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NOTE

DECIDING HYPERGRAPH 2-COLOURABILITY BY *H*-RESOLUTION

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Abstract. Deciding whether a hypergraph is 2-colourable is a computational problem which contains the satisfiability problem in propositional calculus as a special case. We present a method for deciding the 2-colourability of hypergraphs. This method which we call H-resolution is closely related to resolution of boolean formulas and the relation between the two is investigated.

1. Introduction

A hypergraph $G=(V, E)$ is a set of vertices V and a collection E of nonempty subsets of V called edges. We say that G is *2-colourable* if there is a partition $V=V_1\cup V_2$ such that for every $A\in E$, $A\cap V_1\neq\emptyset$, $A\cap V_2\neq\emptyset$. Deciding whether a hypergraph is 2-colourable is NP-complete [2, p. 211]. The question of deciding satisfiability of Boolean formulas in CNF is a special case of it.

Let $F=\bigwedge_{i=1}^{k} c_i$ be a Boolean formula in CNF with variables $x_1,\ldots,x_n$. Each clause c_i has the form

$$c_i=\bigvee_{\alpha\in A_i} x_\alpha \vee \bigvee_{\beta\in B_i} \bar{x}_\beta.$$

We associate with F a hypergraph $G=G_F$ with vertex set $V=\{x_1,\ldots,x_n\}\cup\{\bar{x}_1,\ldots,\bar{x}_n\}\cup\{f\}$ where f is an element not in $\{x_i \mid 1\leqslant i\leqslant n\}$. The edge set of G consists of all edges $X_i=\{x_i, \bar{x}_i\}$ $(n\geqslant i\geqslant 1)$ and all edges

$$Y_i=\{x_\alpha \mid \alpha\in A_i\}\cup\{\bar{x}_\beta \mid \beta\in B_i\}\cup\{f\} \quad (k\geqslant i\geqslant 1).$$

Notice that F is satisfiable iff G is 2-colourable. Given a truth assignment which satisfies F, we associate with it a 2-colouring $V=V_1\cup V_2$ as follows: If x_i is true in the assignment then $x_i\in V_1$, $\bar{x}_i\in V_2$ and vice versa if x_i is false. The element f belongs

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to V_2. Now, for each i, $\{x_i, \bar{x}_i\}$ intersects both V_1 and V_2. An edge Y_j which corresponds to a clause meets V_2 on f and meets V_1 because it has a true variable.

On the other hand given a 2-colouring $V = V_1 \cup V_2$ with say $f \in V_2$. We assign true to each x_i in V_1 and false to those in V_2. This is consistent because the edges $\{x_i, \bar{x}_i\}$ meet both V_1 and V_2. The edge of every clause meets V_1 on an element other than f and so every clause is satisfied. This construction will be useful in discussing the relation between resolution and our procedure for deciding 2-colourability.

2. The basic operation

Note first that if $G = (V, E)$ is a hypergraph and $G_1 = (V, E_1)$ is another hypergraph on the same vertex set and $E_1 \supset E$ then G_1 is 2-colourable only if G is. Suppose now that $B \in E$ is an edge and $\{A_x \mid x \in B\} \subseteq E$ is a collection of edges indexed by the vertices of B, which satisfy

$$A_x \cap B = \{x\} \quad \textit{for every } x \in B.$$

Denote $C = (\bigcup_{x \in B} A_x) \setminus B = \bigcup_{x \in B} (A_x \setminus \{x\})$, let $E_1 = E \cup \{C\}$ and $G_1 = (V, E_1)$. Now if $G = (V, E)$ is 2-colourable, then so is G_1, for if $V = V_1 \cup V_2$ is a 2-colouring of G then it can fail to be a 2-colouring of G_1 only in that $V_1 \supset C$, say. But then, since $V_1 \not\supset A_x$ we must have $x \in V_2$ for all $x \in B$, i.e. $V_2 \supset B$ which is a contradiction.

We say that C is the results of performing an *H-resolution* of $\{A_x \mid x \in B\}$ *over* B. It is said to be *useful* if C contains no edge of G.

We thus conclude that G_1 is 2-colourable iff G is. If by repeated application of the above mentioned operation we create an edge of size one then this proves that G is not 2-colourable. Our main result is that this is a necessary and sufficient condition. Before we state and prove our theorem, let us remark that if the above mentioned C contains any of the edges in E then introducing C into E gives no new information, for if in a hypergraph $G = (V, E)$ the edges $P, Q \in E$ satisfy $P \supset Q$ then omitting P does not change the status of G with respect to 2-colourability. Now we are ready to state and prove the following theorem.

