Theoretical Computer Science 38 (1985) 167-192
North-Holland

PERIODIC CHARACTER SEQUENCES WHERE IDENTIFYING TWO CHARACTERS STRICTLY REDUCES THE PERIOD*

Masatosi IMORI and Hisao YAMADA**
Faculty of Science, University of Tokyo, Bunkyô-Ku Hongô, Tokyo 113, Japan

Communicated by M. Nivat
Received January 1983
Revised December 1984

Abstract. Let A_m be an m element alphabet, where $m \geqslant 2$. It is established that, among all periodic sequences in A_m, there exist those whose primitive period must reduce under every homomorphic map induced by $\tau: A_m \to A_{m-1}$ and that, for $m \geqslant 3$, their primitive period is equal to, or greater than ${}_mC_2 \cdot \mathrm{PP}({}_mC_2)$, where $\mathrm{PP}(k)$ is the product of the first k prime numbers.

1. Introduction

In this article we shall show that there are periodic sequences whose primitive period must become smaller if we reduce the size of the alphabet of their constituent letters, by mapping the alphabet onto another of smaller size, regardless of the choices of the map. Then we shall obtain the greatest lower bound of the primitive periods of the periodic sequences having such period-reducing property.

Our interest in studying these properties arose in conjunction with the study of the action of one-dimensional cellular automata on periodic sequences [3]. Since each cell of a cellular automaton decides its next state based on the state information of a fixed number of neighbourhood cells, there is an intrinsic connection between the behavior of one-dimensional cellular automata and periodic sequences, and what a cellular automaton is or is not capable of doing with periodic sequences is one of the basic properties of cellular automata.

The problem investigated in the present article is a property of degenerate cellular automata in that the number of neighbourhood cells is reduced to only one. However, this property not only serves as the foundation of the behaviors of general cellular automata in conjunction with periodic sequences as we showed in [5], but is also

* A part of this work was presented as an invited paper at the 4th Australian Computer Science Conference, University of Queensland, Australia, May 11-13, 1981.
** The second author was partially supported, 1971-1972, in the early stage of this work at the Moore School of Electrical Engineering, University of Pennsylvania, by the US Army Research Office DA-31-124-ARO(D)-98 and by the National Science Foundation GP-5561.

0304-3975/85/$3.30

of a theoretical interest on its own right. In the present article we shall confine our treatment only to the special case mentioned above.

In recent years interest in theoretical aspects of uniform arrays seems to have declined in spite of the fact that most of what have been shown in the theory so far appear to have solid substance and beauty in them. The reason for such a waning may be that theoretical treatment of cellular spaces is tedious and difficult because it demands concurrent processing of an unbounded number of parallel elements. In order to develop a well coordinated theory of cellular spaces in all directions, we feel that systematic examination of this parallelism from the elementary structure upward is first needed, at the most fundamental level of which lies the study of parallel actions of maps, namely the case for the neighbourhood size of one, although this obvious fact is often overlooked in the study of cellular space. Our present work is an attempt into such a direction.

Some basic definitions and a formal statement of our main results are first given. Let A_m be a finite nonempty *alphabet* of m elements called *letters*, and let $\mathbb{Z}$ be the set of all integers. A *sequence* in A_m is a map $c:\mathbb{Z}\to A_m$, which represents a two way infinite sequence of letters. The image of $i\in\mathbb{Z}$ under c is written as $c(i)$ and is referred to as the ith letter of the sequence. Similarly, for any $S\subset\mathbb{Z}$, we denote by $c(S)$ the set of the images of S, etc. Let $A_m^{\mathbb{Z}}$ denote the set of all possible sequences in A_m, i.e. $A_m^{\mathbb{Z}}=\{c\mid c:\mathbb{Z}\to A_m\}$. By a *pattern* we mean a partition of $\mathbb{Z}$. For any $a_i\in A$ (in $c(\mathbb{Z})$), let $B_i=c^{-1}(a_i)$, then $B_i\neq\emptyset$ and $B_i\cap B_j=\emptyset$ if $i\neq j$, and $\bigcup_{a_i\in c(\mathbb{Z})} B_i=\mathbb{Z}$. Hence c^{-1} defines a partition $\Pi(c)$ on $\mathbb{Z}$, called *pattern c*. B_i is called a *block* of $\Pi(c)$. Conversely, for a given m block partition $\Pi^{(m)}$ of $\mathbb{Z}$ (i.e. a pattern), there exists a sequence $c\in A_m^{\mathbb{Z}}$ such that $\Pi(c)=\Pi^{(m)}$, which is unique up to the permutation of letters in A_m. Hence, in the following sections we will often study sequences in terms of their partitions. However, we on occasions retain nomenclatures from sequences.

Whenever no confusions result, the use of subscripts and superscripts may be omitted, e.g., we use A for A_m, B for B_i, Π for $\Pi^{(m)}$, etc.

For any $S\subset\mathbb{Z}$ and $k\in\mathbb{Z}$, we let $S+k$ denote $\{x+k\mid x\in S\}$, which is a *translation* of S by k. Let the *period set* of c be defined by $P(c)=\{k\mid(\forall z\in\mathbb{Z})\ (c(z+k)=c(z))\}$. Similarly the period set of B, by $P(B)=\{k\mid B+k=B\}$. Whenever it is convenient, we denote by t_k a translation by k, which is for $z\mapsto z+k$. Define the period set $P(\Pi)$ of a pattern Π by $\{k\mid(\forall B_i\in\Pi)(t_k(B_i)=B_i)\}$. Clearly, for any sequence $c\in A^{\mathbb{Z}}$, $P(\Pi(c))=P(c)$.

For each $c\in A^{\mathbb{Z}}$, define the *primitive period*, denoted by $\pi(c)$, as the smallest positive element of $P(c)$. If such $\pi(c)$ does not exist, i.e. $P(c)=\{0\}$, we may let $\pi(c)$ be ω, the smallest transfinite ordinal. Similarly, the primitive period $\pi(B)$ of B and the primitive period $\pi(\Pi)$ of Π are defined. It is easy to see that the next remark is valid.

Remark 1.1. If $\pi(c)\neq\omega$, then the primitive period divides all the elements of the period set.

Let $\tilde{A}^{\mathbb{Z}} = \{c \mid c \in A^{\mathbb{Z}} \text{ and } \pi(c) \neq \omega\}$. The elements of $\tilde{A}^{\mathbb{Z}}$ are called *periodic sequences*.

For $m \geq 2$, let $\tau: A_m \to A_{m-1}$ be a surjection. Without loss of generality, we shall assume that $A_{m-1} \subset A_m$. It is naturally extended to $\tau: A_m^{\mathbb{Z}} \to A_{m-1}^{\mathbb{Z}}$, that is,

$$(\forall c \in A_m^{\mathbb{Z}})(\forall z \in \mathbb{Z})(c' = \tau(c) \Rightarrow c'(z) = \tau(c(z))).$$

Our main results are that

$$\text{(i)} \quad (\forall m \geq 2)(\exists c \in \tilde{A}_m^{\mathbb{Z}})(\forall \tau: A_m \to A_{m-1})(\pi(\tau(c)) < \pi(c)),$$

and that

$$\text{(ii)} \quad (\forall m \geq 3)(\forall c \in \tilde{A}_m^{\mathbb{Z}})[(\forall \tau: A_m \to A_{m-1})\,(\pi(\tau(c)) < \pi(c)) \Rightarrow \pi(c) \geq {}_mC_2 \cdot \mathrm{PP}({}_mC_2)],$$

where $\mathrm{PP}(k)$ is the product of the first k prime numbers.

After being established, our results may not appear so surprising because the maps we use can look at only one letter of a periodic sequence at a time, so to speak, yet the primitive period of the periodic sequence may be of any length. However, our proof turned out to be lengthy. It is perhaps because our proof uses only an elementary part of number theory [2]. On the other hand, our search in various branches of mathematics has failed to uncover any previous studies on the properties of periodic sequences which are pertinent to our problem.

As it was already stated, the map $\tau: A_m \to A_{m-1}$ we used constitute a very special case of cellular automata where the neighbourhood of a constituent cell is the cell itself. Although it is a degenerate case, it nevertheless is a valid cellular automaton, and the general case is studied by Yamada and Imori [5].

2. Some properties of periodic sequences

For a finite set $\{k_1, \ldots, k_n\} \subset \mathbb{Z}$, we denote by $(k_1, \ldots, k_n)$ the greatest common divisor (gcd), and by $[k_1, \ldots, k_n]$ the least common multiple (lcm). By $k_1, \ldots, \hat{k}_t, \ldots, k_n$, we denote sequence $k_1, \ldots, k_{t-1}, k_{t+1}, \ldots, k_n$ (i.e. the sequence of k_1 through k_n except k_t), etc. For $k_1, k_2 \in \mathbb{Z}$, let $k_1 \mid k_2$ denote 'k_1 divides k_2'.

