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Note on the game chromatic index of trees☆

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Abstract

We study edge coloring games defining the so-called game chromatic index of a graph. It has been reported that the game chromatic index of trees with maximum degree $\Delta=3$ is at most $\Delta+1$. We show that the same holds true in case $\Delta\geqslant 6$, which would leave only the cases $\Delta=4$ and 5 open.

Keywords: Game theory; Chromatic index

1. Introduction

We consider edge coloring games defined as follows. Two players, A (Alice) and B (Bob) are given a graph $G=(V,E)$ with maximum degree Δ and a number k of colors. A and B move in turns. Each move consists in (feasibly) coloring a yet uncolored edge. (Passing is not allowed, though one may allow it in the special cases we study here.)

Recall that the chromatic index $\chi'(G)$ is the smallest number k such that G can be edge colored with k colors. By Vizing's Theorem, this is either Δ or $\Delta+1$. (cf. [7,8,1]).

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We will be mainly interested here in the *maker–breaker* variant of the edge coloring game, where A seeks to achieve a completely colored graph G (and B tries to prevent this). The *game chromatic index* of G is the smallest number $k(\geqslant \chi'(G))$ of colors for which A has a winning strategy. Cai and Zhu [3] studied this parameter for certain classes of graphs.

There is a corresponding notion of *game chromatic number*, defined similarly by node coloring games, introduced by Bodlaender [2], which has received considerable attention in the literature (cf. [4–6]).

Remark. To be precise, the game chromatic index (number) may depend on who is to make the first move. It is generally assumed that A moves first, but this requirement is irrelevant for the graphs (trees with $\Delta \geqslant 6$) we study here.

The game chromatic index has first been studied by Cai and Zhu [3], who show that trees with maximum degree Δ have game chromatic index at most $\Delta + 2$ (cf. also Section 2) and mention that the class of trees with maximum degree 3 has game chromatic index at most $\Delta+1=4$. Our main result is the proof of this statement for $\Delta \geqslant 6$ and that the bound $\Delta+1$ is sharp (Theorems 3.1 and 3.2). Assuming the aforementioned results on $\Delta = 3$ to hold, only the cases $\Delta = 4$ and 5 remain open.

We remark that there is also another natural version of the coloring game in which the role of A and B are symmetric: a player loses as soon as (s)he cannot make any feasible move when it is his (her) turn, although there are still some uncolored edges. In case the graph is completely edge-colored, this situation is considered as a *tie*. Clearly, if A has a winning strategy for the maker–breaker variant (with respect to a given number of colors), she can avoid losing the game in the second variant.

2. An outline of A's strategy

It is natural to consider forests rather than trees. So assume $F = T_1 \cup \ldots \cup T_r$ is a forest with tree *components* T_s and maximum degree $\Delta \geqslant 6$. There are $k = \Delta+1$ colors.

Suppose $e \in T_s$ is the first edge that receives a color. We then split T_s into the two (unique) largest subtrees T_s' and T_s'' of T_s satisfying $T_s' \cap T_s'' = \{e\}$. So both T_s' and T_s'' contain a colored copy of e. We may then consider the game as being continued in the resulting forest with $r + 1$ components.

Continuing this way, we arrive after $t \geqslant 0$ moves at a forest with $r + t$ components. Each component T_s has a certain number $\gamma = \gamma(T_s)$ of colored (*leaf*) *edges*, i.e., edges incident with a leaf of T_s. We refer to such a component also as a *γ-component* ($\gamma \geqslant 0$).

Basically, A seeks to maintain $\gamma \leqslant \Delta$ for all components. Note that a (γ+1)-component can be created only by splitting a γ-component into a (γ+1)-component and a (harmless) 1-component. As soon as B creates a (Δ+1)-component, A can split this into two components with $\gamma \leqslant \Delta$ each by coloring a suitable edge e separating the $\Delta+1$ colored edges (since Δ is the maximum degree). This procedure is much in the spirit of the maker's strategy in the node coloring game on trees (cf. [2]).

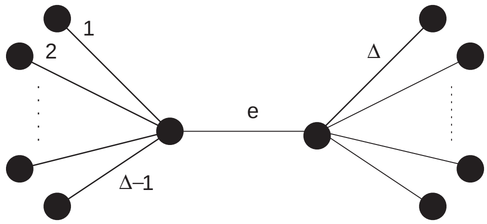

Fig. 1.

