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Many-to-many two-disjoint path covers in restricted hypercube-like graphs

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ABSTRACT

A Disjoint Path Cover (DPC for short) of a graph is a set of pairwise (internally) disjoint paths that altogether cover every vertex of the graph. Given a set S of k sources and a set T of k sinks, a many-to-many k-DPC between S and T is a disjoint path cover each of whose paths joins a pair of source and sink. It is classified as *paired* if each source of S must be joined to a designated sink of T, or *unpaired* if there is no such constraint. In this paper, we show that every m-dimensional restricted hypercube-like graph with at most $m-3$ faulty vertices and/or edges being removed has a paired (and unpaired) 2-DPC joining arbitrary two sources and two sinks where $m \geqslant 5$. The bound $m-3$ on the number of faults is optimal for both paired and unpaired types.

1. Introduction

An interconnection network is frequently modeled as a graph in which the vertices and edges represent nodes and links, respectively. Since node and/or link failure is inevitable in a large network, fault tolerance is essential to the network performance. One of the central issues in the study of interconnection networks is to detect (vertex-)disjoint paths, which is naturally related to routing among nodes and fault tolerance of the network [17,25].

Disjoint path is one of the fundamental notions in graph theory from which many properties of a graph can be deduced [2,25]. A *disjoint path cover* (DPC for short) of a graph is a set of pairwise (internally) disjoint paths that collectively cover every vertex of the graph. The disjoint path cover problem finds applications in many areas such as software testing, database design, and code optimization [1,27]. In addition, the problem is concerned with applications where full utilization of network nodes is important [32].

Let G be an undirected graph. For a set of k sources $S = \{s_1, s_2, \ldots, s_k\}$ and a set of k sinks $T = \{t_1, t_2, \ldots, t_k\}$ such that $S \cap T = \emptyset$, a *many-to-many k-DPC* is a disjoint path cover composed of k paths each of which joins a pair of source and sink. It partitions the vertex set $V(G)$ into k subsets. The many-to-many k-DPC is called *paired* if each source s_i should be joined to a specific sink t_i, whereas it is called *unpaired* if each source s_i can be freely joined to a sink t_j under an arbitrary bijection σ from S to T where $t_j = \sigma(s_i)$. The other two possible k-disjoint path covers are of one-to-many type joining $S = \{s\}$ and $T = \{t_1, t_2, \ldots, t_k\}$, and of one-to-one type joining $S = \{s\}$ and $T = \{t\}$, which are clearly understandable. For more discussion, refer to [23,32].

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Definition 1. A graph G is called *f-fault paired* (resp. *unpaired*) *k-disjoint path coverable* if $f + 2k \leqslant |V(G)|$ and G has a paired (resp. unpaired) k-DPC joining an arbitrary set S of k sources and a set T of k sinks in $G \setminus F$ for any fault set F where $S \cap T = \emptyset$ and $|F| \leqslant f$.

An f-fault *paired* k-disjoint path coverable graph is, by definition, f-fault *unpaired* k-disjoint path coverable. Given S and T in a graph G, it is NP-complete to determine if there exists a one-to-one, one-to-many, or many-to-many k-DPC joining S and T for any fixed $k \geqslant 1$ [32,33]. The disjoint path cover problems have been studied for graphs such as hypercubes [5–7,10,13,19,24], recursive circulants [20,21,32,33], and hypercube-like graphs [18,22,28,33], cube of a connected graph [29,30], and k-ary n-cubes [35,37]. Necessary conditions for a graph G to be f-fault many-to-many k-disjoint path coverable have been established in terms of its connectivity $\kappa(G)$ and its minimum degree $\delta(G)$ [32,33], as shown below.

Lemma 1. (a) *If a graph G with $|V(G)| \geqslant f + 2k + 1$ is f-fault* unpaired *k ($\geqslant 2$)-disjoint path coverable, then $f + k \leqslant \delta(G) - 1$ [33].*
(b) *If a graph G is f-fault* paired *k-disjoint path coverable, then $f + 2k \leqslant \kappa(G) + 1$ [32].*

Meanwhile, Restricted Hypercube-Like graphs (RHL graphs for short) [31] are a subset of nonbipartite hypercube-like graphs that have received much attention over the recent decades. For example, crossed cubes [12], Möbius cubes [8], twisted cubes [14], multiply twisted cubes [11], Mcubes [36], and generalized twisted cubes [4] are all RHL graphs. An m-dimensional RHL graph, which will be defined in the next section, has 2^m vertices. It is an m-regular graph of connectivity m.

Every m-dimensional RHL graph with $m \geqslant 3$ is known to be (a) f-fault *unpaired* k-disjoint path coverable for any f and $k \geqslant 1$ subject to $f + k \leqslant m - 2$ [28], and (b) f-fault *paired* k-disjoint path coverable for any f and $k \geqslant 2$ subject to $f + 2k \leqslant m$ [33]. The bound $m - 2$ on $f + k$ for the unpaired type and the bound m on $f + 2k$ for the paired type respectively are one less than the optimal bounds of the necessary conditions of Lemma 1. It is still an open problem whether the optimal bounds can be achieved for all RHL graphs.

The problem has been partially solved in the sense that recursive circulants have the optimal bounds. Note that every odd-dimensional recursive circulant $G(2^m, 4)$ is included in RHL graphs (while not every even-dimensional recursive circulant is). Every m-dimensional recursive circulant $G(2^m, 4)$ with $m \geqslant 5$ is known to be (a) f-fault *unpaired* k-disjoint path coverable for any f and $k \geqslant 2$ subject to $f + k \leqslant m - 1$ [20], and (b) f-fault *paired* k-disjoint path coverable for any f and $k \geqslant 2$ subject to $f + 2k \leqslant m + 1$ [21].

In this paper, we achieve the optimal bounds of the necessary conditions of Lemma 1 for all RHL graphs where $k = 2$. In other words, we prove our main theorem that *every m-dimensional RHL graph is $(m - 3)$-fault* paired *2-disjoint path coverable where $m \geqslant 5$*. This leads to the fact that the graph is also $(m - 3)$-fault *unpaired* 2-disjoint path coverable. The bound $m - 3$ on the number of faults is the maximum possible for both paired and unpaired types.

Our contribution can also be seen as a generalization of fault-hamiltonicity of RHL-graphs, discovered in [31], that every m-dimensional RHL graph is $(m - 3)$-fault hamiltonian-connected, where a graph is said to be *hamiltonian-connected* if every pair of vertices are joined by a hamiltonian path. Note that a paired (or unpaired) 2-disjoint path coverable graph is always hamiltonian-connected [32]. To be precise, a graph G has a hamiltonian path from s to t passing through a prescribed edge (x, y), where $\{x, y\} \cap \{s, t\} = \emptyset$ and x is required to be visited before y, if and only if G has a paired 2-DPC joining the (s, x) and (y, t) pairs (i.e., $s_1 = s$, $t_1 = x$, $s_2 = y$, and $t_2 = t$). If the order in which the two end-vertices of the prescribed edge (x, y) are encountered during traversal of a hamiltonian path from s to t does not matter, it suffices to employ an unpaired 2-DPC joining $S = \{s, t\}$ and $T = \{x, y\}$ (instead of the paired one).

The rest of this paper is organized as follows. We give preliminaries in Section 2. Sections 3 and 4 are then devoted to a proof of our main theorem. Finally, we conclude in Section 5.

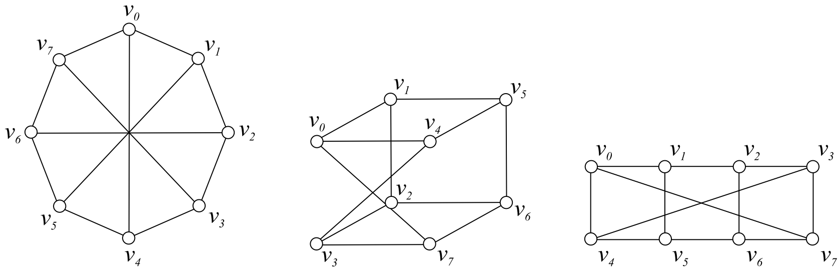
2. Preliminaries

A 3-dimensional RHL graph is isomorphic to recursive circulant $G(8, 4)$ that has a vertex set $\{v_i\colon\ 0 \leqslant i \leqslant 7\}$ and an edge set $\{(v_i, v_j)\colon\ i + 1 \text{ or } i + 4 \equiv j \pmod 8\}$. The 3-dimensional RHL graph is also isomorphic to a 3-dimensional twisted cube TQ_3 or a Möbius ladder with four spokes [26] shown in Fig. 1. An m-dimensional RHL graph, $m \geqslant 4$, is recursively defined with a graph operation $\oplus$. Given two graphs G_0 and G_1 with the same number of vertices and a bijection ϕ from $V(G_0)$ to $V(G_1)$, we denote by $G_0 \oplus_\phi G_1$ the graph whose vertex set is $V(G_0) \cup V(G_1)$ and edge set is $E(G_0) \cup E(G_1) \cup \{(v, \phi(v))\colon\ v \in V(G_0)\}$. To simplify the notation, we often omit the bijection ϕ from $\oplus_\phi$ when it is clear in the context.