Patterns can be considered to be equivalence relations over $\mathbb{Z}$, which enables to define the inclusion and the intersection of two patterns. These relations are defined through maps which represent patterns. Suppose maps c_a and c_b represents patterns Π_a and Π_b respectively, then the patterns satisfy $\Pi_a \subseteq \Pi_b$ if and only if $c_a(x) = c_a(y) \Rightarrow c_b(x) = c_b(y)$ for any $x, y \in \mathbb{Z}$. Similarly patterns Π, Π_a and Π_b satisfy $\Pi = \Pi_a \cap \Pi_b$ if and only if $c(x) = c(y) \Leftrightarrow c_a(x) = c_a(y)$ and $c_b(x) = c_b(y)$ for any $x, y \in \mathbb{Z}$, where maps c, c_a and c_b represent patterns Π, Π_a and Π_b respectively. For patterns Π, Π_a and Π_b, let $\pi = \pi(\Pi)$, $\pi_a = \pi(\Pi_a)$ and $\pi_b = \pi(\Pi_b)$.

Proposition 2.1. *For $S \subset \mathbb{Z}$, let $\bar{S} = \mathbb{Z} - S$. Then, including the case of $\pi(S) = \omega$,*

(i) $P(S) = P(\bar{S})$,

(ii) $\Pi_a \subseteq \Pi_b$ implies $\pi_b | \pi_a$,
(iii) $\Pi = \Pi_a \cap \Pi_b$ implies $\pi = [\pi_a, \pi_b]$.

Proof. (i) is obvious from the definition of the period.

(ii) Suppose maps c_a and c_b represent patterns Π_a and Π_b. $\pi_a \in P(\Pi_a)$ gives $c_a(x+\pi_a) = c_a(x)$ for any $x \in \mathbb{Z}$. Since $\Pi_a \subseteq \Pi_b$, $c_b(x+\pi_a) = c_b(x)$ which shows $\pi_a \in P(\Pi_b)$. So $\pi_b | \pi_a$.

(iii) Suppose c, c_a and c_b represent patterns Π, Π_a and Π_b respectively. Then $\Pi \subseteq \Pi_a$ and $\Pi \subseteq \Pi_b$ from $\Pi = \Pi_a \cap \Pi_b$. Then $\pi_a | \pi$, $\pi_b | \pi$ and $[\pi_a, \pi_b] | \pi$. Now π_a, $\pi_b | [\pi_a, \pi_b]$, so $c_a(x+[\pi_a, \pi_b]) = c_a(x)$ and $c_a(x+[\pi_a, \pi_b]) = c_b(x)$. Now we have $c(x+[\pi_a, \pi_b]) = c(x)$ from the definition, which shows $[\pi_a, \pi_b] \in P(\Pi)$ and $\pi | [\pi_a, \pi_b]$. Hence $\pi = [\pi_a, \pi_b]$. □

For a pattern $\Pi = (B_1, \ldots, B_m)$, denote $\pi(\Pi)$ by π, and $\pi(B_i)$ by π_i, $1 \leq i \leq m$.

Proposition 2.2. $\pi = [\pi_1, \ldots, \pi_m]$.

We note the following slight generalization of an elementary property of numbers [2, p. 24].

Remark 2.3. For any $z_0, z_1, \ldots, z_n \in \mathbb{Z}$, there exist $r_1, r_2, \ldots, r_n \in \mathbb{Z}$ such that

$$z_0(z_1, \ldots, z_n) = r_1 z_1 + r_2 z_2 + \cdots + r_n z_n.$$

For $\Pi^{(m)} = (B_1, \ldots, B_m)$, let $\Pi^{(m)}_{(i,j)} = (B_i \cup B_j, B_1, \ldots, \hat{B}_i, \ldots, \hat{B}_j, \ldots, B_m)$, $\pi = \pi(\Pi^{(m)})$ and $\pi_{(i,j)} = \pi(\Pi^{(m)}_{(i,j)})$, then clearly $\pi_{(i,j)} | \pi$. Let $\pi / \pi_{(i,j)}$ be the factor with which the primitive period reduces upon the unification of B_i and B_j. Then the following holds.

Proposition 2.4. (i) *If $i \neq u$ or $j \neq v$, then $[\pi_{(i,j)}, \pi_{(u,v)}] = \pi$, and*
(ii) *the members of $\{\pi / \pi_{(i,j)} | i \neq j\}$ are relatively prime in pairs.*

Proof. (i) $\Pi = \Pi_{(i,j)} \cap \Pi_{(u,v)}$ because $i \neq j$ and $u \neq v$. Then the result follows from Proposition 2.1(iii).

(ii) From $[\pi_{(i,j)}, \pi_{(u,v)}] = \pi$, it immediately follows that $\pi / \pi_{(i,j)}$ and $\pi / \pi_{(u,v)}$ are relatively prime. □

Let $t_k : \mathbb{Z} \to \mathbb{Z}$ be a translation by $k \in \mathbb{Z}$. Then the following holds.

Proposition 2.5

$$(\forall \Pi)[(\forall B_i \in \Pi)(\exists t_{k_i})\,(t_{k_i}(B) \subseteq B_i) \Leftrightarrow \pi(\Pi) \neq \omega].$$

Proof. (i) Left to right: take $k = k_1 k_2 \ldots k_m$, then $t_k(B_i) \subseteq B_i$. $\mathbb{Z} = \bigcup_{i=1}^{m} B_i$, hence $\bigcup_{i=1}^{m} t_k(B_i) = t_k(\mathbb{Z}) = \mathbb{Z}$, which leads to $t_k(B_i) = B_i$.

(ii) Right to left: t_k for any $k \in P(\Pi)$ has the desired property. □

Corollary 2.6

$$(\forall \Pi)(\forall B_i \in \Pi)(\forall t_k)\ [\pi_i \neq \omega \text{ and } t_k(B_i) \subseteq B_i \Rightarrow t_k(B_i) = B_i].$$

Proof. Suppose $t_k(B_i) \subsetneq B_i$, then $t_k(t_k(B_i)) = t_{2k}(B_i) \subseteq t_k(B_i) \subsetneq B_i$. Apply t_k for π_i times. Then $t_{k \cdot \pi_i}(B_i) \subsetneq B_i$. Yet $t_{k \cdot \pi_i}(B_i) = B_i$, because $\pi_i = \pi(B_i)$; a contradiction. □

Denote $\pi_{ij} = \pi(B_i \cup B_j)$. Note that, in general, $\pi_{ij} \neq \pi_{(i,j)} = \pi(\Pi_{(i,j)})$.

Proposition 2.7. *With respect to $\Pi^{(m)}$, let $B_i, B_j \in \Pi^{(m)}$ be such that $B_i \neq B_j$. Then*

$$(\forall t_{k_i}, t_{k_j}, t_{k_{ij}})[t_{k_i}(B_i) \subseteq B_i, t_{k_j}(B_j) \subseteq B_j, t_{k_{ij}}(B_i \cup B_j) \subseteq B_i \cup B_j]$$
$$\Rightarrow (\pi_{ij} = \omega \Leftrightarrow [\pi_i, \pi_j] = \omega).$$

Proof. Assume $\pi_{ij} \neq \omega$. Then Corollary 2.6 gives $t_{k_{ij}}(B_i \cup B_j) = B_i \cup B_j$, hence Proposition 2.1(i) gives $t_{k_{ij}}(\overline{B_i \cup B_j}) = \overline{B_i \cup B_j}$ and $\pi(\overline{B_i \cup B_j}) = \pi_{ij}$. Hence, applying Proposition 2.5 to $(B_i, B_j, \overline{B_i \cup B_j})$, and using Proposition 2.2, we obtain $[\pi_i, \pi_j, \pi_{ij}] \neq \omega$, hence $\pi_i \neq \omega \neq \pi_j$, and finally $[\pi_i, \pi_j] \neq \omega$. The converse that $[\pi_i, \pi_j] \neq \omega \Rightarrow \pi_{ij} \neq \omega$ is trivially established. □

It should be noted that the converse does not hold, that is, there are cases where $\pi_i = \pi_j = \pi_{ij} = \omega$, yet there do not exist translations to satisfy the premise. Their construction is left as an excercise.

