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# New results of exhaustive search in the game Amazons

Raymond Georg Snatzke

*Friedrich-Schiller-Universität Jena, Institut für Angewandte Mathematik, Jena 07740, Germany*

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**Abstract**

Amazons is a young, abstract, strategic, two-player game, in which the first player unable to move loses. We present a database for small Amazons positions, which for every position holds the canonical combinatorial game theory values, its thermograph and the corresponding move for every canonical option.

Such a database is useful to find values and structures in games that were unknown before and hard or impossible to find or verify by hand. In Amazons we were able to prove the existence of nimbers, of fractions down to $\frac{1}{64}$ and of various infinitesimals, but these results also suggest that there is no easy construction for most of these values. The database also demonstrates how complex canonical forms in Amazons can be and that many Amazons positions have properties and values that are totally counterintuitive.


*Keywords:* Combinatorial game theory; Canonical values; Thermography; Exhaustive search; Amazons

## 1. Introduction

In 1997 Berlekamp proposed to analyze the game Amazons with the means of combinatorial game theory. Our approach was to study Amazons by creating a large database of Amazons positions and their game theoretic values. First results were presented at the Combinatorial Game Theory Workshop at Berkeley in 2000, see [8]. This article describes the continuation of this work and further results.

*E-mail address:* rgsnatzke@aol.com (R.G. Snatzke).

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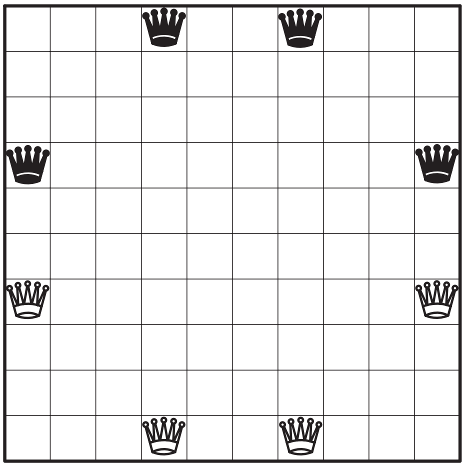

Fig. 1. Amazons starting position.

### 1.1. Amazons

Amazons was invented in 1988 by the Argentinian Walter Zamkauskas. It is a strategic, abstract game for two players, played on a board of 10 times 10 squares. The starting position can be seen in Fig. 1.

The players move alternately, White starts. Each move consists of moving an amazon and shooting an arrow with that same amazon. Amazon and arrow move like a Chess queen, i.e. diagonally, vertically or horizontally as far as the player wants and no object, amazon or arrow, blocks its path. Neither amazon nor arrow jump over any other piece, nor is any piece captured. There is an infinite supply of arrows. Arrows remain on the square where they are placed and are never moved again. Therefore the board is slowly filled with arrows, one arrow per move. The first player unable to move loses.

With these simple rules Amazons offers abundant tactical possibilities and strategic depth. There is now also an active Amazons programming scene and an Amazons program tournament is a fixed part of the yearly Mind Sports Olympiad [6].

### 1.2. Combinatorial game theory

Combinatorial game theory has its historic roots in the study of the game Nim [4]. Some decades ago the theory was generalized and applied to partizan games, too [3], i.e. games in which the players have different moves available.

For combinatorial game theory to apply we need a finite game for two players, who alternate moves. The first player unable to move loses. Additionally, the game must not contain any chance elements and it has to be a perfect information game. Amazons fulfills all of these conditions.

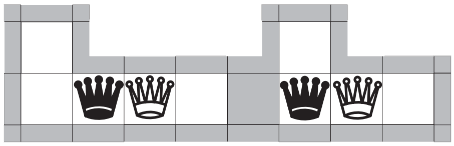

Fig. 2. A sum of two Amazons positions.

Combinatorial game theory is especially powerful in analyzing sums of independent games. In such a sum the player to move may choose in which of the games he will make his next move. Quite often it is easy to decide which move is best for a particular game, but very difficult to see in which game to move next. In these cases combinatorial game theory offers practical exact solutions to decide in which game a move is most urgent.

If the exact solution is too complex and difficult, one has powerful heuristics at one's disposal, especially thermography. Berlekamp introduced the concept of an *enriched environment*, in which thermography practically is the exact solution [1].

Some games automatically split up in several independent subgames, which can be analyzed on their own. The endgame of Go is a good example. The arrows in Amazons also have a tendency to split the board into several independent rooms. These can then be examined individually. Therefore the application of combinatorial game theory to Amazons is not only possible but also useful.