Definition 2. (See [31].) A graph that belongs to RHL_m is called an *m-dimensional RHL graph* where

- $RHL_3 = \{G(8, 4)\}$, and
- $RHL_m = \{G_0 \oplus_\phi G_1\colon\ G_0, G_1 \in RHL_{m-1},\ \phi \text{ is a bijection from } V(G_0) \text{ to } V(G_1)\}$ for $m \geqslant 4$.

Every m-dimensional RHL graph, $m \geqslant 3$, is nonbipartite and has 2^m vertices of degree m. It can be easily verified by induction on m that the graph has no triangle (cycle of length three) and there exist at most two common neighbors for



(a) $G(8,4)$ (b) TQ_3 (c) Möbius ladder

Fig. 1. The 3-dimensional RHL graph.

any pair of vertices in the graph. Construction of paired DPCs in RHL graphs was suggested in [32] and improved in [33] as follows.

Lemma 2. *(See [33].) Every m-dimensional RHL graph, $m \geqslant 3$, is f-fault* paired *k-disjoint path coverable for any f and $k \geqslant 2$ subject to $f + 2k \leqslant m$.*

The disjoint path cover of a graph is naturally related to its hamiltonian properties. For instance, a hamiltonian path between two distinct vertices in a graph G is in fact a 1-DPC, irrespective of its type, of G joining the vertices. By definition, a graph of order at least three has a one-to-many 2-DPC for any $S = \{s\}$ and $T = \{t_1, t_2\}$ if and only if it is hamiltonian-connected. Also, a graph has a one-to-one 2-DPC for any $S = \{s\}$ and $T = \{t\}$ if and only if it is hamiltonian. The hamiltonian properties of RHL graphs were studied in [31] as shown below, where a graph G is said to be *f-fault hamiltonian-connected* (resp. *hamiltonian*) if any pair of vertices are joined by a hamiltonian path (resp. there exists a hamiltonian cycle) in $G \setminus F$ for any fault set F where $|F| \leqslant f$. For more discussion, refer to, for example, [15,16,34].

Lemma 3. *(See [31].) Every m-dimensional RHL graph, $m \geqslant 3$, is $(m-3)$-fault hamiltonian-connected and is $(m-2)$-fault hamiltonian.*

Using a many-to-many disjoint path cover, constructions of a hamiltonian path/cycle passing through prescribed edges were suggested in [29,32,33]. It was shown that if G is f-fault paired k $(\geqslant 2)$-disjoint path coverable, then for any fault set F where $|F| \leqslant f$, graph $G \setminus F$ has a hamiltonian path between arbitrary two vertices s and t that passes through any sequence of $k-1$ pairwise nonadjacent edges $((x_1, y_1), (x_2, y_2), \dots, (x_{k-1}, y_{k-1}))$ in the specified order where $s \neq x_i, y_i$ and $t \neq x_i, y_i$ for all $1 \leqslant i \leqslant k-1$. The s–t hamiltonian path passes through each edge (x_i, y_i) in the direction from x_i to y_i. For the problem of hamiltonian path/cycle through prescribed edges, refer to [3,9].

Hereafter, a disjoint path cover whose type is not specified is assumed to be paired. We denote by k-DPC$[\{(s_1, t_1), \dots, (s_k, t_k)\} \mid G, F]$ a k-DPC joining $S = \{s_1, s_2, \dots, s_k\}$ and $T = \{t_1, t_2, \dots, t_k\}$ in $G \setminus F$ where $S \cap T = \emptyset$. Thus, 1-DPC$[\{(v, w)\} \mid G, F]$ is a hamiltonian path between two vertices v and w in $G \setminus F$. In a *generalized* k-DPC$[\{(s_1, t_1), \dots, (s_k, t_k)\} \mid G, F]$, we allow any source s_i to be identical to its sink t_i. If $s_i = t_i$, then the s_i–t_i path in the generalized k-DPC is necessarily one-vertex path. A generalized 2-DPC$[\{(s_1, t_1), (s_2, t_2)\} \mid G, F]$ can be derived from one of the following three DPCs unless $s_1 = t_1$ and $s_2 = t_2$:

- 2-DPC$[\{(s_1, t_1), (s_2, t_2)\} \mid G, F]$ if $s_1 \neq t_1$ and $s_2 \neq t_2$,
- 1-DPC$[\{(s_1, t_1)\} \mid G, F \cup \{s_2\}]$ if $s_1 \neq t_1$ and $s_2 = t_2$, and
- 1-DPC$[\{(s_2, t_2)\} \mid G, F \cup \{s_1\}]$ if $s_1 = t_1$ and $s_2 \neq t_2$.

Both of the sources and sinks are called *terminals*. A vertex v is called to be *free* if it is neither a fault nor a terminal. An edge (v, w) is called to be *free* if it is nonfaulty and both v and w are free. Graphs G_0 and G_1 are called the *components* of $G_0 \oplus G_1$. For a vertex v in a component G_i, we denote by $\bar{v}$ the vertex adjacent to v in the other component G_{1-i}, for $i = 0, 1$.

3. Paired 2-DPCs in RHL graphs

In this section, we prove our main theorem by induction on m; however, three exceptional cases of the proof will be deferred to the next section. The induction hypothesis is that both components G_0 and G_1 of an m-dimensional RHL graph $G_0 \oplus G_1$ are $(m-4)$-fault paired 2-disjoint path coverable for $m \geqslant 6$. Sometimes, we will employ Lemma 2. Another useful fact from Lemma 3 is that both G_0 and G_1 are $(m-4)$-fault hamiltonian-connected and $(m-3)$-fault hamiltonian for $m \geqslant 5$.

In case when a single component G_i contains all the $m-3$ faults, we need some stronger properties, stated in Lemma 4, than the aforementioned property that $G_i \setminus F$ has a hamiltonian cycle. For a graph G with a hamiltonian cycle C, a nonfaulty edge (x, y) of G is called an *x-chord* or *y-chord* w.r.t. (with respect to) C if $x, y \in V(C)$ and $(x, y) \notin E(C)$. A path in a graph is represented as a sequence of vertices.

Lemma 4. *Suppose that a graph G_i of RHL_{m-1}, $m \geqslant 5$, has a fault set F where $|F| = m-3$. Let C_h be a hamiltonian cycle of $G_i \setminus F$, and u and v be arbitrary two vertices on the cycle C_h.*

(a) *$G_i \setminus F$ has a u-chord or a v-chord w.r.t. C_h unless $m = 5$ and $p = 2$ where p is the number of faulty common neighbors of u and v.*
(b) *Let C_h be represented by (u, x, v, y, P) for some subpath P. Then, $G_i \setminus F$ has a u-chord different from (u, y) w.r.t. C_h if there is no v-chord w.r.t. C_h.*

Proof. Among the $m-1$ edges incident to u, there exist $m-3$ candidates for u-chords excluding the two edges of C_h. Similarly, there also exist $m-3$ candidates for v-chords. The total number of candidates for u-chords and v-chords is $2m-6$ if $(u, v) \notin E(G) \setminus E(C_h)$; otherwise, the total number is $2m-7$. Observe that a single faulty edge excludes at most one edge from the candidates; a single faulty vertex excludes one edge from the candidates if it is a neighbor of u or v, but not both; however, a single faulty vertex excludes two edges from the candidates if it is a common neighbor of u and v. Keep in mind that G_i has no triangle and any pair of vertices of G_i have at most two common neighbors.