Proposition 2.8. *With respect to $\Pi^{(m)}$, let $B_i, B_j \in \Pi^{(m)}$ be such that $B_i \neq B_j$ and $[\pi_i, \pi_j] \neq \omega$. Every translation $t: \mathbb{Z} \to \mathbb{Z}$ such that $t(B_i \cup B_j) \subseteq B_i \cup B_j$ satisfies $t(B_i) \subseteq B_i$ and $t(B_j) \subseteq B_j$ if and only if $\pi_{ij} = [\pi_i, \pi_j]$.*

Proof. From $[\pi_i, \pi_j] \neq \omega$, $\pi_{ij} \neq \omega$. Let $t: z \mapsto z + \pi_{ij}$. Then $(B_i \cup B_j) + \pi_{ij} = B_i \cup B_j$ by definition. On the other hand, $B_i + \pi_{ij} \subseteq B_i$ by the given assumption, hence $B_i + \pi_{ij} = B_i$ by the use of $(B_i, B_j, \overline{B_i \cup B_j})$ and Proposition 2.1(i), and $t_{\pi_{ij}}$ and Corollary 2.6. Similarly, $B_j + \pi_{ij} = B_j$. Hence, $\pi_i | \pi_{ij}$ and $\pi_j | \pi_{ij}$, or $[\pi_i, \pi_j] | \pi_{ij}$. On the other hand, $B_i + [\pi_i, \pi_j] = B_i$ and $B_j + [\pi_i, \pi_j] = B_j$ gives $(B_i \cup B_j) + [\pi_i, \pi_j] = B_i \cup B_j$, or $\pi_{ij} | [\pi_i, \pi_j]$. Hence $\pi_{ij} = [\pi_i, \pi_j]$. The converse is easily established. □

Let $\Pi^{(m)} = (B_1, \ldots, B_m)$, $\pi_i = \pi(B_i)$, $1 \leq i \leq m$. Denote $g = (\pi_1, \ldots, \pi_m) \neq \omega$, and further let $\mathbb{Z}_u = \{z \mid z = u \bmod g\}$, $0 \leq u \leq g-1$; and $\Pi_g = \{\mathbb{Z}_u\}_{u=0}^{g-1}$ which is a partition of $\mathbb{Z}$. We call $\mathbb{Z}_u$ a *channel* of Π_g with respect to *modulus* g and *residue* u. We also say that $\mathbb{Z}$ is constituted by the *multiplexing* of channels in Π_g, and Π_g is *associated* with $\mathbb{Z}$.

Proposition 2.9. *Given $\Pi^{(m)}$ and its associated channels Π_g, if $B_i \cap \mathbb{Z}_u \neq \emptyset$ for some $B_i \in \Pi^{(m)}$ and $\mathbb{Z}_u \in \Pi_g$, then $\pi(B_i \cap \mathbb{Z}_u) \mid \pi_i$.*

Proof. Let $\Pi_a^{(2)} = (B_i, \bar{B}_i)$ and $\Pi' = \Pi_a^{(2)} \cap \Pi_g$. Then $B_i \cap \mathbb{Z}_u$ is one of the blocks in Π', which gives $\pi(B_i \cap \mathbb{Z}_u) \mid \pi(\Pi')$ and $\pi(\Pi') = [\pi(\Pi_a^{(2)}), \pi(\Pi_g)]$ from Proposition 2.1(iii). Hence, from $\pi(\Pi_a^{(2)}) = \pi_i$ and $\pi(\Pi_g) \mid \pi_i$, $\pi(\Pi') = \pi_i$ and the result follows. □

3. Patterns with three components

First we note a few properties of integers that we use.

Remark 3.1. For any $z_1, \ldots, z_m$, d is a common divisor of $z_1, \ldots, z_m$ if any only if $d \mid (z_1, \ldots, z_m)$ [2, p. 25].

Remark 3.2. If $z_1, \ldots, z_m$ are nonzero integers, and if $d_1 = z_1$, $d_2 = (d_1, z_2), \ldots, d_m = (d_{m-1}, z_m)$, then $d_m = (z_1, \ldots, z_m)$ [2, p. 26].

Remark 3.3. For any $z_1, z_2, z_3 \in \mathbb{Z}$, z_3 is prime to $z_1 z_2$ if and only if z_3 is prime to both z_1 and z_3 [2, p. 32].

Remark 3.4 (*Chinese remainder theorem*). If positive integers $z_1, \ldots, z_m$ are relatively prime in pairs and if $k_1, \ldots, k_m$ are any given integers, then the m congruences $x = k_i \bmod z_i$, $1 \leq i \leq m$, have a common solution which is unique up to modulo $z_1 z_2 \ldots z_m$ [2, p. 55].

For $\Pi^{(3)}$, we note that $\pi_{ij} = \pi_{(i,j)}$, if $i \neq j$. From now on, we will often use this fact without explicitly stating it.

Theorem 3.5. *For a given $\Pi^{(3)}$, let $r_i = \pi / \pi_i$, $i = 1, 2, 3$, and $g = (\pi_i, \pi_2, \pi_3)$; then*

(i) *r_i are pairwise relatively prime,*
(ii) *$\pi_1 = g r_2 r_3$, $\pi_2 = g r_1 r_3$ and $\pi_3 = g r_1 r_2$, and*
(iii) *$g > 1$.*

Proof. (i) From Proposition 2.1(i), $\pi_1 = \pi_{23}$, $\pi_2 = \pi_{13}$ and $\pi_3 = \pi_{12}$. Hence, from Proposition 2.4(ii), r_i are pairwise relatively prime.

(ii) Since r_i are pairwise relatively prime, $[r_1, r_2, r_3] = [[r_1, r_2], r_3] = [r_1 r_2, r_3]$. But $r_1 r_2$ and r_3 are again pairwise relatively prime by Remark 3.3. Hence $[r_1, r_2, r_3] = r_1 r_2 r_3$. From $\pi / r_i = \pi_i$, we have $r_i \mid \pi$, hence $[r_1, r_2, r_3] \mid \pi$ or $r_1 r_2 r_3 \mid \pi$. Putting $\pi = d r_1 r_2 r_3$, we have $\pi_i = \pi / r_i = d r_1 r_2 r_3 / r_i$. To show $d = g$, we let, in Remark 3.2, $d_1 = \pi_1 = d r_2 r_3$,

$d_2=(d_1, \pi_2)=(dr_2r_3, dr_1r_3)=dr_3$ and $d_3=(d_2, \pi_3)=(dr_3, dr_1r_2)=d(r_1r_2, r_3)=d$, hence $d=(\pi_1, \pi_2, \pi_3)=g$.

(iii) Finally, to show $g>1$, assume $g=1$. Then $\pi_1=r_2r_3$, $\pi_2=r_1r_3$ and $\pi_3=r_1r_2$. For an arbitrary $z_1\in B_1$, let $S_3(z_1)=\{z \mid z_1=z \bmod r_3\}$. Since r_1 and r_2 are relatively prime, by Remark 2.3 we can write $S_3(z_1)=\{z \mid (\exists u, v\in \mathbb{Z})(z=z_1+(ur_1+vr_2)r_3)\}=\{z \mid (\exists k_1, k_2\in \mathbb{Z})(z=z_1+k_1\cdot \pi_1+k_2\cdot \pi_2)\}$. Then, from $z_1+k_1\cdot \pi_1\in B_1$ for any k_1, and $(\forall z\in B_2)(z-k_2\cdot \pi_2\in B_2)$ for any k_2, we conclude that $S_3(z_1)\subseteq B_1\cup B_3=\bar{B}_2$. Through symmetric arguments, we also have, for any $z_2\in B_2$, $S_1(z_2)=\{z \mid z_2=z \bmod r_1\}\subseteq B_1\cup B_2=\bar{B}_3$, and for any $z_3\in B_3$, $S_2(z_3)=\{z \mid z_3=z \bmod r_2\}\subseteq B_2\cup B_3=\bar{B}_1$. Since r_1, r_2 and r_3 are relatively prime in pairs, we conclude, by the Chinese Remainder Theorem (Remark 3.4), that $S_1(z_2)\cap S_2(z_3)\cap S_3(z_1)\neq\emptyset$, or $\bar{B}_1\cap\bar{B}_2\cap\bar{B}_3\neq\emptyset$. But $B_1\cup B_2\cup B_3=\mathbb{Z}$, hence $\bar{B}_1\cap\bar{B}_2\cap\bar{B}_3=\emptyset$. Hence $g>1$. □

We note in passing that this theorem does not generalize to patterns with a greater number of components. For example, take $\Pi^{(4)}=(B_1,\ldots, B_4)$ such that

$$B_1=\{z \mid z=0 \bmod 15\},$$

$$B_2=\{z \mid z=1 \bmod 21\},$$

$$B_3=\{z \mid z=2 \bmod 35\},$$

$$B_4=\mathbb{Z}-(B_1\cup B_2\cup B_3).$$

Then, it can be shown that B_1, B_2, B_3 and B_4 indeed make up a pattern, and $\pi_1=15$, $\pi_2=21$, $\pi_3=35$ and $\pi_4=105$. Clearly $r_1=7$, $r_2=5$, $r_3=3$ and $r_4=1$ and they are relatively prime in pairs but $\pi_i=r_1r_2r_3r_4/r_i$, and $g=1$.

Theorem 3.6. *For any $\Pi^{(3)}$ and its associated channels Π_g,*

$$(\forall \mathbb{Z}_u\in \Pi_g)(\exists B_i\in \Pi^{(3)})(B_i\cap \mathbb{Z}_u=\emptyset).$$

Proof. For $i=1,2,3$, assume $B_i\cap\mathbb{Z}_u\neq\emptyset$, and let $S_{iu}=B_i\cap\mathbb{Z}_u$, and let $h_u:\mathbb{Z}_u\to\mathbb{Z}: z\mapsto(z-u)/g$. Since $S_{1u}\cup S_{2u}\cup S_{3u}=\mathbb{Z}_u$, $h_u(\mathbb{Z}_u)$ forms a pattern $(h_u(S_{1u}), h_u(S_{2u}), h_u(S_{3u}))$. By Proposition 2.9, $\pi(S_{iu})\mid\pi_i$, and also clearly $\pi(S_{iu})/\pi(h_u(S_{iu}))=g$. Hence $\pi(h_u(S_{iu}))\mid(\pi_i/g)$. Since $(\pi_1/g, \pi_2/g, \pi_3/g)=1$ by Remark 3.1, it follows that $(\pi(h_u(S_{1u})), \pi(h_u(S_{2u})), \pi(h_u(S_{3u})))=1$, also by Remark 3.1. This contradicts Theorem 3.5(iii) when $(h_u(S_{1u}), h_u(S_{2u}), h_u(S_{3u}))$ is considered as a pattern. □

Corollary 3.7. *For any pattern $\Pi^{(3)}$, each element $\mathbb{Z}_u$ of its associated channels Π_g contains elements from at most two blocks of $\Pi^{(3)}$.*

This property does not, in general, hold for $\Pi^{(m)}$ if $m>3$, as we shall see in Section 9.