Theorem 2.1. (i) *Let $G = (V, E)$ be a hypergraph and let C be an edge produced by an H-resolution in G. Then G is 2-colourable iff $G \cup C$ is 2-colourable.*

(ii) *If no useful H-resolution can be performed in G, then G is 2-colourable iff all edges of G have size $\geqslant 2$.*

(iii) *If G is not 2-colourable and has no edges of size $\leqslant 1$, then a useful resolution may be found in polynomial time.*

Proof. (i) Was shown above.

(ii) If edges of size $\leqslant 1$ exist, then clearly G is not 2-colourable. To prove the converse consider a maximal subset $W \subseteq V$ such that W contains no edge of G.

If for every $A \in E$ we have $A \cap W \neq \emptyset$, then set $V_1 = W$, $V_2 = V \backslash W$ to 2-colour G. If this is not the case, then there is an edge B with $B \cap W = \emptyset$. Consider any $x \in B$. Since W was chosen maximal, $W \cup \{x\}$ already contains an edge, say A_x. Since $B \cap W = \emptyset$ and $W \supset A_x \backslash \{x\}$, it follows that $A_x \cap B = \{x\}$.

Now $(\bigcup_{x \in B} A_x) \backslash B = \bigcup (A_x \backslash \{x\})$ contains no edge because

$$W \supset \bigcup (A_x \backslash \{x\}),$$

and W contains no edge.

(iii) The above procedure is clearly polynomial in the size G. □

The relationship between resolution and H-resolution is described in the following theorem.

Theorem 2.2. *Let F be a* CNF *formula on n variables and let $G = G_F$ be the corresponding hypergraph.*

(i) *F is satisfiable iff G is 2-colourable.*

(ii) *If F can be shown non-satisfiable in k resolutions, then G can be shown non 2-colourable in k H-resolutions.*

(iii) *If G can be shown non 2-colourable in k H-resolutions, then F can be shown non-satisfiable in at most $2kn$ resolutions.*

(iv) *Given a hypergraph G on n vertices, it can be decided in time $O(n^{2.5})$ whether or not $G = G_F$ for some* CNF *formula F.*

Proof. (i) Was shown above.

(ii) If C_1, C_2 can be resolved at x to yield C, then $C_1 \cup f$ and $C_2 \cup f$ can be H-resolved over $\{x, \bar{x}\}$ to yield $C \cup f$. (Here and below no distinction is made between a clause and the set of its literals.) If in k resolution steps the empty clause is produced showing that F is not satisfiable, then in k H-resolution steps the edge $\{f\}$ is produced in F showing that G is not 2-colourable.

(iii) Here we want to show that every H-resolution in G can be simulated in at most $2n$ resolution steps in F as follows.

Lemma 2.3. *Let F be a* CNF *formula on variables $x_1, \ldots, x_n$ in which all clauses $x_i \vee \bar{x}_i$ appear $(i = 1, \ldots, n)$. Let $G = G_F$ be the corresponding hypergraph. If G can be transformed to a hypergraph G' in t H-resolutions, then in no more than $2tn$-resolutions F can be transformed into a formula F' such that*

(a) *For every $A \in E(G')$ with $f \in A$ there is a clause C of F' with $C \subseteq A$.*

(b) *For every $A \in E(G')$ with $f \notin A$ there are clauses C_1, C_2 of F with C_1, $\hat{C}_2 \subseteq A$ where $\hat{C}$ is the edge containing the negations of all literals in C.*

Before we prove the lemma let us show how it implies (iii). First, the assumption that all clauses $x_i \vee \bar{x}_i$ are in F is no restriction. The addition of these clauses to a CNF formula does not change its status with respect to satisfaction. Also as we shall see the proof makes no essential use of these clauses and they are assumed to exist only to simplify some formulations.

Now consider a proof for the non 2-colourability of G which uses t H-resolutions. We perform at most $2tn$ resolutions on F as in the lemma. The proof ends when an edge of size one is produced in G which shows that G is not 2-colourable. There are two cases, either this edge is $\{f\}$ or it is $\{x\}$ for some literal x. In the first case, by (a) of the lemma the empty clause is produced in F. In the second case, by (b) either the empty clause is produced or the clauses x and $\bar{x}$ are produced for some variable x. But then they can be resolved to yield the empty clause. In every case F is proved unsatisfiable.