Suppose for the first case that $(u, v) \notin E(G) \setminus E(C_h)$. At most $|F| + p$ edges are eventually excluded from the $2m-6$ candidates (where p is the number of faulty common neighbors of u and v). Thus, the number of remaining candidate chords is at least $(2m-6) - (|F| + p) = (2m-6) - (m-3+p) = m-3-p$. The number is at least one unless $m = 5$ and $p = 2$ since $m \geqslant 5$ and $p \leqslant 2$. Suppose for the second case that $(u, v) \in E(G) \setminus E(C_h)$. There exists no common neighbor of u and v since G_i has no triangle. Therefore, at most $|F|$ edges are eventually excluded from the $2m-7$ candidates. As a result, $(2m-7) - |F| = (2m-7) - (m-3) = m-4 > 0$ for every $m \geqslant 5$. Lemma 4(a) is proved.

Suppose that C_h is (u, x, v, y, P) and there is no v-chord w.r.t. C_h. By Lemma 4(a), there exists a u-chord. Suppose the u-chord is (u, y); otherwise, we are done. Then, u and v have two nonfaulty common neighbors, x and y. Furthermore, all the $m-3$ faults are adjacent or incident to v since no v-chord exists. These imply that no fault is adjacent or incident to u, since every common neighbor of u and v is nonfaulty and $(u, v) \notin E(G)$. Thus, the number of the u-chords is at least $m-3 \geqslant 2$, which means that there exists another u-chord different from (u, y). Therefore, Lemma 4(b) is also proved. □

Now, we are ready to prove our main theorem. We do not explicitly separate the base step of $m = 5$ from the inductive step of $m \geqslant 6$ to avoid repetition. The three exceptional cases, deferred to the next section, will occur only in the base step of $m = 5$.

Theorem 1. *Every m-dimensional RHL graph is $(m-3)$-fault* paired 2*-disjoint path coverable where $m \geqslant 5$.*

Proof. Let $G_0 \oplus G_1$ be an m-dimensional RHL graph where $G_0, G_1 \in RHL_{m-1}$ and $m \geqslant 5$. For a virtual faulty edge set F', a 2-DPC of $G_0 \oplus G_1 \setminus (F \cup F')$ is also a 2-DPC of $G_0 \oplus G_1 \setminus F$. Thus, by treating arbitrary $m-3-|F|$ nonfaulty edges as virtually faulty, we assume that

$$|F| = m-3.$$

F_0 and F_1 denote the fault sets in G_0 and G_1, respectively. F_2 denotes the set of faulty edges between G_0 and G_1. Then, $F = F_0 \cup F_1 \cup F_2$. Let $f_0 = |F_0|$, $f_1 = |F_1|$, and $f_2 = |F_2|$ so that $f = |F| = f_0 + f_1 + f_2 = m-3$. We also denote the number of source–sink pairs in G_i by k_i where $i = 0, 1$, and the number of source–sink pairs between G_0 and G_1 by k_2. Then, $k = k_0 + k_1 + k_2 = 2$. We assume without loss of generality (wlog) that

$$k_0 \geqslant k_1 \text{ and if } k_0 = k_1, \quad f_0 \geqslant f_1.$$

Furthermore, it is assumed that

- $s_1, s_2, t_1, t_2 \in V(G_0)$ if $k_0 = 2$,
- $s_1, s_2, t_1 \in V(G_0)$ and $t_2 \in V(G_1)$ if $k_0 = k_2 = 1$,
- $s_1, t_1 \in V(G_0)$ and $s_2, t_2 \in V(G_1)$ if $k_0 = k_1 = 1$, and
- $s_1, s_2 \in V(G_0)$ and $t_1, t_2 \in V(G_1)$ if $k_2 = 2$.

We will construct a 2-DPC$[\{(s_1, t_1), (s_2, t_2)\} \mid G_0 \oplus G_1, F]$ for any sets F, S, and T where $|S| = |T| = 2$, $S \cap T = \emptyset$, and $|F| = m-3$. There are three cases depending on the distribution of faults.

Case 1. $f_0 = f = m-3$.

There exists a hamiltonian cycle C_h in $G_0 \setminus F_0$ by Lemma 3. We have four subcases.

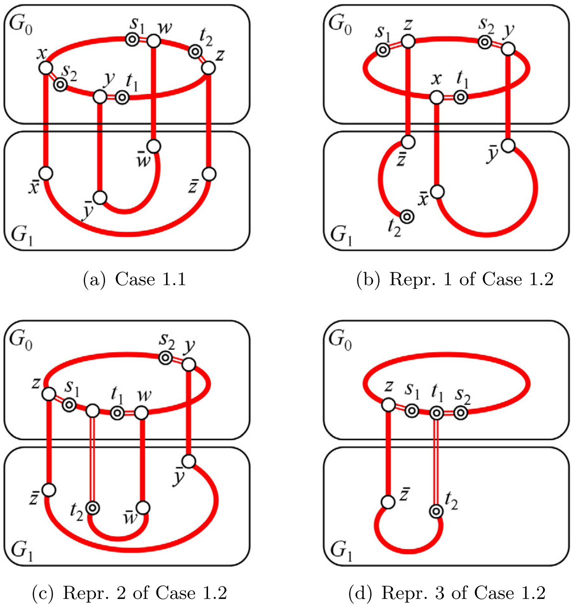


Fig. 2. Illustrations of Cases 1.1 and 1.2 in the proof of Theorem 1.

Case 1.1. $k_0 = 2$.

The hamiltonian cycle C_h can be divided into four disjoint subpaths. For example, let C_h be $(s_1, P_x, x, s_2, P_y, y, t_1, P_z, z, t_2, P_w, w)$. Then, the cycle C_h can be divided into subpaths (s_1, P_x, x), (s_2, P_y, y), (t_1, P_z, z), and (t_2, P_w, w). The subpath (s_1, P_x, x) is a one-vertex path (s_1) if $s_1 = x$, which means that s_1 is adjacent to s_2 in C_h. Each of the subpaths (s_2, P_y, y), (t_1, P_z, z), and (t_2, P_w, w) may also be a one-vertex path. Even if the order of the terminals in C_h is different from that in the aforementioned example, we can always extract four disjoint paths from C_h. As shown in Fig. 2(a), it suffices to merge the four paths of G_0 and a 2-DPC of G_1 to obtain the final 2-DPC of $G_0 \oplus G_1$. For the aforementioned example, we must use a 2-DPC$[\{(\bar{x}, \bar{z}), (\bar{y}, \bar{w})\} \mid G_1, \emptyset]$ with the edges $(x, \bar{x})$, $(y, \bar{y})$, $(z, \bar{z})$, and $(w, \bar{w})$. The existence of 2-DPC in G_1 is due to Lemma 2.

Case 1.2. $k_0 = k_2 = 1$.

The hamiltonian cycle C_h of $G_0 \setminus F_0$ can be expressed in one of the following three representations by traversing it in the reverse order if necessary. Vertices u and v are used instead of s_1 and t_1 such that $\{u, v\} = \{s_1, t_1\}$.

Repr. 1: $C_h = (u, P, x, v, P', y, s_2, P'', z)$ where $\bar{x}, \bar{y} \neq t_2$. Each of the subpaths (u, P, x), (v, P', y), and (s_2, P'', z) may be a one-vertex path. As shown in Fig. 2(b), it suffices to merge C_h and a generalized 2-DPC$[\{(\bar{x}, \bar{y}), (\bar{z}, t_2)\} \mid G_1, \emptyset]$ with $(x, \bar{x})$, $(y, \bar{y})$, and $(z, \bar{z})$ and discard (x, v), (y, s_2), and (z, u). The generalized 2-DPC exists by Lemma 2 if $\bar{z} \neq t_2$, and by Lemma 3 otherwise.

Repr. 2: $C_h = (u, \bar{t_2}, v, w, P, y, s_2, P', z)$ where v, w, y, and s_2 are all distinct. As shown in Fig. 2(c), it suffices to merge C_h and a 2-DPC$[\{(\bar{y}, \bar{z}), (\bar{w}, t_2)\} \mid G_1, \emptyset]$ with $(w, \bar{w})$, $(y, \bar{y})$, and $(z, \bar{z})$ and discard (v, w), (y, s_2), and (z, u). The existence of 2-DPC in G_1 is due to Lemma 2.

Repr. 3: $C_h = (u, v, s_2, P, z)$ where $\bar{v} = t_2$. As shown in Fig. 2(d), it suffices to merge C_h and 1-DPC$[\{(\bar{z}, t_2)\} \mid G_1, \emptyset]$ with $(z, \bar{z})$ and discard (v, s_2) and (z, u). The 1-DPC of G_1 exists by Lemma 3.

Case 1.3. $k_0 = k_1 = 1$.

The hamiltonian cycle C_h of $G_0 \setminus F_0$ can be expressed in one of the following three representations.