4. Translation on patterns

For any $\Pi^{(m)}$ and its $g=(\pi_1,\ldots,\pi_m)$, we have $g\,|\,\pi_i$, $1\leq i\leq m$. Hence we see that every translation $t:\mathbb{Z}\to\mathbb{Z}$ such that $t(B_i)=B_i$ for all i also gives $t(\mathbb{Z}_u)=\mathbb{Z}_u$ for all channels $\mathbb{Z}_u$ in Π_g. Furthermore, the following holds.

Proposition 4.1. *With respect to $\Pi^{(3)}$ and its associated Π_g, if a translation $t:\mathbb{Z}\to\mathbb{Z}$ is such that $t(B_i\cup B_j)=B_i\cup B_j$ for some B_i, $B_j\in\Pi^{(3)}$, $B_i\neq B_j$, then $t(\mathbb{Z}_u)=\mathbb{Z}_u$ for each $\mathbb{Z}_u\in\Pi_g$.*

Proof. Assume $t_r(B_i\cup B_j)=B_i\cup B_j$. By Proposition 2.1(i), $t_r(B_k)=B_k$, where B_k is the third block of $\Pi^{(3)}$. Hence $r\in P(B_k)$, or $\pi_k|r$. By Theorem 3.5(ii), $\pi_k=gr_ir_j$, hence $g\,|\,r$, or $r\in P(\mathbb{Z}_u)$. Thus $t_r(\mathbb{Z}_u)=\mathbb{Z}_u$. □

For any B_i, $B_j\in\Pi^{(3)}$, $i\neq j$, t_r with $r\in P(B_i\cup B_j)$ clearly satisfies the premise of the proposition. Note that if the premise is weakened to $t(B_i\cup B_j)\subseteq B_i\cup B_j$, then the proof does not hold, because $t(B_i\cup B_j)\neq B_i\cup B_j$ may happen when $\pi(\Pi)=\omega$. But for this case Π_g is not defined because g is not.

Proposition 4.2. *Given $\Pi^{(3)}$ and its associated Π_g. The following two conditions are equivalent for any B_i, $B_j\in\Pi^{(3)}$, $i\neq j$:*

(i) $\pi_{ij}<[\pi_i,\pi_j]$.

(ii) *There is at least one $\mathbb{Z}_u\in\Pi_g$ such that*

$$B_i\cap\mathbb{Z}_u\neq\emptyset \quad \text{and} \quad B_j\cap\mathbb{Z}_u\neq\emptyset.$$

Proof. (i)⇒(ii). Assume that B_i and $B_j\in\Pi^{(3)}$ satisfy condition (i). Let the third block be denoted by B_k, then $P(B_i\cup B_j)=P(B_k)$ by Proposition 2.1(i), or $\pi_{ij}=\pi_k$. Since $\pi_{ij}<[\pi_i,\pi_j]$ by the premise, there exists, by Proposition 2.8, a translation $t:(B_i\cup B_j)\to(B_i\cup B_j)$ such that $t(z_i)\notin B_i$ for some $z_i\in B_i$, or $t(z_j)\notin B_j$ for some $z_j\in B_j$, say the former. For this z_i, then, $t(z_i)\in B_j$. On the other hand, by Proposition 4.1, any translation $t:\mathbb{Z}\to\mathbb{Z}$ such that $t(B_i\cup B_j)=B_i\cup B_j$ also gives $t(\mathbb{Z}_u)=\mathbb{Z}_u$ for each $\mathbb{Z}_u\in\Pi_g$. Hence there exists some $\mathbb{Z}_u\in\Pi_g$ such that $z_i\in\mathbb{Z}_u$ and $t(z_i)\in\mathbb{Z}_u$, or $B_i\cap\mathbb{Z}_u\neq\emptyset$ and $B_j\cap Z_u\neq\emptyset$.

(ii)⇒(i). The contradiction is led from the assumption that B_i and B_j satisfying $\pi_{ij}=[\pi_i,\pi_j]$ also satisfy condition (ii). The third block is denoted by B_k, then $\pi_k=\pi_{ij}$. From the assumption, $\pi_{ij}=[\pi_i,\pi_j]$ and $g=(\pi_k,\pi_i,\pi_j)=(\pi_{ij},\pi_i,\pi_j)=([\pi_i,\pi_j],\pi_i,\pi_j)=(\pi_i,\pi_j)$. From the assumption, there exist channel $\mathbb{Z}_u$ in Π_g such that $\mathbb{Z}_u\cap B_i\neq\emptyset$ and $\mathbb{Z}_u\cap B_j\neq\emptyset$. Then there exist z_i and $z_j\in\mathbb{Z}_u$ such that $z_i\in B_i$, $z_j\in B_j$ and $z_i\neq z_j$. Since z_i, $z_j\in\mathbb{Z}_u$, there exist an $m\in\mathbb{Z}$ such that $z_i=z_j+m\cdot g$. By Remark 2.3, we can write $g=k_1\cdot\pi_i+k_2\cdot\pi_j$ for some k_1 and $k_2\in\mathbb{Z}$. Then $z_i=z_j+mg=z_j+m(k_1\cdot\pi_i+k_2\cdot\pi_j)$. Namely $z_i-m\cdot k_1\cdot\pi_i=z_j+m\cdot k_2\cdot\pi_j$. Hence $z_i\in B_i$ and $z_i-m\cdot k_i\cdot\pi_i\in B_i$. Similarly, $z_j+m\cdot k_2\cdot\pi_j\in B_j$. Then $B_i\cap B_j\neq\emptyset$, which contradicts $B_i\cap B_j=\emptyset$. □

Corollary 4.3. *Given $\Pi^{(3)}$ and its associated channel Π_g, if there does not exist a $\mathbb{Z}_u \in \Pi_g$ such that $\mathbb{Z}_u \cap B_i \neq \emptyset$ and $Z_u \cap B_j \neq \emptyset$. Then $\pi_{ij} = [\pi_i, \pi_j]$.*

Note that there exist $\Pi^{(3)}$ which do not satisfy the premise of Proposition 4.2(i). One example is the one with $B_i = \{z \mid z = i-1 \bmod 3\}$, $i = 1, 2, 3$, for which $g = 3$ and $r_1 = r_2 = r_3 = 1$ in Theorem 3.5. On the other hand, the case where $\pi_{ij} < [\pi_i, \pi_j]$ for some i and j and $\pi_{ij} = [\pi_i, \pi_j]$ for other i and j can readily be constructed by the use of Theorem 3.5; for example, $g = 2$, $r_1 = 1$, $r_2 = 3$ and $r_3 = 2$. The sequence with "1 1 2 1 1 3 2 1 1 1 2 3' as its subsequence for the primitive period is such an example.

Combining the results so far obtained for channels of certain $\Pi^{(3)}$, we have the following.

Corollary 4.4. *Let $\Pi^{(3)}$ be such that $\pi_{ij} < [\pi_i, \pi_j]$ for some B_i and B_j. Then there exists a channel $\mathbb{Z}_u$ in Π_g such that*

(i) $\mathbb{Z}_u \subseteq B_i \cup B_j$, *but* $\mathbb{Z}_u \not\subseteq B_i$ *nor* $\mathbb{Z}_u \not\subseteq B_j$,

(ii) $(\forall t)[t(B_i) = B_i \Rightarrow t(\mathbb{Z}_u) = \mathbb{Z}_u]$ *and*

$(\forall t)[t(B_j) = B_j \Rightarrow t(\mathbb{Z}_u) = \mathbb{Z}_u]$,

(iii) $(\forall t)[t(B_i \cup B_j) = (B_i \cup B_j) \Rightarrow t(\mathbb{Z}_u) = \mathbb{Z}_u]$.

Proof. Noting that $g \mid \pi_i$ and $\mathbb{Z}_u \in \Pi_g$, we have (ii) for all $\mathbb{Z}_u$. From Proposition 4.1, we have (iii) for all $\mathbb{Z}_u$. Hence, using Proposition 4.2, we may pick an appropriate $\mathbb{Z}_u$ to satisfy (i). □

Up to now, we have used the channel partition Π_g of $\mathbb{Z}$, which is uniquely defined once a pattern $\Pi^{(3)}$ is given. We now begin to construct objects which are a generalization of channels of $\Pi^{(3)}$ characterized by Corollary 4.4.