Fig. 2 shows a sum of two Amazons positions. Here combinatorial game theory can provide an exact solution. The left position has value $1 = \{0|\}$, the right position has value $-\frac{1}{2} = \{-1|0\}$. The expression in brackets describes the different canonical move options of player Left (Black) and Right (White), e.g. in the right position Black can move to $-1$ and White to 0. By the way, combinatorial game theory traditionally assigns positive numbers to black and negative numbers to white. We will content ourselves with these few sentences as an introduction to combinatorial game theory.

## 2. Database

Our idea was to build a database of small Amazons positions and their game theoretic values, then to search for structures within the database. With Berlekamp's theoretic work about Amazons positions on boards of size $n \times 2$ [2] we also had a tool to check the correctness of our results.

### 2.1. Construction

The database was built bottom-up. Within a given maximal game board every possible smaller game board up to a certain number of living squares, i.e. unblocked by arrows, was constructed. For example, in order to stick to the work of Berlekamp we started with a maximal game board of size $11 \times 2$. Within this frame we constructed all possible shapes of smaller game boards up to 22 living squares, i.e. all smaller boards that fit within the $11 \times 2$ restriction. Within other restrictions we could not reach the maximum, e.g. within the restrictions of $5\times 4$ we only constructed all game boards up to 12 squares.

Every shape was included only once. All mirrored, turned or moved copies were represented by just one game board. The only other condition for inclusion of a shape was that the game board had to be at least diagonally connected. Unconnected game boards were considered as two different game boards.

For every game board thus found we then constructed every possible position with one black and one white amazon or just one single black amazon. We eliminated all positions where the places of the black and white amazons were simply exchanged.

Every position with no empty squares left has value 0, because neither player can move. When we know the values of all positions with $n$ empty squares, we can determine the value of a position with $n+1$ empty squares by playing every possible move from this position for both players. As every move kills a square, we are now reduced to a position with a maximum of $n$ empty squares, whose value we already know. The values that result from all possible moves are then combined according to the rules of combinatorial game theory to form the value of our position in question.

In addition to the game theoretic value, we also included in our database the thermograph of every position and the actual moves that lead to all different subgames of the game theoretic value. While the game theoretic value of a position is sufficient to determine the exact outcome of any sum of games, the thermograph is merely a heuristic to determine which game in the sum is the most urgent. The advantage of the thermograph is that it is easier to compute and that one can see the temperature of a game, i.e. the urgency, at a glance and also the mean of the game, its projected value one can expect.

In some cases we also computed the values for two amazons versus one or for two amazons versus two.

In essence this is a bottom-up brute-force calculate-all approach. The algorithm described above is straightforward and so are most of the algorithms which attack the subproblems. Only in few special cases more elaborate means of problem solving were needed. The biggest problem were time and space constraints, as the number of possible positions grows exponentially with the number of live squares.

### 2.2. Scope

The complete scope of the database can be seen in Table 1.

Positions with amazons of only one color are necessary only as a basis for all other positions. They offer some challenges, as these positions can be defective, see [7]. But as all such positions are integers, their game theoretic aspects are not very interesting.

Altogether the database contains 66,214,767 positions with game theoretic values, thermographs and moves. There is some redundancy in the database, because smaller positions, which are the basis for all positions on bigger game boards, are included anew for every set of restrictions, e.g. the starting position with just two squares and one black amazon exists for restrictions $11\times 2$, $6\times 3$, $5\times 4$ and $10\times 10$.

As the data contained in Table 1 is the absolute maximum our workstation could handle with 1.5 GB RAM and the given program, there will be no further enlargements

Table 1
Scope of the database

| Restrictions | No. of Amazons (b, w) | Max. No. of live connected squares | No. of positions |
|---|---|---|---|
| $11\times 2$ | 1, 0 | 22 | 937,163 |
| $11\times 2$ | 1, 1 | 22 | 6,212,539 |
| $11\times 2$ | 2, 0 | 18 | 6,141,318 |
| $11\times 2$ | 2, 1 | 12 | 7,383,708 |
| $6\times 3$ | 1, 0 | 18 | 260,379 |
| $6\times 3$ | 1, 1 | 18 | 1,262,552 |
| $6\times 3$ | 2, 0 | 12 | 1,011,507 |
| $6\times 3$ | 2, 1 | 11 | 6,053,802 |
| $6\times 3$ | 2, 2 | 10 | 5,570,592 |
| $5\times 4$ | 1, 0 | 12 | 871,063 |
| $5\times 4$ | 1, 1 | 12 | 4,194,977 |
| $10\times 10$ | 1, 0 | 10 | 8,822,106 |
| $10\times 10$ | 1, 1 | 9 | 4,850,948 |
| $10\times 10$ | 2, 0 | 9 | 4,850,948 |
| $10\times 10$ | 2, 1 | 8 | 3,506,154 |
| $10\times 10$ | 2, 2 | 8 | 4,285,011 |