Repr. 1: $C_h = (s_1, x, P, y, t_1, P')$ where x and y are distinct and moreover $\bar{x}$ or $\bar{y}$ is free. If one of $\bar{x}$ and $\bar{y}$ is a terminal, let wlog $\bar{x}$ be s_2. It suffices to merge C_h and a generalized 2-DPC$[\{(s_2, \bar{x}), (\bar{y}, t_2)\} \mid G_1, \emptyset]$ with the edges $(x, \bar{x})$ and $(y, \bar{y})$ and discard the edges (s_1, x) and (y, t_1). The generalized 2-DPC exists by Lemma 2 if $\bar{x} \neq s_2$, and by Lemma 3 otherwise.

Repr. 2: $C_h = (s_1, t_1, \sigma_2, P, \tau_2)$ where $\{\bar{\sigma_2}, \bar{\tau_2}\} = \{s_2, t_2\}$. There exists an s_1-chord or t_1-chord w.r.t. C_h by Lemma 4(a) since s_1 and t_1 have no common neighbor. Assume wlog that an s_1-chord (s_1, w) exists. Then, $w \notin \{t_1, \sigma_2, \tau_2\}$ and C_h can be represented by $(s_1, t_1, \sigma_2, P_w, w, z, P_z, \tau_2)$. Notice that $z \neq \tau_2$; otherwise, G_0 would have a triangle (s_1, w, τ_2), which is a contradiction. It suffices to merge C_h and a 1-DPC$[\{(\bar{\sigma_2}, \bar{z})\} \mid G_1, \{\bar{\tau_2}\}]$ with the edges (s_1, w), $(z, \bar{z})$, and $(\tau_2, \bar{\tau_2})$ and discard the edges (s_1, t_1), (w, z), and (τ_2, s_1).

Repr. 3: $C_h = (s_1, x, t_1, \sigma_2, P, \tau_2)$ where $\{\bar{\sigma_2}, \bar{\tau_2}\} = \{s_2, t_2\}$. There exists an x-chord or σ_2-chord w.r.t. C_h by Lemma 4(a) since x and σ_2 have a nonfaulty common neighbor t_1. Suppose for the first case that there exists an x-chord (x, w). Then, $w \notin \{s_1, t_1, \sigma_2, \tau_2\}$. In addition, there exists a vertex $z \notin \{w, \sigma_2, \tau_2\}$ such that C_h can be represented by

$(s_1, x, t_1, \sigma_2, P_w, w, z, P_z, \tau_2)$ or $(s_1, x, t_1, \sigma_2, P_z, z, w, P_w, \tau_2)$. Let wlog C_h be the former one. It suffices to merge C_h and a 1-DPC$[\{(\bar{\sigma_2}, \bar{z})\} \mid G_1, \{\bar{\tau_2}\}]$ with the edges (x, w), $(z, \bar{z})$, and $(\tau_2, \bar{\tau_2})$ and discard the edges (x, t_1), (w, z), and (τ_2, s_1). Suppose for the second case that there exists no x-chord but a σ_2-chord (σ_2, w). Then, $w \notin \{x, t_1\}$; moreover, w is not s_1 by Lemma 4(b) even though w might be τ_2. In addition, C_h can be represented by $(s_1, x, t_1, \sigma_2, P_z, z, w, P_w, \tau_2)$ for some $z \notin \{\sigma_2, w\}$. It suffices to merge C_h and a 1-DPC$[\{(\bar{\sigma_2}, \bar{z})\} \mid G_1, \{\bar{\tau_2}\}]$ with the edges (σ_2, w), $(z, \bar{z})$, and $(\tau_2, \bar{\tau_2})$ and discard the edges (t_1, σ_2), (z, w), and (τ_2, s_1).

Case 1.4. $k_2 = 2$.

The hamiltonian cycle C_h of $G_0 \setminus F_0$ can be expressed in one of the following four representations.

Repr. 1: $C_h = (s_1, s_2, P)$. Then, for some distinct vertices x and y, C_h can be represented by $(s_1, s_2, P_x, x, y, P_y)$ where $\{\bar{x}, \bar{y}\} \cap \{t_1, t_2\} = \emptyset$. It suffices to merge C_h and a 2-DPC$[\{(\bar{x}, t_2), (\bar{y}, t_1)\} \mid G_1, \emptyset]$ with the edges $(x, \bar{x})$ and $(y, \bar{y})$ and discard (s_1, s_2) and (x, y).

Repr. 2: $C_h = (s_1, P, x, s_2, P', y)$ where $\bar{x} \neq t_2$, $\bar{y} \neq t_1$, and $\{\bar{x}, \bar{y}\} \neq \{t_1, t_2\}$. It suffices to merge C_h and a generalized 2-DPC$[\{(\bar{x}, t_1), (\bar{y}, t_2)\} \mid G_1, \emptyset]$ with $(x, \bar{x})$ and $(y, \bar{y})$ and discard (x, s_2) and (y, s_1).

Repr. 3: $C_h = (s_i, \tau_i, s_j, P, \tau_j)$ where $\{i, j\} = \{1, 2\}$, $\bar{\tau_i} = t_i$, and $\bar{\tau_j} = t_j$. Let $i = 1$ and $j = 2$ wlog. Then, $C_h = (s_1, \tau_1, s_2, P, \tau_2)$ where $\bar{\tau_1} = t_1$ and $\bar{\tau_2} = t_2$. There exists a τ_1-chord or τ_2-chord by Lemma 4(a). Suppose that there exists a τ_1-chord (τ_1, w). Then $w \notin \{s_1, s_2, \tau_2\}$; moreover, C_h can be represented by $(s_1, \tau_1, s_2, P_z, z, w, P_w, \tau_2)$ for some $z \neq s_2, w$. It suffices to merge C_h and a 1-DPC$[\{(\bar{z}, t_2)\} \mid G_1, \{t_1\}]$ with the edges (τ_1, t_1), (τ_1, w) and $(z, \bar{z})$ and discard the edges (s_1, τ_1), (τ_1, s_2), and (z, w). Suppose that there exists no τ_1-chord but a τ_2-chord (τ_2, w). Then $w \notin \{s_1, \tau_1\}$; furthermore, $w \neq s_2$ by Lemma 4(b). Thus, C_h can be represented by $(s_1, \tau_1, s_2, P_w, w, z, P_z, \tau_2)$ for some $z \neq w, \tau_2$. It suffices to merge C_h and a 1-DPC$[\{(\bar{z}, t_2)\} \mid G_1, \{t_1\}]$ with the edges (τ_1, t_1), (τ_2, w), and $(z, \bar{z})$ and discard the edges (τ_1, s_2), and (w, z), and (τ_2, s_1).

Repr. 4: $C_h = (s_1, \tau_1, P, \tau_2, s_2, u, P')$ where $u \notin \{s_1, s_2\}$, $\bar{\tau_1} = t_1$, and $\bar{\tau_2} = t_2$. There exists a τ_1-chord or τ_2-chord by Lemma 4(a) unless $m = 5$ and $p = 2$ where p is the number of faulty common neighbors of τ_1 and τ_2. The exceptional case that $m = 5$ and $p = 2$ will be dealt with later in Lemma 14 of Section 4.2. Assume wlog that a τ_1-chord (τ_1, w) exists. We have three subcases depending on the location of w. In the first subcase of $w = s_2$, it suffices to merge C_h and a 1-DPC$[\{(\bar{u}, t_1)\} \mid G_1, \{t_2\}]$ with the edges (τ_1, s_2), (τ_2, t_2), and $(u, \bar{u})$ and discard the edges (s_1, τ_1), (τ_2, s_2), and (s_2, u). In the second subcase that w is on the subpath (P, τ_2) of C_h, C_h can be represented by $(s_1, \tau_1, P_z, z, w, P_w, \tau_2, s_2, u, P')$, where the subpath (w, P_w, τ_2) may be a one-vertex path (τ_2). It suffices to merge C_h and a 2-DPC$[\{(\bar{u}, t_1), (\bar{z}, t_2)\} \mid G_1, \emptyset]$ with the edges (τ_1, w), $(z, \bar{z})$, and $(u, \bar{u})$ and discard the edges (s_1, τ_1), (z, w), and (s_2, u). In the final subcase that w is on (u, P'), C_h can be represented by $(s_1, \tau_1, P, \tau_2, s_2, u, P_w, w, z, P_z)$, where (u, P_w, w) may be a one-vertex path (u). It suffices to merge C_h and a 2-DPC$[\{(\bar{z}, t_1), (\bar{u}, t_2)\} \mid G_1, \emptyset]$ with the edges (τ_1, w), $(u, \bar{u})$, and $(z, \bar{z})$ and discard the edges (s_1, τ_1), (s_2, u), and (w, z).