For a given pattern $\Pi^{(3)}$ such that $\pi_{ij} < [\pi_i, \pi_j]$ for a fixed choice of B_i and B_j, we define a family Φ_{ij} of certain subsets of $\mathbb{Z}$ such that $F \in \Phi_{ij}$, $F \subseteq \mathbb{Z}$, if and only if

(i) $F \subseteq B_i \cup B_j$, but $F \not\subseteq B_i$ nor $F \not\subseteq B_j$,

(ii) $(\forall t)[t(B_i) = B_i \Rightarrow t(F) = F]$ and

$(\forall t)[t(B_j) = B_j \Rightarrow t(F) = F]$,

(iii) $(\forall t)[t(B_i \cup B_j) = B_i \cup B_j \Rightarrow t(F) = F]$.

The $\mathbb{Z}_u \in \Pi_g$ of Corollary 4.4 clearly satisfies the above conditions, hence $\Phi_{ij} \neq \emptyset$. Φ_{ij} is ordered set with respect to the set inclusion relation. Furthermore, $\pi(F) \mid g$ for any element F of Φ_{ij} and, if $F_u, F_v \in \Phi_{ij}$, then $F_u \cup F_v \in \Phi_{ij}$. Hence, Φ_{ij} is a finite semilattice with respect to the set-union operation. The unique maximal element of Φ_{ij} is called the *frame* of Φ_{ij}, denoted by F_{ij}. Then $\pi(F_{ij}) \mid g$. We have the following result.

Proposition 4.5. *$\Phi_{ij} \neq \emptyset$ and $\pi(F_{ij}) \mid g$.*

Proposition 4.6. *Let F_{ij} be the frame of Φ_{ij}, and let $t: \mathbb{Z} \to \mathbb{Z}$ be a translation such that $t(B_i \cup B_j) = B_i \cup B_j$, then*

(i) $(\forall z_i \in B_i)[z_i \notin F_{ij} \Rightarrow t(z_i) \in B_i]$,

(ii) $(\forall z_j \in B_j)[z_j \notin F_{ij} \Rightarrow t(z_j) \in B_j]$.

Proof. Assume (i) does not hold. Then there exists a $z_i \in B_i$ such that $z_i \notin F_{ij}$ and $t(z_i) \in B_j$. Let Π_g be the channels of $\Pi^{(3)}$ as before, then, by Proposition 4.1, there exists a $\mathbb{Z}_u \in \Pi_g$ such that $z_i \in \mathbb{Z}_u$ and $t(z_i) \in \mathbb{Z}_u$. For this $\mathbb{Z}_u$, $\mathbb{Z}_u \cap B_i \neq \emptyset$ and $\mathbb{Z}_u \cap B_j \neq \emptyset$, and it follows that $\mathbb{Z}_u = (\mathbb{Z}_u \cap B_i) \cup (\mathbb{Z}_u \cap B_j)$ by Corollary 3.7. Also, this $\mathbb{Z}_u$ clearly satisfies the conditions that it be a member of Φ_{ij}, as in Corollary 4.4. Hence $\mathbb{Z}_u \in \Phi_{ij}$, and we also have $\mathbb{Z}_u \cup F_{ij} \in \Phi_{ij}$. However, the particular $z_i \in \mathbb{Z}_u$ is not in F_{ij}, hence $\mathbb{Z}_u \cup F_{ij} \neq F_{ij}$. This contradicts the premise that F_{ij} is maximal. Part (ii) is similarly proved. □

In the above treatment of frames, the set Π_g of channels, defined in terms of $\Pi^{(3)}$, played an essential role. However, if we take any $\Pi^{(m)} = (B_1, \ldots, B_i, \ldots, B_j, \ldots, B_m)$ such that $\pi_{ij} < [\pi_i, \pi_j]$ for a fixed pair of i and j, then $\Pi'(i,j) = (B_i, B_j, \overline{B_i \cup B_j})$, called the *derived* (three component) *pattern*, is a three component pattern still with $\pi_{ij} < [\pi_i, \pi_j]$ for those i and j. Hence, $\Phi'_{ij} \neq \emptyset$ for this derived pattern and its frame $F'_{ij} \in \Phi'_{ij}$ may be studied just as before, in terms of channels $\Pi'_g(i,j)$ for this derived pattern. Now, for a given pattern Π, if a translation $t: \mathbb{Z} \to \mathbb{Z}$ is such that $t(B_i) = B_i$ for all $B_i \in \Pi$, then we say t is a *block identity translation* of Π, and denote it by $t: \Pi \to \Pi$. Clearly, t is not unique for a given Π.

Proposition 4.7. *Given a $\Pi^{(m)}$ such that $\pi_{ij} < [\pi_i, \pi_j]$, let $\Pi^{(m)}_{(u,v)} = (B_u \cup B_v, B_1, \ldots, \hat{B}_u, \ldots, \hat{B}_v, \ldots, B_m)$ as before, $F'_{ij} \in \Phi'_{ij}$ be the frame of the derived pattern $\Pi'_{(i,j)}$, and t be a translation. Then,*

(i) $(\forall t: \Pi^{(m)} \to \Pi^{(m)})\ t(F'_{ij}) = F'_{ij}$,

(ii) $(\forall t: \Pi^{(m)}_{(u,v)} \to \Pi^{(m)}_{(u,v)})\ t(F'_{ij}) = F'_{ij}$.

Proof. (i) is obvious from the definition of a frame. To show (ii), we have to consider the following two cases. (1) $\{u, v\} = \{i, j\}$: For this case, $t(B_i \cup B_j) = B_i \cup B_j$. Then $t(F'_{ij}) = F'_{ij}$ from $\pi(F'_{ij}) \mid \pi(B_i \cup B_j)$. (2) $\{u, v\} \neq \{i, j\}$: For this case, for at least one of B_i and B_j, say B_i, $t(B_i) = B_i$. Then $t(F'_{ij}) = F'_{ij}$ from $\pi(F'_{ij}) \mid \pi(B_i)$. □

By the use of Corollary 4.3, note that the following holds.

Corollary 4.8. *Let $\Pi^{(3)}$ be such that $\pi_{ij} < [\pi_i, \pi_j]$ for some $B_i, B_j \in \Pi^{(3)}$. Then there exists a channel $\mathbb{Z}_u \in \Pi^{(3)}_g$ such that $\mathbb{Z}_u \subseteq B_i \cup B_j$ and*

$$(\forall t: \Pi^{(3)} \to \Pi^{(3)})\ t(\mathbb{Z}_u) = \mathbb{Z}_u.$$

5. Prime patterns

Let $\Pi(c)=(B_1,\ldots,B_m)$, $m\geqslant 2$, be the pattern of a sequence $c\in\tilde{A}_m^{\mathbb{Z}}$, and let $\Pi_{(i,j)}(c)=(B_i\cup B_j, B_1,\ldots,\hat{B}_i,\ldots,\hat{B}_j,\ldots,B_m)$. If $\pi(\Pi_{(i,j)}(c))<\pi(\Pi(c))=\pi(c)$ for *all i* and *j*, then we say *c* is a *prime sequence* and $\Pi(c)$ is a *prime pattern.* Note that if a prime sequence exists for each $m\geqslant 2$, then our first result stated in the Introduction is established.

Since $\#\tilde{A}_1^{\mathbb{Z}}=1$ and, for $c\in\tilde{A}_1^{\mathbb{Z}}$, $\pi(c)=1$, the following is trivially established.

Proposition 5.1. *Each $c\in\tilde{A}_2^{\mathbb{Z}}$ such that $\pi(c)>1$ is a prime sequence.*

As we have done so far, we continue to study sequences in terms of their patterns.

Proposition 5.2. *If $\Pi^{(m)}$ is a prime pattern, $m\geqslant 2$, then $\pi_{ij}<[\pi_i,\pi_j]$ for all i and j, $i\neq j$, where $\pi_{ij}=\pi(B_i\cup B_j)$.*

Proof. That $\pi_{ij}\leqslant[\pi_i,\pi_j]$ is obvious. Also, it is clear that the proposition holds for $m=2$. Let $m\geqslant 3$, and assume $\Pi^{(m)}$ is prime and $\pi_{ij}=[\pi_i,\pi_j]$, $i<j$. Then,

$$\begin{aligned}\pi_{(i,j)}&=[\pi_{ij},\pi_1,\ldots,\hat{\pi}_i,\ldots,\hat{\pi}_j,\ldots,\pi_m]\\&=[[\pi_i,\pi_j],\pi_1,\ldots,\hat{\pi}_i,\ldots,\hat{\pi}_j,\ldots,\pi_m]\\&=\pi,\end{aligned}$$

which is contrary to the premise that $\Pi^{(m)}$ is a prime pattern. □

Proposition 5.3. *If $\Pi^{(3)}$ is a prime pattern, then* (i) *each channel in its associated channels Π_g contains elements from at most two blocks of $\Pi^{(3)}$, and also* (ii) *for any two blocks B_i and B_j of $\Pi^{(3)}$ there exists at least one channel in Π_g which contains elements from two blocks of $\Pi^{(3)}$. Hence,* (iii) *$g\geqslant 3$.*

Proof. The proof is immediate. (i) follows from Corollary 3.7, (ii) from Propositions 4.2 and 5.2, and (iii) from (ii) and ${}_3C_2=3$. □

Besides the above, many properties we have shown so far also hold for a prime $\Pi^{(3)}$ as a special case, such as Corollary 4.4 and Propositions 4.5 through 4.7 and Corollary 4.8. Especially useful in what follows is Proposition 4.7.

