantees that

$$\{\{J(\mathscr{S})\} : \mathscr{S} \subset \mathscr{A}'(n) \quad \text{and} \quad |\mathscr{S}| = \lceil |a(n)|/2 \rceil\}$$

is an anti-chain. This set has size $\binom{a(n)+1}{\lceil a(n)/2 \rceil}$ which, by Sterling's approximation, is about $2^{1+a(n)}/\sqrt{a(n) \cdot \pi/2} \geqslant 2^{a(n)}/\sqrt{a(n)}$. $\square$

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