Case 2. $f_1 = f = m - 3$.

There is a hamiltonian cycle C_h in $G_1 \setminus F_1$ from Lemma 3. We have only two subcases since we assume that $k_0 \geqslant k_1$ and moreover $f_0 \geqslant f_1$ whenever $k_0 = k_1$.

Case 2.1. $k_0 = 2$.

Suppose $m \geqslant 7$ for the first case. Then, there exists a pair of free vertices u and v in G_0 such that $(\bar{u}, \bar{v})$ is an edge of C_h. Since G_0 is paired 3-disjoint path coverable by Lemma 2, there exists a 3-DPC$[\{(s_1, u), (v, t_1), (s_2, t_2)\} \mid G_0, \emptyset]$. It suffices to merge the 3-DPC and C_h with $(u, \bar{u})$ and $(v, \bar{v})$ and discard $(\bar{u}, \bar{v})$. Suppose $m = 6$ for the second case. We claim that there exists a pair of terminal u and free vertex v in G_0 such that $(\bar{u}, \bar{v})$ is an edge of C_h. Since G_0 has four terminals and G_1 has three faults, there exists a terminal u such that $\bar{u}$ is nonfaulty. Let $C_h = (x_1, x_2, \ldots, x_q)$ for some $q \geqslant 2^5 - 3 = 29$, and $\bar{u}$ be x_3 wlog. If $\bar{x_2}$ is not a terminal, it suffices to pick up the pair $(u, \bar{x_2})$; similarly, if $\bar{x_4}$ is not a terminal, it suffices to pick up $(u, \bar{x_4})$. Now assume that both $\bar{x_2}$ and $\bar{x_4}$ (as well as $\bar{x_3}$) are terminals. Then, $\bar{x_1}$ or $\bar{x_5}$, say $\bar{x_1}$, is not a terminal. It suffices to pick up a pair of terminal $\bar{x_2}$ and free vertex $\bar{x_1}$. Thus, the claim is proved. Assume wlog that s_1 is such terminal u of the claim. It suffices to merge C_h and a 2-DPC$[\{(v, t_1), (s_2, t_2)\} \mid G_0, \{s_1\}]$ with $(s_1, \bar{s_1})$ and $(v, \bar{v})$ and discard $(\bar{s_1}, \bar{v})$. The existence of the 1-fault 2-DPC in G_0 is due to Lemma 2. The last case of $m = 5$ will be dealt with later in Lemma 15 of Section 4.2.

Case 2.2. $k_0 = k_2 = 1$.

Let the hamiltonian cycle C_h of $G_1 \setminus F_1$ be (t_2, x, P, y) where $t_2 \neq x, y$. Suppose $\{\bar{x}, \bar{y}\} \neq \{s_1, t_1\}$ for the first case. We assume wlog that $\bar{x} \notin \{s_1, t_1\}$. Then, it suffices to merge C_h and a generalized 2-DPC$[\{(s_1, t_1), (s_2, \bar{x})\} \mid G_0, \emptyset]$ with the edge $(\bar{x}, x)$ and discard the edge (t_2, x). Suppose $\{\bar{x}, \bar{y}\} = \{s_1, t_1\}$ for the second case. There exists an x-chord or y-chord w.r.t. C_h by Lemma 4(a). Assume wlog that an x-chord (x, w) exists. Then, $w \notin \{t_2, y\}$; moreover, C_h can be represented by $(t_2, x, P_z, z, w, P_w, y)$ for some $z \neq x, w$. It suffices to merge C_h and a generalized 2-DPC$[\{(s_1, t_1), (s_2, \bar{z})\} \mid G_0, \emptyset]$ with the edges $(\bar{z}, z)$ and (x, w) and discard the edges (t_2, x) and (z, w).

Case 3. $f_0 < f$ and $f_1 < f$.

We have four subcases depending on the distribution of terminals: $k_0 = 2$, $k_0 = k_2 = 1$, $k_0 = k_1 = 1$, and $k_2 = 2$. Suppose $k_0 = 2$ for the first subcase. Then, there exists a 2-DPC$[\{(s_1,t_1),(s_2,t_2)\} \mid G_0, F_0]$ unless $m = 5$ and $f_0 = 1$. The 2-DPC exists by the induction hypothesis if $m \geqslant 6$, and by Lemma 2 if $m = 5$ and $f_0 = 0$. The exceptional case that $m = 5$ and $f_0 = 1$ will be dealt with later in Lemma 16 of Section 4.2. A path in the 2-DPC of G_0 has an edge (u, v) such that both $(u, \bar{u})$ and $(v, \bar{v})$ are free. It suffices to merge the 2-DPC of G_0 and a 1-DPC$[\{(\bar{u}, \bar{v})\} \mid G_1, F_1]$ with the edges $(u, \bar{u})$ and $(v, \bar{v})$ and discard the edge (u, v). Suppose $k_0 = k_2 = 1$ for the second subcase. Unless $m = 5$ and $f_0 = 1$, there exists a 2-DPC$[\{(s_1,t_1),(s_2,x)\} \mid G_0, F_0]$ for some vertex x such that $(x, \bar{x})$ is free. The exceptional case that $m = 5$ and $f_0 = 1$ is deferred to Lemma 16. It suffices to merge the 2-DPC of G_0 and a 1-DPC$[\{(\bar{x}, t_2)\} \mid G_1, F_1]$ with edge $(x, \bar{x})$. In the third subcase of $k_0 = k_1 = 1$, it suffices to merge a 1-DPC$[\{(s_1,t_1)\} \mid G_0, F_0]$ and a 1-DPC$[\{(s_2,t_2)\} \mid G_1, F_1]$. Suppose $k_2 = 2$ for the last subcase. Unless $m = 5$ and $f_0 = 1$, there exists a 2-DPC$[\{(s_1,x),(s_2,y)\} \mid G_0, F_0]$ for some vertices x and y such that $(x, \bar{x})$ and $(y, \bar{y})$ are free. The exceptional case is deferred to Lemma 16. It suffices to merge the 2-DPC of G_0 and a 2-DPC$[\{(\bar{x},t_1),(\bar{y},t_2)\} \mid G_1, F_1]$ with edges $(x, \bar{x})$ and $(y, \bar{y})$. The 2-DPC of G_1 exists by the induction hypothesis if $m \geqslant 6$, and by Lemma 2 if $m = 5$ and $f_1 = 0$. We do not have to consider the situation that $m = 5$ and $f_1 = 1$ since it arises only in the deferred case that $m = 5$ and $f_0 = 1$. This completes the entire proof. □

Corollary 1. *Every m-dimensional RHL graph is $(m-3)$-fault* unpaired 2-*disjoint path coverable where $m \geqslant 5$.*

Corollary 2. *Suppose that a graph G in RHL_m has a fault set F where $m \geqslant 5$ and $|F| \leqslant m-3$. Then, the graph $G \setminus F$ has a hamiltonian path between any two vertices s and t that passes through an arbitrary prescribed edge (x, y) in the direction from x to y provided $\{s,t\} \cap \{x,y\} = \emptyset$.*

4. Three exceptional cases

We first study several properties on DPCs of 4-dimensional RHL graphs in Section 4.1, and then, utilizing them, deal with the three exceptional cases of the proof of Theorem 1 in Section 4.2.

4.1. Properties of RHL_4

The DPC properties of RHL_4 are addressed in Lemmas 5 through 13. All the lemmas given in this subsection, except Lemma 12, were verified by computer programs that exhaustively searched (generalized) DPCs in the 4-dimensional RHL graphs mostly on the basis of depth-first-search.

Every graph in RHL_4 is 1-fault hamiltonian-connected by Lemma 3; however, none is 2-fault hamiltonian-connected. The following lemma shows that given two faults, there exist at least two nonfaulty vertices that have a hamiltonian path to any other nonfaulty vertex.