Let $\Pi^{(m)}$, $m\geqslant 3$, be a prime pattern, let $\Pi'(i,j)=(B_i, B_j, \overline{B_i\cup B_j})$ be the derived pattern with respect to B_i and B_j, and let $F'_{ij}\in\Phi'_{ij}$ be the frame of $\Pi'(i,j)$, all for each pair of i and j, $1\leqslant i<j\leqslant m$. Such definitions are justified for prime patterns because of Propositions 4.7 and 5.2. Since $F'_{ij}\subseteq\mathbb{Z}$, we may define the primitive period of F'_{ij} as before, and denote it by $f_{ij}=\pi(F'_{ij})$. Let f be the lcm of all f_{ij}, i.e.

$$f=[f_{12},f_{13},\ldots,f_{1m},f_{23},f_{24},\ldots,f_{2m},\ldots,f_{m-1,m}]=\operatorname{lcm}\{f_{ij}\mid 1\leqslant i,j\leqslant m\}.$$

Let Π_f be the partition of $\mathbb{Z}$ such that $\Pi_f = (\mathbb{Z}_0, \mathbb{Z}_1, \dots, \mathbb{Z}_{f-1})$, where $\mathbb{Z}_u = \{z \mid z = u \bmod f\}$. We shall call Π_f *f-channels*, and the previously defined Π_g *g-channels*. As we shall see, this Π_f becomes a generalization of Π_g, as was promised in the preceding section.

Proposition 5.4. *For a given prime pattern $\Pi^{(m)}$ and any $\Pi^{(m)}_{(i,j)}$ for it, let $t: \Pi^{(m)}_{(i,j)} \to \Pi^{(m)}_{(i,j)}$ be a block identity translation, then $t: \Pi_f \to \Pi_f$, and $f \mid \pi_{(i,j)}$. Similarly, any $t: \Pi^{(m)} \to \Pi^{(m)}$ is also $t: \Pi_f \to \Pi_f$, and $f \mid \pi$.*

Proof. Since $t(B_i \cup B_j) = B_i \cup B_j$, $t(F'_{ij}) = F'_{ij}$ by the definition of F'_{ij}, thus the displacement of t is divisible by f_{ij}. Furthermore, for any $B_u \in \Pi_{(i,j)}$ such that $u \notin \{i, j\}$, we have $t(B_u) = B_u$ and the definition of F'_{uv} (or F'_{vu}) again gives $t(F'_{uv}) = F'_{uv}$ (or $t(F'_{vu}) = F'_{vu}$). Hence, the displacement of t must be divisible by all f_{uv}, or $t: \Pi^{(m)}_f \to \Pi^{(m)}_f$. Finally, $\pi_{(i,j)} \mid \pi$ gives the rest. □

Note that, if $\Pi^{(3)}$ satisfies $\pi_{ij} < [\pi_i, \pi_j]$, then a t such that $t: \Pi^{(3)}_{(ij)} \to \Pi^{(3)}_{(ij)}$ also has the property $t: \Pi^{(3)}_g \to \Pi^{(3)}_g$ because $\Pi^{(3)}$ satisfies the premise of the above proposition by Corollary 4.4. But this is what Corollary 4.8 states, hence Proposition 5.4 for $\Pi^{(m)}$ corresponds to Corollary 4.8 for $\Pi^{(3)}$.

Proposition 5.5. *For fixed, otherwise arbitrary, choices of i and j, let $\mathbb{Z}_{k_1}, \mathbb{Z}_{k_2}, \dots, \mathbb{Z}_{k_q} \in \Pi_f$, $0 \le k_1 < k_2 < \cdots < k_q \le f-1$, be all and only those such that $\mathbb{Z}_{k_r} \cap F'_{ij} \ne \emptyset$, $1 \le r \le q$, then*

(i) *each $\mathbb{Z}_{k_r} \subseteq F'_{ij}$, and $F'_{ij} = \mathbb{Z}_{k_1} \cup \mathbb{Z}_{k_2} \cup \cdots \cup \mathbb{Z}_{k_q}$, and*

(ii) *each $\mathbb{Z}_{k_r}$ contains elements from at most two blocks of $\Pi^{(m)}$.*

Proof. $\pi(\mathbb{Z}_u) = f$ for each $\mathbb{Z}_u \in \Pi_f$, and also $f_{ij} \mid f$. Furthermore, since $\mathbb{Z}_u = \{kf + u \mid k \in \mathbb{Z}\}$, if $F'_{ij} \cap \mathbb{Z}_u \ne \emptyset$ for some $\mathbb{Z}_u$, then $\mathbb{Z}_u \subseteq F'_{ij}$. Since Π_f is a partition of $\mathbb{Z}$, each element of F'_{ij} is in some $\mathbb{Z}_u \in \Pi_f$. Statement (ii) follows from the definition of F'_{ij}. □

Note that statement (ii) is a generalization of Corollary 3.7 on $\Pi^{(3)}_g$ to $\Pi^{(m)}_f$.

Let $K_{ij} = \{k_r \mid \mathbb{Z}_{k_r} \in \Pi_f \text{ and } \mathbb{Z}_{k_r} \cap F'_{ij} \ne \emptyset\}$, then the above proposition states that $\bigcup_{k_r \in K_{ij}} \mathbb{Z}_{k_r} = F'_{ij}$. Proposition 4.2 on $\Pi^{(3)}_g$ generalizes to $\Pi^{(m)}_f$ of a prime pattern as follows.

Theorem 5.6. *Let $\Pi^{(m)}$ be a prime pattern. Then for each pair B_i, $B_j \in \Pi^{(m)}$, there exists at least one $\mathbb{Z}_u$ in Π_f of $\Pi^{(m)}$ such that $\mathbb{Z}_u \cap B_i \ne \emptyset$ and $\mathbb{Z}_u \cap B_j \ne \emptyset$ simultaneously.*

Proof. Take a block identity translation $t: \Pi^{(m)}_{(i,j)} \to \Pi^{(m)}_{(i,j)}$, then $t(B_u) = B_u$ for all B_u such that $u \ne i$ nor $u \ne j$. Next, let $F'_{ij} \in \Phi'_{ij}$ be the frame of the derived $\Pi'(i, j)$, then, from $t(B_i \cup B_j) = B_i \cup B_j$ and Proposition 4.6, we have $t(B_i - F'_{ij}) \subseteq B_i$ and $t(B_j -$

$F'_{ij}) \subseteq B_j$. Thirdly, assume that only $\mathbb{Z}_{k_r} \subseteq B_i$ or $\mathbb{Z}_{k_r} \subseteq B_j$ holds for all $k_r \in K_{ij}$ of Π_f, then, since $F'_{ij} \subseteq B_i \cup B_j$, we have $(\forall \mathbb{Z}_{k_r} \in \Pi_f)[\mathbb{Z}_{k_r} \cap F'_{ij} \neq \emptyset \Rightarrow \mathbb{Z}_{k_r} \subseteq B_i$ or $\mathbb{Z}_{k_r} \subseteq B_j]$. Hence, if $\mathbb{Z}_{k_r} \cap (B_i \cap F'_{ij}) \neq \emptyset$ then $\mathbb{Z}_{k_r} \subseteq B_i \cap F'_{ij}$ by Proposition 5.5; or $B_i \cap F'_{ij}$ is partitioned into elements from Π_f, and so is $B_j \cap F'_{ij}$. Since $t(\mathbb{Z}_{k_r}) = \mathbb{Z}_{k_r}$ for all $\mathbb{Z}_{k_r}$ in F'_{ij} by Proposition 5.4, we conclude that $t(B_i \cap F'_{ij}) = B_i \cap F'_{ij} \subseteq B_i$, and $t(B_j \cap F'_{ij}) \subseteq B_j$.

Taking three results together, $t(B_u) = B_u$ for all $B_u \in \Pi^{(m)}$. This means that any period of $\Pi^{(m)}_{(i,j)}$ is also a period of $\Pi^{(m)}$ or $\pi(\Pi^{(m)}_{(i,j)}) = \pi(\Pi^{(m)})$. But this contradicts the premise that $\Pi^{(m)}$ is a prime pattern, and the assumption is false. □

This is a generalization of Proposition 5.3(ii) from $\Pi^{(3)}_g$ to $\Pi^{(m)}_f$.

Let $\mathbb{Z}_{(i,j)}$ denote an element of Π_f such that $\mathbb{Z}_{(i,j)} \cap B_i \neq \emptyset$, $\mathbb{Z}_{(i,j)} \cap B_j \neq \emptyset$ and $\mathbb{Z}_{(i,j)} \subseteq B_i \cup B_j$, whose existence is proved in the above theorem, although it is not necessarily unique. Then, the following holds.