of the database in the near future. It is certainly conceivable to produce larger databases of Amazons positions if only the thermographs for every position are stored, an approach that is followed by Tegos [10] at the University of Alberta.

## 3. Results

Only the game positions within the restrictions $11\times 2$ and a maximum of one black and one white amazon were evaluated and discussed in [8]. Here we will present some interesting discoveries in the parts of the database recently calculated.

### 3.1. Fractions

Some combinatorial games are numbers. A number represents an advantage of moves for Black, if it is positive, and for White, if it is negative. It may not be obvious at once, but the players can also have an advantage of a fraction of a move. Fig. 2 shows an example. The justification to evaluate a position with $-\frac{1}{2}$ is that exactly two copies of such a position are needed to offset a position with value 1, i.e. $-\frac{1}{2}-\frac{1}{2}+1=0$. Whoever moves first in this sum loses.

Fractions in general are of the form $x/2^n=\{(x-1)/2^n|(x+1)/2^n\}$, where $x/2^n$ is canceled down as far as possible. This means that Left (Black) can move to $(x-1)/2^n$ and Right to $(x+1)/2^n$. Before our automated search the fraction with the largest denominator—canceled down as far as possible—known in Amazons was $\frac{1}{4}$. In [8] a position with value $-\frac{1}{16}$ was presented.

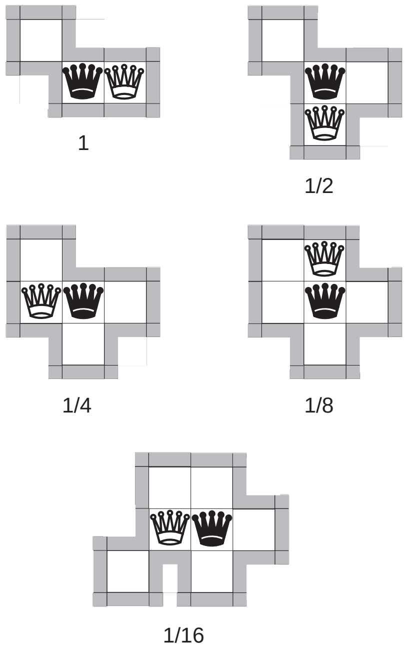


Fig. 3. Simplest construction of a $\frac{1}{16}$.

Our enlarged database offers a $\frac{1}{16}$ on a smaller game board. The construction of this position is shown in Fig. 3.

Interestingly, there is no way to add another square to the $\frac{1}{16}$ in Fig. 3 to obtain a $\frac{1}{32}$. Nevertheless, positions with denominator 32 and even 64 do exist. Fig. 4 shows one example.

Altogether the database contains eight positions with denominator 64. All of these positions have value $\frac{1}{64}$. They are listed in Fig. 5.

All positions with value $\frac{1}{64}$ follow the same pattern. White's amazon circles around the black amazon or around a hole in the middle of the game board and shoots off a distant square. Black on the other hand can move to 0 from these positions. Despite this seemingly simple scheme up to now no position with a value whose denominator is 128 or more has been found. Amazons is even more resilient when one tries to find a construction for positions with values $1/2^n$ in general. None has been found yet. Early in 2003 a construction for Amazons position with values of arbitrary dyadic fractions has been found [8a].

### 3.2. Nimbers

Nimbers are the combinatorial games which were studied first. They describe the positions of the game Nim, which was analyzed as early as 1902 by Charles Bouton [4].