Lemma 5. *Let $G \in RHL_4$ have a fault set F, $|F| = 2$. Then, there exists a subset X of nonfaulty vertices, $|X| \geqslant 2$, such that for each $x \in X$, there exists a* 1-DPC$[\{(x,y)\} \mid G, F]$ *for any nonfaulty vertex $y \neq x$.*

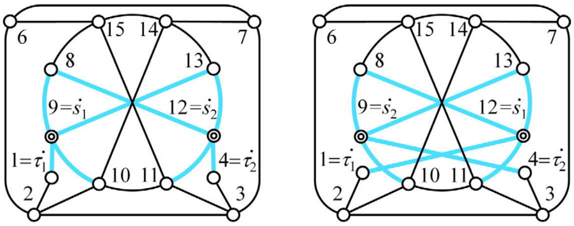
Every graph in RHL_4 is (paired) 2-disjoint path coverable by Lemma 2; however, none except one graph is 1-fault 2-disjoint path coverable.[1] The following lemma shows that given a single fault and three terminals s_1, t_1, and s_2, there exists a generalized 2-DPC joining pairs (s_1,t_1) and (s_2,x) for some nonfaulty vertex x.

Lemma 6. *Let $G \in RHL_4$ have a fault set F, $|F| = 1$, and three terminals s_1, t_1, and s_2 be given in $G \setminus F$. Then, there exists a subset X of nonfaulty vertices, $|X| \geqslant 3$, such that for each $x \in X$, there exists a generalized* 2-DPC$[\{(s_1,t_1),(s_2,x)\} \mid G, F]$.

No graph in RHL_4 is (1-fault) 3-disjoint path coverable by Lemma 1(b). Lemmas 7 and 8 show that given a fault set F, $|F| \leqslant 1$, and four terminals s_1, t_1, s_2, and t_2, we can always pick up two nonfaulty vertices x and y such that there exists a generalized 3-DPC: one path of the generalized 3-DPC joins s_i and t_i, and the other two join $\{s_j, t_j\}$ and $\{x, y\}$, where $\{i, j\} = \{1, 2\}$.

Lemma 7. *Let four terminals s_1, t_1, s_2, and t_2 be given in $G \in RHL_4$. Then, for any vertex x in G (whether it is a terminal or not), there exists a subset Y_x of vertices (depending on x), $|Y_x| \geqslant 3$, such that for each $y \in Y_x$, at least one of the following four DPCs exists where $F = \emptyset$:*

[1] The unique 4-dimensional RHL graph that is 1-fault 2-disjoint path coverable is a graph $G_0 \oplus_\phi G_1$ under a bijection ϕ such that $\phi(v_i) = w_{3i}$ for every i, where $V(G_0) = \{v_0, v_1, \ldots, v_7\}$, $V(G_1) = \{w_0, w_1, \ldots, w_7\}$, v_i is adjacent to both v_{i+1} and v_{i+4} for every i, and similarly for w_i. Here, all arithmetic on the indices of vertices is done modulo 8.



(a) H_a (b) H_b

Fig. 3. Graphs H_a and H_b.

- *a generalized* 3-DPC$[\{(s_1,t_1),(s_2,x),(y,t_2)\} \mid G,F]$,
- *a generalized* 3-DPC$[\{(s_1,t_1),(s_2,y),(x,t_2)\} \mid G,F]$,
- *a generalized* 3-DPC$[\{(s_1,x),(y,t_1),(s_2,t_2)\} \mid G,F]$, *and*
- *a generalized* 3-DPC$[\{(s_1,y),(x,t_1),(s_2,t_2)\} \mid G,F]$.

Lemma 8. *Let $G \in RHL_4$ have a fault set F, $|F|=1$, and four terminals s_1, t_1, s_2, and t_2 be given in $G \setminus F$. Then, there exists a subset X of nonfaulty vertices (whether terminals or not), $|X| \geqslant 2$, such that for each $x \in X$, there exists a subset Y_x of nonfaulty vertices, $|Y_x| \geqslant 2$, such that for each $y \in Y_x$, at least one of the four DPCs of Lemma 7 exists.*

Hereafter, we are concerned with 4-dimensional RHL graphs with two terminals s_1 and s_2 given. We introduce the notions of *good*, *excellent*, and *perfect* vertices.

Definition 3. Let $G \in RHL_4$ have a fault set F, and two terminals s_1 and s_2 be given in $G \setminus F$. For a free vertex x, we let Y_x be the set of free vertices such that any $y \in Y_x$ admits both a 2-DPC$[\{(s_1,x),(s_2,y)\} \mid G,F]$ and a 2-DPC$[\{(s_1,y),(s_2,x)\} \mid G,F]$. Then, x is said to be *good*, *excellent*, and *perfect*, respectively, if $|Y_x| \geqslant 4$, $|Y_x| \geqslant 8$, and $|Y_x| = |V(G) \setminus (F \cup \{s_1,s_2,x\})|$.

Lemma 9. *Let $G \in RHL_4$ have a fault set F, $|F|=1$, and two terminals s_1 and s_2 be given in $G \setminus F$.*

(a) *$G \setminus F$ has at least eight excellent vertices.*
(b) *If $G \setminus F$ has exactly eight excellent vertices, then* (i) *all the free vertices are good,* (ii) *at least two free vertices are perfect, and* (iii) *there exists a subset Y of free vertices, $|Y| \geqslant 4$, such that for each $y \in Y$, there exists a generalized 2-DPC$[\{(s_1,s_1),(s_2,y)\} \mid G,F]$ (and symmetrically, there exists a subset Y' of free vertices, $|Y'| \geqslant 4$, such that for each $y \in Y'$, there exists a generalized 2-DPC$[\{(s_1,y),(s_2,s_2)\} \mid G,F]$).*

The remaining part of this subsection is concerned with the first exceptional case of the proof of Theorem 1 where $f_0 = f = 2$ and $k_2 = 2$. Two graphs H_a and H_b introduced below are useful to describe the component G_0 of a 5-dimensional RHL graph $G_0 \oplus G_1$ in the first exceptional case. Each of the two graphs H_a and H_b has 14 vertices and 24 edges as shown in Fig. 3. The four vertices of each are labeled with $\dot{s}_1$, $\dot{s}_2$, $\dot{\tau}_1$, and $\dot{\tau}_2$ so that $\{\dot{s}_1,\dot{s}_2\} = \{9,12\}$, $\dot{\tau}_1 = 1$, and $\dot{\tau}_2 = 4$. The two graphs have a structural similarity that $H_a \setminus \{\dot{s}_1,\dot{s}_2\}$ is isomorphic to $H_b \setminus \{\dot{s}_1,\dot{s}_2\}$.

Lemma 10. *Let $G_0 \in RHL_4$ have a fault set F composed of two vertices. Given two sources s_1 and s_2 and two free vertices τ_1 and τ_2 in $G_0 \setminus F$, suppose that there exists no pair of vertices u and v in $G_0 \setminus F$ such that $u \neq \tau_2$, $v \neq \tau_1$, $\{u,v\} \neq \{\tau_1,\tau_2\}$, and a generalized 2-DPC$[\{(s_1,u),(s_2,v)\} \mid G_0,F]$ exists. Then, $G_0 \setminus F$ is isomorphic to H_a or H_b under a mapping ρ such that $\rho(s_1) = \dot{s}_1$, $\rho(s_2) = \dot{s}_2$, $\rho(\tau_1) = \dot{\tau}_1$, and $\rho(\tau_2) = \dot{\tau}_2$.*

Lemma 11. *For each triple $(u,v,\{x,y\})$ of the following 26 ones, H_a has a 3-DPC$[\{(\dot{s}_1,u),(\dot{s}_2,v),(x,y)\} \mid H_a,\emptyset]$ and $\{u,v,x,y\} \cap \{\dot{s}_1,\dot{s}_2,\dot{\tau}_1,\dot{\tau}_2\} = \emptyset$:* $(2,6,\{8,13\})$, $(2,8,\{11,13\})$, $(2,10,\{8,13\})$, $(2,13,\{7,8\})$, $(6,7,\{8,13\})$, $(6,13,\{8,10\})$, $(7,3,\{8,13\})$, $(7,8,\{10,13\})$, $(7,10,\{8,13\})$, $(7,13,\{8,15\})$, $(8,3,\{6,13\})$, $(8,6,\{13,14\})$, $(8,7,\{11,13\})$, $(8,10,\{7,13\})$, $(8,13,\{10,11\})$, $(10,8,\{6,13\})$, $(10,11,\{8,13\})$, $(11,3,\{8,13\})$, $(11,6,\{8,13\})$, $(11,8,\{13,14\})$, $(11,13,\{6,8\})$, $(13,3,\{8,10\})$, $(13,6,\{8,11\})$, $(13,8,\{6,7\})$, $(13,10,\{8,15\})$, *and* $(13,11,\{7,8\})$.