Corollary 5.7. *If* $\{i, j\} \neq \{u, v\}$ *then* $\mathbb{Z}_{(i,j)} \neq \mathbb{Z}_{(u,v)}$ *regardless of the choices among possible* $\mathbb{Z}_{(i,j)}$ *and* $\mathbb{Z}_{(u,v)}$.

Proof. The proof clearly follows from the fact that $\{B_i, B_j\} \neq \{B_u, B_v\}$ in which $\mathbb{Z}_{(i,j)}$ and $\mathbb{Z}_{(u,v)}$ are defined. □

Corollary 5.8. *For a prime pattern* $\Pi^{(m)}$, $f \geq {}_mC_2$.

Proof. Π_f is the f block partition of $\mathbb{Z}$, and among f blocks are included by Theorem 5.6 all $\mathbb{Z}_{(i,j)}$ for different choices of $\{i, j\}$. Hence, by Corollary 5.7, there are at least ${}_mC_2$ blocks in Π_f, i.e. the number of possible choices of two blocks from the m blocks of $\Pi^{(m)}$. □

This is a generalization of Proposition 5.3(iii) from $\Pi^{(3)}$ to $\Pi^{(m)}$.

We may make a straightforward observation on a property of nonprime patterns here.

Proposition 5.9. *Let* $\Pi^{(m)}$, $m \geq 3$, *be a pattern such that* $\pi(\Pi^{(m)})$ *is a prime number, then* $\Pi^{(m)}$ *is not a prime pattern.*

Proof. Since $\pi = [\pi_1, \pi_2, \ldots, \pi_m]$ is a prime number p, say the kth, all π_i must be the kth prime number, or $\pi_i = p$. Hence $\pi_{(i,j)}$ is also the kth prime number for any i and j. □

6. Periods of prime patterns

Let $\Pi^{(m)}$ be an m block prime pattern, and let Π_f be its associated f-channels of $\mathbb{Z}$ as defined before. Also, for $i \neq j$, let $t_{ij}: \mathbb{Z} \to \mathbb{Z}$ be a translation such that (i) $t_{ij}(B_i) = B_i$

and $t_{ij}(B_j)=B_j$ for B_i, $B_j \in \Pi^{(m)}$, that (ii) $t_{ij}(\mathbb{Z}_u)=\mathbb{Z}_u$ for all $\mathbb{Z}_u \in \Pi_f$, and that (iii) it is with the minimum positive displacement among all such translations. Denote the displacement by d_{ij}. Then clearly $\pi_i \mid d_{ij}$, $\pi_j \mid d_{ij}$ and $f \mid d_{ij}$. Furthermore, the following holds.

Proposition 6.1. (i) $d_{ij} \mid \pi$, *and*
(ii) $\pi=[d_{ij}, \pi_{(i,j)}]$.

Proof. (i) $t: \Pi^{(m)} \to \Pi^{(m)}$ gives $t: \Pi_f \to \Pi_f$ by Proposition 5.4. Hence the definition of t_{ij} gives $d_{ij} \mid \pi$.

(ii) Let $r=[d_{ij}, \pi_{(i,j)}]$, then $\pi_{(i,j)} \mid \pi$ and (i) gives $r \mid \pi$. On the other hand, consider translation $t_r: \mathbb{Z} \to \mathbb{Z}$. From $\pi_{(i,j)} \mid r$, $t_r(B_k)=B_k$ for all $k \notin \{i,j\}$. Also from $d_{ij} \mid r$, $t_r(B_i)=B_i$ and $t_r(B_j)=B_j$, hence $t_r: \Pi^{(m)} \to \Pi^{(m)}$ and $\pi \mid r$. □

Let

$$\begin{aligned}\delta &= [d_{12}, d_{13}, \ldots, d_{1m}, d_{23}, d_{24}, \ldots, d_{2m}, \ldots, d_{m-1,m}] \\ &= \operatorname{lcm}\{\{d_{ij} \mid 1 \le i,j \le m,\ i \ne j\}.\end{aligned}$$

Proposition 6.2. $\delta=\pi$.

Proof. By Proposition 6.1(ii) we have $\pi=[d_{ij}, \pi_{(i,j)}]$ for every pair $\{i,j\}$ such that $i \ne j$. Now, take the lcm's of the both sides of such identities for all possible pairs $\{i,j\}$, $i \ne j$. Then $\pi=[\delta, \operatorname{lcm}\{\pi_{(i,j)} \mid 1 \le i,j \le m,\ i \ne j\}]$. But $\operatorname{lcm}\{\pi_{(i,j)} \mid 1 \le i,j \le m,\ i \ne j\}=\pi$ by Proposition 2.4(i), hence $\pi=[\delta, \pi]$, or $\delta \mid \pi$. On the other hand, from the definition of t_{ij}, t_δ is a block-identity translation $\Pi^{(m)} \to \Pi^{(m)}$, hence $\pi \mid \delta$. □

Let $s_{ij}=d_{ij}/f$, the displacement of t_{ij} measured in terms of the number of equivalence classes in Π_f, and let $q_{ij}=\pi/\pi_{(i,j)}$, the factor by which the primitive period of $\Pi^{(m)}$ reduces when B_i and B_j are made into one block, which already appeared in Proposition 2.4(ii).

Proposition 6.3. $s_{ij}/q_{ij}=(s_{ij}, \pi_{(i,j)}/f)$.

Proof

$$\begin{aligned}q_{ij} &= \pi/\pi_{(i,j)} = [s_{ij}f, \pi_{(i,j)}]/\pi_{(i,j)} \\ &= \{s_{ij}f \cdot \pi_{(i,j)}/(s_{ij}f, \pi_{(i,j)})\}/\pi_{(i,j)} \\ &= s_{ij}f/(s_{ij}f, \pi_{(i,j)}).\end{aligned}$$

But $f \mid \pi_{(i,j)}$ from Proposition 5.4, hence

$$q_{ij}=s_{ij}/(s_{ij}, \pi_{(i,j)}/f). \qquad \square$$

Corollary 6.4. $q_{ij} \cdot f \mid \pi$.

Proof. $q_{ij} \cdot f = (\pi \cdot f)/\pi_{(ij)} = \pi/(\pi_{(ij)}/f)$. $\square$

Let PP(k) be the product of the first k prime numbers, then the following holds.

Theorem 6.5. *For a prime pattern* $\Pi^{(m)}$, $\pi \geq {}_mC_2 \cdot \mathrm{PP}({}_mC_2)$.

Proof. From Corollary 6.4, we have $q_{ij}f \mid \pi$ and it follows that $\mathrm{lcm}\{q_{ij}f \mid i \neq j\} \mid \pi$. Since the members of $\{q_{ij} \mid i \neq j\} = \{\pi/\pi_{(i,j)} \mid i \neq j\}$ are pairwise relatively prime by Proposition 2.4(ii), and $f \geq {}_mC_2$ by Corollary 5.8, $\mathrm{lcm}\{q_{ij}f \mid i \neq j\} = f \cdot \mathrm{lcm}\{q_{ij} \mid i \neq j\} \geq {}_mC_2 \cdot \mathrm{PP}({}_mC_2)$. Hence, $\pi \geq {}_mC_2 \cdot \mathrm{PP}({}_mC_2)$. $\square$

Before the construction of patterns, we prepare some properties of free monoids [4]. Take a set of alphabets A, which generates a free monoid A^* whose elements are called words. Word b_1 is called *conjugate* to word b_2 when there exist words b' and $b'' \in A^*$ such that $b_1 = b'b''$ and $b_2 = b''b'$. Associated with word $b \in A^*$, a periodic sequence $c_b : \mathbb{Z} \to A$ is constructed by concatinating b repeatedly. It is easy to see that the following remark is valid.

Remark 6.6. Suppose periodic sequences c_1 and c_2 are associated with words b_1 and b_2 respectively. Then there exist $t \in \mathbb{Z}$ such that $c_1(z+t) = c_2(z)$ for any $z \in \mathbb{Z}$ if and only if words b_1 and b_2 are conjugate.

We now construct an m block pattern $\Pi^{(m)}$ as follows. Specific examples will be given in Sections 8 and 9. For an integer $w > 1$, partition $\mathbb{Z}$ into w blocks, $\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_w$ such that $\mathbb{Z}_u = \{z \mid z = u-1 \bmod w\}$, $1 \leq u \leq w$, which will be called *channels*. Note that, for a later convenience, $\mathbb{Z}_u$ are indexed starting from 1, not from 0. Let $h_u : \mathbb{Z}_u \to \mathbb{Z} : z \mapsto (z-u+1)/w$, which is a bijection. We denote $\mathbb{Z}'_u = h_u(\mathbb{Z}_u)$ and call it the *compressed channel* of $\mathbb{Z}_u$ by h_u. Clearly, $\mathbb{Z}'_u = \mathbb{Z}$, hence notation $\mathbb{Z}'_u$ is used for the purpose of identifying that the origin of $\mathbb{Z}'_u$ is $\mathbb{Z}_u$. The notion of compressed channel was already used before in the proof of Theorem 3.6 without an explicit definition.