Proof of Lemma 2.3. By induction on t. For $t=0$ the statement holds by the relationship between F and G_F. So it suffices to show that if a hypergraph G and a formula F satisfy the lemma and if

$$A=\bigcup_{x\in B} A_x\backslash B \tag{1}$$

is an edge produced in G by H-resolving the edges $\{A_x \mid x\in B\}$ over the edge B, then in at most $2n$ resolutions either a clause C as in (a) can be produced if $f\in A$ or clauses C_1, C_2 as in (b) can be produced in F if $f\notin A$.

Consider first the case where, in (1), $f\in B$. Here $f\notin A$ and we have to produce in at most $2n$ resolutions clauses C_1, C_2 so that C_1, $\hat{C}_2\subseteq A$. Since $f\in A_f$ by (a) of Lemma 2.3 and the induction hypothesis, there is a clause C_1 in F with $C_1\subseteq A_f\backslash\{f\}\subseteq A$ so it is only C_2 we have to consider.

For every $x\in B\backslash f$ there is a clause C_x in F with $\hat{C}_x\subseteq A_x$. If for some $x\in B\backslash f$ we have $\bar{x}\notin C_x$ then $\hat{C}_x\subseteq A_x\backslash x\subseteq A$ and so we may choose $C_2=C_x$ for this x. Let us assume then that for every $x\in B$, $\bar{x}\in C_x$ holds. Now by assumption there is an F-clause D with $\hat{D}\subseteq B\backslash\{f\}$. Let $D=\{y_1,\dots,y_m\}$, and let $P_i=C_{\bar{y}_i}$ $(i=1,\dots,m)$. Resolve D with P_1, at y_1 and then resolve the result with P_2 at y_2 and so on. Let C_2 be the outcome of this sequence of resolutions. Then

$$\hat{C}_2\subseteq A$$

as we wanted.

Secondly, consider the case where in (1), $f\notin B$. For every $x\in B$ there is a clause C_x of F with $C_x\subseteq A_x$. If, for some $x\in B$, $x\notin C_x$, then $C=C_x\subseteq A_x\backslash\{x\}\subseteq A$. Otherwise, if for all $x\in B$, $x\in C_x$ holds then we use (b) of the lemma to deduce that there is an F-clause D with $\hat{D}\subseteq B$. Let $D=\{y_1,\dots,y_m\}$ and $P_i=C_{\bar{y}_i}$. Resolve $\hat{D}$ with P_1 at y_1, then resolve the result with P_2 at y_2 and continue until C_m. The resulting clause $C\subseteq A$.

This is all we need if $f\in A$. So we only have to deal now with the case where $f\in A$, $f\notin B$ where we have to produce a clause C_2 with $\hat{C}_2\subseteq A$. But then for every $x\in B$, there is a clause C_x with $\hat{C}_x\subseteq A_x$. If for some $x\in B$, $\bar{x}\notin C_x$, then C_x will do for C_2. Otherwise we use an F-clause $D\subseteq B$ and resolve C_x $(x\in D)$ in order with D as we did before, thus producing C_2 in at most n resolutions.

This completes the proof of the lemma as well as (iii).

(iv) Let $G=(V, E)$ be a hypergraph and consider a graph $H=(V, E_2)$ on the same vertex set with edges $E_2=\{A\in E \mid |A|=2\}$. It is quite easy to see that $G=G_F$ for some formula F iff the vertices of V can be paired into x, $\bar{x}$ pairs such that all edges $\{x, \bar{x}\}$ exist in H. In other words H has a perfect matching. But the existence of a perfect matching in a graph of order n can be decided in time $O(n^{2.5})$ (see [3]). □

Since it is NP-hard to decide whether a hypergraph is 2-colourable it is not to be expected that short proofs for non 2-colourability exist [2, p. 158]. Haken [1] recently solved a long-standing open problem by showing that for a class of non satisfiable boolean formulas PF_n which have $O(n^2)$ variables and $O(n^3)$ clauses every proof of their unsatisfiability takes at least C^n resolutions for some constant $C>1$. From part (c) of Theorem 2.2 it follows that any proof of the non 2-colourability of G_{PF_n} takes at least $O(C_1^n)$ or some $C_1>1$.

The relation between H-resolution and extended resolution is not yet understood. Also we do not know of similar procedures to determine chromatic numbers of hypergraphs when this number is larger than 2.

References

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