Lemma 12. *For each $(u,v,\{x,y\})$ of the 26 triples of Lemma 11, H_b also has a* 3-DPC$[\{(\dot{s}_1,u),(\dot{s}_2,v),(x,y)\} \mid H_b,\emptyset]$ *and* $\{u,v,x,y\} \cap \{\dot{s}_1,\dot{s}_2,\dot{\tau}_1,\dot{\tau}_2\} = \emptyset$.

Proof. It holds that $\{u,v,x,y\} \cap \{\dot{s}_1,\dot{s}_2,\dot{\tau}_1,\dot{\tau}_2\} = \emptyset$ since $\{\dot{s}_1,\dot{s}_2,\dot{\tau}_1,\dot{\tau}_2\}$ of H_b is equal to that of H_a. Suppose that there is a 3-DPC$[\{(\dot{s}_1,u),(\dot{s}_2,v),(x,y)\} \mid H_a,\emptyset]$ in H_a for some triple $(u,v,\{x,y\})$. Then, the path starting from $\dot{s}_1$ in the 3-DPC definitely includes subpath $(9,1,2)$ since $\dot{\tau}_1$ of degree two should be an intermediate vertex of a path in the DPC. Similarly, the path starting from $\dot{s}_2$ includes subpath $(12,4,3)$. If the two subpaths $(9,1,2)$ and $(12,4,3)$ respectively are replaced with $(12,1,2)$ and $(9,4,3)$, the resulting DPC is indeed a 3-DPC$[\{(\dot{s}_1,u),(\dot{s}_2,v),(x,y)\}\mid H_b,\emptyset]$ of H_b for the same triple $(u,v,\{x,y\})$. □

Definition 4 is also concerned with the first exceptional case of the proof of Theorem 1 where $f_0=f=2$ and $k_2=2$; thus, two sinks t_1 and t_2 are given in the component G_1 of a 5-dimensional RHL graph $G_0\oplus G_1$.

Definition 4. For a graph $G_1\in RHL_4$ with two sinks t_1 and t_2 given, a triple $(u,v,\{x,y\})$ with $\{u,v,x,y\}\cap\{t_1,t_2\}=\emptyset$ is called to be *successful* if at least one of the following four DPCs exists:

- a 3-DPC$[\{(t_1,u),(t_2,x),(v,y)\}\mid G_1,\emptyset]$,
- a 3-DPC$[\{(t_1,u),(t_2,y),(v,x)\}\mid G_1,\emptyset]$,
- a 3-DPC$[\{(t_2,v),(t_1,x),(u,y)\}\mid G_1,\emptyset]$, and
- a 3-DPC$[\{(t_2,v),(t_1,y),(u,x)\}\mid G_1,\emptyset]$.

Lemma 13. *Let G_0 satisfy the conditions of Lemma 10 (i.e., its fault set F is composed of two vertices; two sources s_1 and s_2 and two free vertices τ_1 and τ_2 are given in $G_0\setminus F$; there exists no pair of vertices u and v in $G_0\setminus F$ such that $u\neq\tau_2$, $v\neq\tau_1$, $\{u,v\}\neq\{\tau_1,\tau_2\}$, and a generalized 2-DPC$[\{(s_1,u),(s_2,v)\}\mid G_0,F]$ exists). Let G_1 be another graph in RHL_4 with two sinks t_1 and t_2 given. Then, for every bijection ϕ from $V(G_0)$ to $V(G_1)$ such that $\phi(\tau_1)=t_1$ and $\phi(\tau_2)=t_2$, there exists a triple $(u,v,\{x,y\})$ in G_0 with $\{u,v,x,y\}\cap\{s_1,s_2,\tau_1,\tau_2\}=\emptyset$ such that a 3-DPC$[\{(s_1,u),(s_2,v),(x,y)\}\mid G_0,F]$ exists and its corresponding triple $(\phi(u),\phi(v),\{\phi(x),\phi(y)\})$ of G_1 is successful.*

Lemma 13 was verified by an expedient discussed below, because a straightforward examination of every bijection is extremely time-consuming and practically impossible. Each graph G_0 of Lemma 13 with F being removed is isomorphic to H_a or H_b by Lemma 10 under a mapping ρ such that $\rho(s_1)=\dot{s}_1$, $\rho(s_2)=\dot{s}_2$, $\rho(\tau_1)=\dot{\tau}_1$ and $\rho(\tau_2)=\dot{\tau}_2$. Thus, we can restrict our attention to H_a and H_b although a dozen or so graphs in RHL_4 satisfy the conditions of Lemma 10. Besides, H_a and H_b fortunately have the same set of 26 triples shown in Lemma 11 such that for each triple $(u,v,\{x,y\})$ with $\{u,v,x,y\}\cap\{\dot{s}_1,\dot{s}_2,\dot{\tau}_1,\dot{\tau}_2\}=\emptyset$, there exists a 3-DPC for pairs $(\dot{s}_1,u)$, $(\dot{s}_2,v)$, and (x,y). Thus, we again restrict our attention only to H_a.

Each graph in RHL_4 with two sinks given turned out to have a dominantly large number of successful triples. Thus, we list the triples not successful for each pair of terminals t_1 and t_2 in an arbitrary graph G_1 of RHL_4 such that for each triple $(u,v,\{x,y\})$ with $\{u,v,x,y\}\cap\{t_1,t_2\}=\emptyset$, there exists none of the four DPCs of Definition 4. After that, it suffices to check whether or not there exists a bijection ϕ from $V(H_a)\cup\{0,5\}$ to $V(G_1)$ such that $\phi(\tau_1)=t_1$, $\phi(\tau_2)=t_2$, and all of the 26 triples of Lemma 11 are mapped to the triples of G_1 not successful. No such bijection was detected for any pair of terminals t_1 and t_2 in the graph G_1. As a result, Lemma 13 was verified.

4.2. Lemmas for the exceptional cases

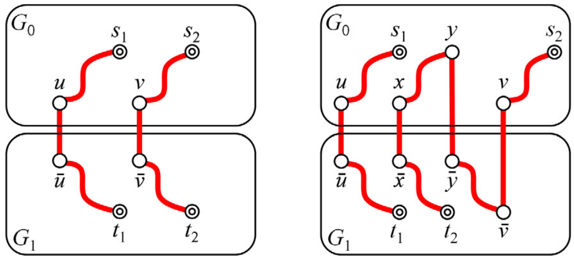
The DPC properties of RHL_4 discussed in Section 4.1 allow us to prove the three exceptional cases of the proof of Theorem 1. All of the exceptional cases are for $m=5$, which will be dealt with one by one in the following lemmas.

Lemma 14. *Every 5-dimensional RHL graph $G_0\oplus G_1$ has a paired 2-DPC when $f_0=f=2$, $k_2=2$, G_0 contains two faulty vertices, and $G_0\setminus F$ has a hamiltonian cycle of the form $(s_1,\bar{t}_1,P,\bar{t}_2,s_2,u,P')$.*

Proof. Suppose that there exists a *desirable* pair of vertices u and v in $G_0\setminus F$ such that $u\neq\bar{t}_2$, $v\neq\bar{t}_1$, $\{u,v\}\neq\{\bar{t}_1,\bar{t}_2\}$, and a generalized 2-DPC$[\{(s_1,u),(s_2,v)\}\mid G_0,F]$ exists. As shown in Fig. 4(a), it suffices to merge the generalized 2-DPC of G_0 and a generalized 2-DPC$[\{(\bar{u},t_1),(\bar{v},t_2)\}\mid G_1,\emptyset]$ with edges $(u,\bar{u})$ and $(v,\bar{v})$. On the contrary, suppose that there exists no such desirable pair in $G_0\setminus F$. Then, G_0 satisfies the conditions of Lemma 10 under the assumption that $\bar{t}_1=\tau_1$ and $\bar{t}_2=\tau_2$. Thus, there exists a triple $(u,v,\{x,y\})$ in $G_0\setminus F$ with $\{u,v,x,y\}\cap\{s_1,s_2,\tau_1,\tau_2\}=\emptyset$, by Lemma 13, such that a 3-DPC$[\{(s_1,u),(s_2,v),(x,y)\}\mid G_0,F]$ exists and its corresponding triple $(\bar{u},\bar{v},\{\bar{x},\bar{y}\})$ of G_1 is successful. So, at least one of the four 3-DPCs of Definition 4 exists. Whichever 3-DPC exists, say a 3-DPC$[\{(t_1,\bar{u}),(t_2,\bar{x}),(\bar{v},\bar{y})\}\mid G_1,\emptyset]$, it suffices to merge the 3-DPC of G_0 and the 3-DPC of G_1 with edges $(u,\bar{u})$, $(v,\bar{v})$, $(x,\bar{x})$, and $(y,\bar{y})$ as shown in Fig. 4(b). □