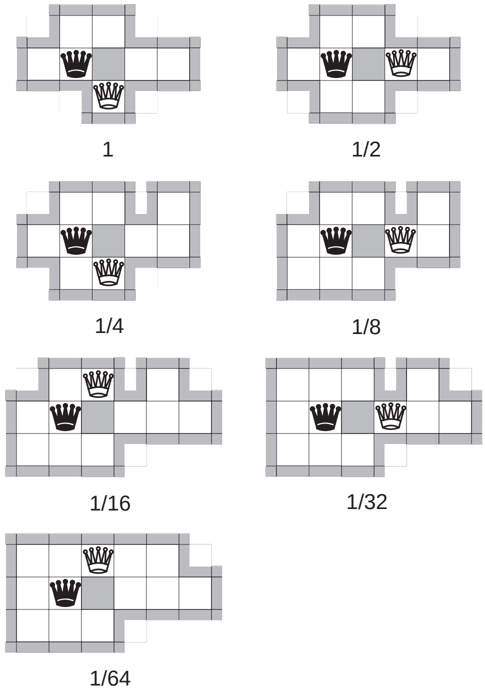


Fig. 4. Construction of a $\frac{1}{64}$.

In fact every neutral game is a nimber. This surprising theorem was discovered independently by Sprague and Grundy in the 1930s [9,5]. In a neutral game both players have the same canonical options from every possible position.

The simplest nimber is $0 = \{|\}$. Both players have no moves available. It is also called $*0$. Beginning with 0 the other nimbers are constructed iteratively: $* = *1 = \{0|0\}$, $*2 = \{0, *1|0, *1\}$ and so forth, in general:

$$*n = \{0, *1, \ldots *(n-1)|0, *1, \ldots *(n-1)\}.$$

Positions with value 0 and $*$ are very common in most combinatorial games, even in games which are not neutral, so called partizan games. But values of $*n, n \geqslant 2$ are a different matter. Even $*2$ poses a problem, as both sides must have exactly the same options, and not only one option each, but more than one. In a partizan game like Amazon, where both sides have different pieces to move, such extraordinarily symmetric positions are hard to imagine, and in many partizan games no higher nimbers than $*$ have been found yet.

But in Amazons other nimbers have been found, see [8]. Fig. 6 shows the smallest Amazon position with value $*2$. Tegos [10] was first to find positions with value $*3$ with only one black and one white amazon. As $*3 = *2 + *$, the next big step would be to discover a $*4$ in Amazons. None is known yet.

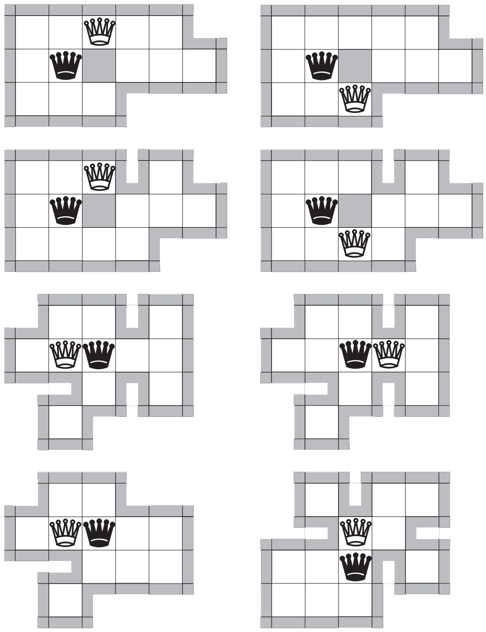

Fig. 5. All positions with value $\frac{1}{64}$.

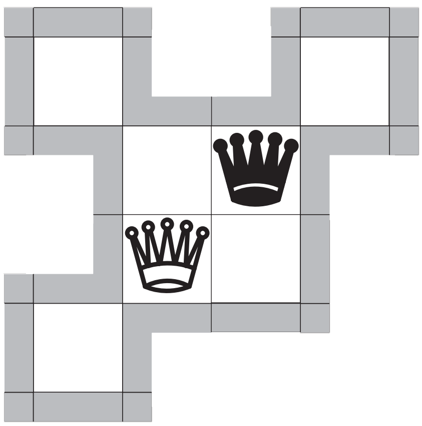

Fig. 6. Smallest position with value $*2$.

But Amazons offers symmetry even where it is totally unexpected. Fig. 7 shows one of the strangest examples of a nimber our database has to offer. Not only are the board and the positions of the amazons not symmetric, not even the number of amazons is equal. And yet Black's amazon in the upper left is not useless. It plays an essential role in this position as it adds to Black's options as soon as the other black amazon