This simple observation implies that the game chromatic index for trees (forests) is at most

$$\Delta + 2.$$

To obtain the (as we will see) correct bound $\Delta+1$, however, A's strategy has to be refined. Indeed, assume that A for example creates a Δ-component as indicated in Fig. 1 below, containing an uncolored edge e with Δ pairwise different colors adjacent to it, $\Delta - 1$ on one side and 1 on the other. We refer to such an edge as a *critical edge*. Then B may assign color $\Delta+1$ to one of the uncolored edges adjacent to e and A has lost the coloring game.

To describe A's strategy in more detail, we introduce some notation. The *induced subtree* of a component T consists of all colored edges together with the smallest uncolored subtree of T joining them. A node v in T is a *base node* if it has degree at least 3 in the induced subtree. We call T a *star* or *star-like* if T has a unique base node v (i.e., the induced subtree is a topological star with center v). Clearly, a γ-*star* is a star with γ colored edges.

A's strategy will ensure that after each of her moves, every γ-component with $\gamma \geqslant 3$ is star-like. Note that this strategy ensures in particular that B has always a feasible move left, unless the graph is already completely colored. Furthermore, A will win, provided she can avoid creating a critical edge.

This observation suggests two things: First, we should concentrate on star components in which the base node has large degree. We therefore call a star component T (and its base node v) *relevant* if v has degree at least 6 in T (not necessarily in the induced subtree).

Second, A should avoid critical edges. Suppose T is a star component with base node v. A colored edge in T is then called *matched* if its color also occurs among the colors on edges incident to v. (So in particular every colored edge incident to v is automatically matched.) For example, the edge colored Δ in Fig. 1 is unmatched. A seeks to minimize the number of unmatched (colored) edges. Ideally, every γ-star should be *completely matched* in the sense that it has no unmatched colored edges.

In Section 3 we show that A can always ensure the following properties to hold after her move:

(S) Every γ-component with $\gamma \geqslant 3$ is a star.

(M) Every relevant γ-star has at most $\max\{5-\gamma, 0\}$ unmatched colored edges.

In particular, since we assume $\Delta \geqslant 6$, A will never create a critical edge (since a star component as in Fig. 1 is relevant).

3. Star moves and split moves

We show that A can maintain (M) and (S) by induction on the number of moves. Hence assume (M) and (S) hold after some move of A. In case they are also valid after the subsequent move of B, then A has an easy job to do: In case there are γ-components with $\gamma \leqslant 2$ and uncolored edges, finding a suitable move is trivial. If all components that still contain uncolored edges are stars, A chooses one of them, say γ-star T with base node v. If all colored edges are incident to v (and hence the number of unmatched edges is zero), she colors an uncolored edge adjacent to v. Then (M) obviously holds for the new $(\gamma+1)$-star. Finally, assume that T has some colored edge e (either matched or unmatched) not incident to v. Then A colors the first edge on the path from v to e (with any feasible color). This leaves $\gamma = \gamma(T)$ unchanged and the number of unmatched edges is certainly not increased. So (M) (and clearly also (S)) remain valid.

Now assume B's move violates (S). So assume B colors an edge in a γ-star T with base node v, thereby creating a new $(\gamma+1)$-component with a new additional base node w of degree 3 in the induced subtree, cf. Fig. 2. (B has just assigned color i to the edge indicated in Fig. 2.)

In this case, A performs a *split move* by coloring an edge on the uncolored path P joining v and w. First note that such a split move is always possible. Indeed, the only case where A cannot feasibly color an edge on P occurs when $P = \{e\}$ consists of a single edge with $k = \Delta+1$ pairwise different colors adjacent to it, $\Delta - 1$ on one side and 2 on the other. But in this case e would have been a critical edge *before* B's move, contrary to our assumption. So A can indeed perform a split move and (S) will be satisfied afterwards. We are to show that the split move can be carried out in such a way that also (M) holds. We distinguish two cases

Case 1: $P = \{e\}$ consists of a single edge.