Lemma 15. *Every 5-dimensional RHL graph $G_0\oplus G_1$ has a paired 2-DPC when $f_1=f=2$ and $k_0=2$.*

Proof. There exists a nonfaulty vertex x in G_1, by Lemma 5, such that a 1-DPC$[\{(x,x')\}\mid G_1,F_1]$ exists for any nonfaulty vertex $x'\neq x$. Then, for the vertex $\bar{x}$ in G_0, there exists at least one vertex $\bar{y}$ in G_0, $y\notin F_1$, such that one of the four generalized 3-DPCs of Lemma 7 exists. Recall that $|F_1|=2$. Whichever generalized 3-DPC exists, say a generalized 3-DPC$[\{(s_1,t_1),(s_2,\bar{x}),(\bar{y},t_2)\}\mid G_0,\emptyset]$, it suffices to merge the generalized 3-DPC of G_0 and a 1-DPC$[\{(x,y)\}\mid G_1,F_1]$ of G_1 with edges $(x,\bar{x})$ and $(y,\bar{y})$. □



(a) A desirable pair (u, v) exists. (b) No desirable pair exists.

Fig. 4. Illustration of the proof of Lemma 14.

Lemma 16. *Every 5-dimensional RHL graph $G_0 \oplus G_1$ has a paired 2-DPC when $f_0 = 1$ and $f_1 + f_2 = 1$.*

Proof. The case of $k_0 = k_1 = 1$ was already covered in the proof of Theorem 1. There remain three cases.

Case 1. $k_0 = 2$.

There exists a pair of nonfaulty vertices x and y in G_0, by Lemma 8, such that one of the four generalized 3-DPCs of Lemma 7 exists in G_0 and $\bar{x}$, $(x, \bar{x})$, $\bar{y}$, and $(y, \bar{y})$ are all nonfaulty. Recall that $f_1 + f_2 = 1$. Whichever generalized 3-DPC exists, it suffices to merge the generalized 3-DPC of G_0 and a 1-DPC$[\{(\bar{x}, \bar{y})\} \mid G_1, F_1]$ with edges $(x, \bar{x})$ and $(y, \bar{y})$.

Case 2. $k_0 = k_2 = 1$.

There exists at least one nonfaulty vertex x in G_0, by Lemma 6, such that $\bar{x} \neq t_2$, $\bar{x} \notin F$, $(x, \bar{x}) \notin F$, and a generalized 2-DPC$[\{(s_1, t_1), (s_2, x)\} \mid G_0, F_0]$ exits. It suffices to merge the generalized 2-DPC of G_0 and a 1-DPC$[\{(\bar{x}, t_2)\} \mid G_1, F_1]$ with edge $(x, \bar{x})$.

Case 3. $k_2 = 2$.

Case 3.1. $f_1 = 0$ $(f_2 = 1)$.

Since $G_0 \setminus F_0$ has at least eight excellent vertices by Lemma 9(a), there is an excellent vertex x of G_0 such that $(x, \bar{x})$ is free. Then, there exists a free vertex $y \neq x$ in G_0, due to Definition 3, such that $(y, \bar{y})$ is free and a 2-DPC$[\{(s_1, x), (s_2, y)\} \mid G_0, F_0]$ exists. It suffices to merge the 2-DPC of G_0 and a 2-DPC$[\{(\bar{x}, t_1), (\bar{y}, t_2)\} \mid G_1, \emptyset]$ in G_1 with edges $(x, \bar{x})$ and $(y, \bar{y})$.

Case 3.2. $f_1 = 1$ $(f_2 = 0)$.

There are 16 nonfaulty edges of the type $(x, \bar{x})$ for $x \in V(G_0)$ since $f_2 = 0$. Suppose for the first case that there exists an edge $(x, \bar{x})$ such that x and $\bar{x}$ are excellent vertices of G_0 and G_1, respectively. Then, there exists a subset Y_x of free vertices, $|Y_x| \geqslant 8$, in G_0 such that for each $y \in Y_x$, a 2-DPC$[\{(s_1, x), (s_2, y)\} \mid G_0, F_0]$ exists. In addition, there exists a subset $Y_{\bar{x}}$ of free vertices, $|Y_{\bar{x}}| \geqslant 8$, in G_1 such that for each $z \in Y_{\bar{x}}$, a 2-DPC$[\{(t_1, \bar{x}), (t_2, z)\} \mid G_1, F_1]$ exists. Since $x \notin Y_x$ and $\bar{x} \notin Y_{\bar{x}}$, there exists a free vertex $y \in Y_x$ where $\bar{y} \in Y_{\bar{x}}$. It suffices to merge a 2-DPC$[\{(s_1, x), (s_2, y)\} \mid G_0, F_0]$ and a 2-DPC$[\{(t_1, \bar{x}), (t_2, \bar{y})\} \mid G_1, F_1]$ with $(x, \bar{x})$ and $(y, \bar{y})$.

Now, suppose for the second case that there exists no such edge $(x, \bar{x})$. Since each of G_0 and G_1 has at least eight excellent vertices by Lemma 9(a), each should have exactly eight excellent vertices. Thus, there exists a perfect vertex x in G_0, by Lemma 9(b), such that $\bar{x} \notin F_1$. Then, for any free vertex $w \neq x$ in G_0, there exists a 2-DPC$[\{(s_1, x), (s_2, w)\} \mid G_0, F_0]$ as well as a 2-DPC$[\{(s_1, w), (s_2, x)\} \mid G_0, F_0]$. We have two subcases depending on whether $\bar{x}$ is a terminal or not. Suppose for the first subcase that $\bar{x} \notin \{t_1, t_2\}$. Then, $\bar{x}$ is a good vertex of G_1 by Lemma 9(b) since G_1 has exactly eight excellent vertices. Thus, there is a subset $Y_{\bar{x}}$ of free vertices, $|Y_{\bar{x}}| \geqslant 4$, in G_1 such that for each $y \in Y_{\bar{x}}$, a 2-DPC$[\{(t_1, \bar{x}), (t_2, y)\} \mid G_1, F_1]$ exists. Since $|Y_{\bar{x}}| \geqslant 4$, it is possible to pick up a vertex y in $Y_{\bar{x}}$ such that $\bar{y}$ is free. It suffices to merge a 2-DPC$[\{(s_1, x), (s_2, \bar{y})\} \mid G_0, F_0]$ and the 2-DPC of G_1 with edges $(x, \bar{x})$ and $(\bar{y}, y)$. Suppose for the second subcase that $\bar{x} \in \{t_1, t_2\}$. Let $\bar{x}$ be t_1 first. Then, there exists a subset Y of free vertices, $|Y| \geqslant 4$, in G_1, by Lemma 9(b), such that for each $y \in Y$, a generalized 2-DPC$[\{(t_1, t_1), (t_2, y)\} \mid G_1, F_1]$ exists. It suffices to pick up y in Y such that $\bar{y}$ is free, and merge a 2-DPC$[\{(s_1, x), (s_2, \bar{y})\} \mid G_0, F_0]$ and the generalized 2-DPC of G_1 with $(x, \bar{x})$ and $(\bar{y}, y)$. If $\bar{x}$ is t_2, then symmetrically, it suffices to merge a 2-DPC$[\{(s_1, \bar{y}), (s_2, x)\} \mid G_0, F_0]$ and a generalized 2-DPC$[\{(t_1, y), (t_2, t_2)\} \mid G_1, F_1]$, for some y, with $(x, \bar{x})$ and $(\bar{y}, y)$. □

5. Conclusion

We proved that every m-dimensional RHL graph, $m \geqslant 5$, is $(m-3)$-fault paired 2-disjoint path coverable, and thus it is also $(m-3)$-fault unpaired 2-disjoint path coverable. The bound $m-3$ on the number of faults is the maximum possible for the m-dimensional RHL graph to be paired (resp. unpaired) 2-disjoint path coverable. It is our conjecture that every m-dimensional RHL graph, $m \geqslant 5$, is (a) f-fault *unpaired* k-disjoint path coverable for any f and $k \geqslant 2$ subject to $f + k \leqslant m - 1$, and (b) f-fault *paired* k-disjoint path coverable for any f and $k \geqslant 2$ subject to $f + 2k \leqslant m + 1$.

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