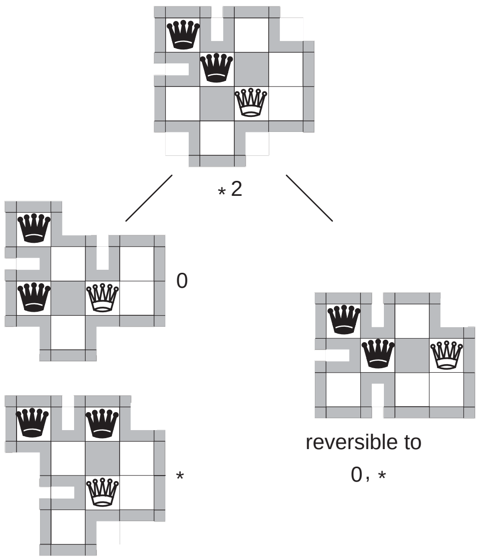


Fig. 7. A strange nimber.

moves away. Without this square and the second black amazon the position would not have value $*2$ but $-\frac{1}{4}$.

White in Fig. 7 needs only one move. In this position White's move has a special property, the so called *reversibility*. It means that Black enhances his positions by answering White's move at once and that it is analytically correct to consider this White/Black exchange as instantaneous. So White's options from this position really are his options after his first move and Black's answer, which is not shown in the figure.

### 3.3. Special cases of interest

There are many other interesting facts and data one can find in our database. Here we just want to give some few examples.

Hunting for records is one possibility. The maximum number of subgames in an Amazons positions in our database is 293,752. One can imagine this to be the game tree that originates from the position where no further simplification (domination, reversibility) is possible.

A record of another sort is shown in Fig. 8. In this position Black has 18, White 15 canonical options. They are depicted in the seven lower diagrams of the figure. In each you can see one move of one of the amazons. The black and white circles mark the possible spots for shooting the arrow after the move.

That an option is canonical means that it cannot be eliminated without altering the value of the positions. To put it in another way: for every one of these canonical options it is possible to find a combinatorial game which, if added to the position in

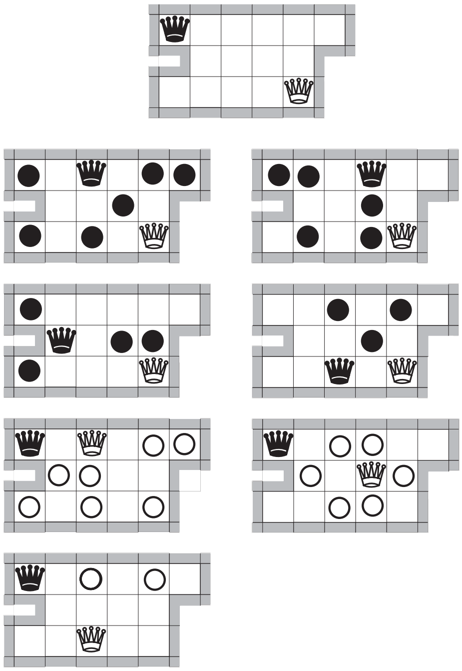

Fig. 8. Maximum number of canonical options in database: 18 left, 15 right.

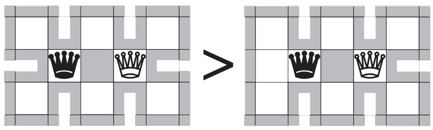

Fig. 9. A counterintuitive relation.

Fig. 8, makes the move that corresponds with this option the one and only winning move—not one of these options can be pruned.

Amazons is a game about territory. Having the own amazons positioned in territories with many empty squares means one has many more moves at one's disposal. Yet sometimes the relation between some Amazons positions is completely counterintuitive.

Fig. 9 shows an example. The symmetric position to the left has value 0, whoever moves first loses. In the right position we add a square next to the black amazon. Yet the value of the position is $\{\{\{1|0\}|0\}|0\}$, a negative infinitesimal, which is won for White no matter who moves first. The reason for this strange behavior is that the optimal moves for both sides more or less result in an exchange of their positions, the white amazon ending up (or threatening to end up) on the left half of the board and vice versa. The additional square next to the black amazons finally only helps White.

## 4. Conclusion

Amazons is a complex game. It is challenging strategically and tactically. Additionally, it is ideal for the application of combinatorial game theory. Amazons offers a plethora of different values, amazing and fascinating positions with unexpected behavior and interesting properties.

Yet it is hard to find regular patterns in Amazons. Exhaustive search and brute force methods, as used in the construction of our database, are useful tools to detect interesting positions, whose evaluation by hand would be next to impossible. Amazons, it seems, is an ideal testbed for experimental mathematics.

Our database as it exists now is not publicly accessible. Its size and programming structure pose serious problems for a possible internet presentation. Up to now the only means to get access to the data is via the author.

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