In this case, A assigns any feasible color to e. This ensures that w has at least one incident colored edge, so (M) holds with respect to the 3-star centered at w. Since γ remains unchanged for the star-component centered at v (and A's move certainly does

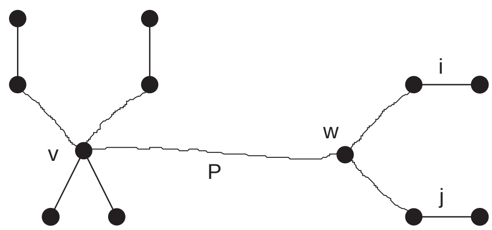

Fig. 2.

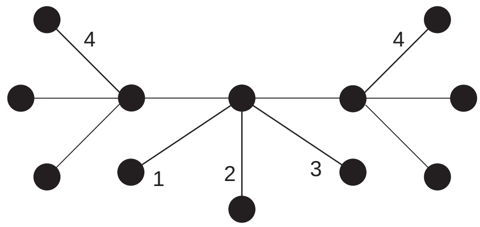

Fig. 3.

not increase the number of unmatched edges in this component), (M) also holds with respect to v.

Case 2: P has at least two edges.

If the edge colored with j in Fig. 2 is not incident with w, then A colors the edge on P incident with w with j. Then (M) holds relative to w and (since nothing changes with respect to v) also for v.

If j in Fig. 2 is incident to w, A colors the edge on P that is incident with v (with any feasible color). This coloration certainly does not increase the number of unmatched edges in the v-component. So (M) holds for v as it did before. It also holds for the 3-star centered at w since j is matched.

Next assume that (S) holds after B's move but (M) does not. So B has just created a (relevant!) new γ'-star T' with base node v' violating (M) (either by creating a new 3-star with 3 unmatched edges or by doing something odd to an existing γ-star with base node v'). In this case A responds by a *star move* (see the definition in the next paragraph), reducing the number of unmatched edges by 2 in case there are at least two such edges and to 0 otherwise. Note that this reaction re-installs (M) since in the worst case B can have raised γ to $\gamma'=\gamma+1$ and introduced one unmatched edge, so there can be at most 2 "too many" unmatched edges in T'.

If T' contains only one unmatched edge e, A colors the first edge on the path from v' to e with any feasible color, thereby reducing the number of unmatched edges in T' to zero. Hence assume there are at least two unmatched edges. If two of them have different colors i and j, then A may color the first edge on the path from v' to the j-colored edge with color i. Otherwise, if all unmatched edges have the same color i, then A colors a non-induced edge adjacent to v' with color i. Note that such an edge necessarily exists: Indeed, since T' has two unmatched edges, one of them must have been present before B's move, implying that B's move was made in a γ-component with $\gamma \leqslant 4$. Hence $\gamma' \leqslant 5$.

We have proved

Theorem 1. *Any tree (forest) with maximum degree $\Delta \geqslant 6$ has game chromatic index at most $\Delta+1$.*

Remark. Although the assumption $\Delta \geqslant 6$ is used only once in the proof, it seems to be crucial for the strategy to work. Consider the tree depicted in Fig. 3 for $\Delta=5$ where B has just colored a second unmatched edge with 4 (the thin lines represent uncolored edges). It is not only impossible for A, here, to reinstall (M) but B has a winning strategy. So A will lose.

The bound in Theorem 1 is easily seen to be sharp

Theorem 2. *For any $\Delta \geqslant 2$ there exists a tree with game chromatic index equal to $\Delta+1$.*

Proof. It suffices to exhibit a tree $T = T_\Delta$ such that A has no winning strategy for T if the number of colors is $k = \Delta$. For $\Delta = 2$ this is trivial: Take any sufficiently long path. (If B is to move first, a path with at least 5 edges is needed.)

For $\Delta \geqslant 3$ we let $T = T_\Delta$ be the unique rooted tree of height 2 with $\Delta+1$ nodes of degree Δ and $\Delta(\Delta - 1)$ leaves. Thus, the root v is incident with Δ "base edges" and each base edge in turn is adjacent to $\Delta - 1$ leaf edges.

For $\Delta = 3$ the claim is straightforward to check. (B can create a "critical edge" after two moves of A, no matter who starts.) The case $\Delta \geqslant 4$ can be solved similarly as follows. In his first moves B colors base edges in such a way that (after his move) each of the remaining uncolored base edges has only uncolored adjacent leaf edges. He proceeds this way as long as (after his move) there are still (at least) two such uncolored base edges left. Then A will have lost in two further steps as in case $\Delta = 3$. $\square$

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