Next, take w compressed, horizontal and two-way infinite channels and lay them down in the order of index u, from $u = 1$ to w, such that the same location $z \in \mathbb{Z}'_u$ for each u is aligned vertically as a column (refer to Fig. 1). Now, place on each of these compressed channels a sequence generated either by (i) the repetition of a single letter of A_m, called a *diluent sequence*, or by (ii) a periodic sequence of letters in A_m, consisting of a primitive period called a *generator sequence* G_u, with the restriction that, if two distinct compressed channels have generator sequences consisting of the same subset of the letters of A_m, then the generator sequences are not conjugate. We shall call by *basic* (m block) *pattern* the sequence obtained from these sequences including at least one generator sequence through reconstituting $\mathbb{Z}$

from the set $\{\mathbb{Z}'_u\}$ of these compressed channels by the reversed procedure of compression, called *multiplexing* or *decompressing* (cf. Fig. 1). Let the lengths of these generator or diluent sequences be $r_1, r_2, \ldots, r_w$, not necessarily distinct. Then the following proposition holds.

$$\mathbb{Z}_1: \quad \ldots ab\underset{G_1}{\underline{\underline{ab}}}ababababababababab\ldots.$$

$$\mathbb{Z}_2: \quad \ldots bb\underset{G_2}{\underline{\underline{abbb}}}abbbabbbabbbabbbab\ldots.$$

$$\mathbb{Z}_3: \quad \ldots cc\underset{G_3}{\underline{\underline{acc}}}accaccaccaccaccacca\ldots.$$

$$\vdots$$

$$\mathbb{Z}_w: \quad \ldots cc\underset{G_w}{\underline{\underline{bcccc}}}bccccbccccbccccbc\ldots.$$

Fig. 1. Compressed channels and generator sequences. (Decompressing gives a basic pattern of $\ldots aaa \ldots bbbc \ldots cabc \ldots cbba \ldots caac \ldots cbbc \ldots b \ldots$, where underlined letters arise from generator sequences $G_1, G_2, \ldots, G_w$).

Proposition 6.7. *The primitive period of the basic pattern is* $\pi = w \cdot [r_1, r_2, \ldots, r_w]$, *and the translation by* π *will map each of the decompressed channels which constitute the basic pattern, onto itself.*

Proof. Let $y = w \cdot [r_1, r_2, \ldots, r_w]$, then it is clear from the construction that $\pi | y$. To show the other way, take a compressed channel which uses more than one letter in it, and focus attention on the letters of its generator sequence, $a_1 a_2 \ldots a_{r_i}$. After decompression, these letters have their respective places in the multiplexed sequence, to be denoted by $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_{r_i}$. Now take $\bar{a}_1$ and take its translation by π, then we naturally find another letter a_1 there, to be denoted by $\bar{a}'_1$. Now view that $\bar{a}'_1$ as one on a certain compressed channel, and denote it by a'_1. Next, take the original a_2 and repeat the same procedure. Since $\bar{a}_1$ and $\bar{a}_2$ are w apart after multiplexing, $\bar{a}'_1$ and $\bar{a}'_2$ are also w apart, hence a'_1 and a'_2 are found on the same compressed channel. Continue the same procedure to the last a_{r_i} and repeat the whole procedure n times, then we have $(a'_1 a'_2 \ldots a'_{r_i})^n$ on the same compressed channel as a subsequence. But $a_1 a_2 \ldots a_{r_i}$ is unique among the generators on the compressed channels by Remark 6.6, hence the compressed channels for $a_1 a_2 \ldots a_{r_i}$ and for $a'_1 a'_2 \ldots a'_{r_1}$ must be the same channel, because n can be arbitrarily large and ultimately the sequence length exceeds that of any other generator sequences. A compressed channel which contains only one kind of letters, i.e. a diluent, is mapped onto itself by t_w. Hence, each compressed channel is mapped onto itself by t, which proves that $y | \tau$. □

Corollary 6.8. *Let Π be a basic m block pattern, and $\Pi_w = (\mathbb{Z}'_1, \mathbb{Z}'_2, \ldots, \mathbb{Z}'_w)$ be the compressed channels, in that order, which are used to generate Π and $\Pi'_w = (\mathbb{Z}'_{i_1}, \mathbb{Z}'_{i_2}, \ldots, \mathbb{Z}'_{i_w})$ be a permutation of Π_w. Then the pattern Π' generated by multiplexing Π'_w is also a basic pattern, having the same primitive period as Π, and t is a block identity translation $\Pi'_w \to \Pi'_w$.*

For $k \geq {}_mC_2$, let $R(k) = \{r_1, r_2, \ldots, r_k\}$ be an arbitrary set of k positive integers which are relatively prime in pairs. Denote by $\mathrm{PR}(k)$ the product of all integers in $R(k)$. Let $A_m = \{a_1, a_2, \ldots, a_m\}$, and order all pairs of distinct letters as (a_1, a_2), $(a_1, a_3), \ldots, (a_1, a_m), (a_2, a_3), (a_2, a_4), \ldots, (a_2, a_m), \ldots, (a_{m-1}, a_m)$. Since k is equal to or greater than the number of such pairs, assign to each $r_v \in R(k)$ a pair (a_i, a_j) surjectively, that is, fix an arbitrary surjection $\{v\} \to \{(i, j)\}$. We shall often use such a correspondence and denote it by $v \mapsto \langle i, j\rangle$, or *by* $v = \langle i, j\rangle$ if it is bijective. Now, let $a_i(a_j)^{r_v - 1}$ be a generator sequence for v, where $v \mapsto \langle i, j\rangle$. Take $(k+d)$ compressed channels, $d \geq 0$, and (i) assign the above k primitive generator sequences to any k of $(k+d)$ compressed channels, and (ii) a diluent letter, not necessarily distinct, to each of the remaining d compressed channels. Finally, fill each compressed channel either by periodic sequences generated by the generator sequence if one is assigned to it by (i) above, or by the single letter sequence of the assigned diluent letter if that is what is assigned to it by (ii) above. The sequence obtained by multiplexing these $k+d$ compressed channels will be called a *d-diluted R-pattern.*

Proposition 6.9. *For a given $R(k)$, a d-diluted R-pattern has the primitive period $\pi = (k+d) \cdot \mathrm{PR}(k)$.*

Proof. A d-diluted R-pattern is a basic pattern, hence $\pi = (k+d) \cdot [r_1, r_2, \ldots, r_k]$ by Proposition 6.7. But $r_1, r_2, \ldots, r_k$ are pairwise relatively prime integers, hence $[r_1, r_2, \ldots, r_k] = \mathrm{PR}(k)$. □

Proposition 6.10. *A d-diluted R-pattern is a prime pattern.*

Proof. Let Π be the given pattern and let $\Pi_{(i,j)}$ be the result of the combination of B_i and B_j through the changes of all a_j in Π to a_i. Then, by observing the compressed channels of Π after the change and by Proposition 6.9, it is clear that

$$\pi(\Pi_{(i,j)}) \leq (k+d)[r_1, \ldots, \hat{r}_{v_1}, \ldots, \hat{r}_{v_2}, \ldots, \hat{r}_{v_b}, \ldots, r_k],$$

where each $v_e \mapsto \langle i, j\rangle$, $1 \leq e \leq b$ and for all such e. The inequality here is due to the fact that $\Pi_{(i,j)}$ may no longer be a basic pattern because, if two generator sequences in some (a_y, a_i) and (a_y, a_j), $y \leq i$, had the same length, then, after the change of a_j to a_i, they will violate the condition required for a basic pattern. Hence, $\pi(\Pi_{(i,j)}) < \pi(\Pi)$ for all i and j, $i \neq j$. □

Propositions 6.9 and 6.10 together give the following theorem.

Theorem 6.11. *If a positive integer z is such that there exist a nonnegative integer d and a set of $k \geq {}_mC_2$ pairwise relatively prime positive integers for some $m \geq 3$ such that $z = (k+d) \cdot \mathrm{PR}(k)$, then there exist m block prime patterns whose primitive period is z.*

Note that the condition on z here is indeed satisfied by π in Theorem 6.5.

7. Minimal prime patterns

Let $k = {}_mC_2$, $m \geq 3$, and take k compressed channels. Also let $P(k)$ be the set of the first k prime numbers. Construct an m block 0-diluted R-pattern using $R(k) = P(k)$, then it is a prime pattern by Proposition 6.10, and we call it an (m block) *minimal* prime pattern.

Theorem 7.1. *An m block minimal prime pattern has the following properties, where $1 \leq i < j \leq m$, and $\mathrm{pr}(k)$ is the k-th prime number*:

(i) $\pi = {}_mC_2 \cdot \mathrm{PP}({}_mC_2)$,

(ii) $f = {}_mC_2$,

(iii) $\mathrm{lcm}\{\pi/\pi_{(i,j)} \mid i<j\} = \mathrm{PP}({}_mC_2)$,

(iv) $F'_{ij} \in \Pi_f$,

(v) $f_{ij} = f$,

(vi) $\pi_i = f \cdot \mathrm{pr}(\langle 1, i\rangle) \cdot \mathrm{pr}(\langle 2, i\rangle) \cdot \ldots \cdot \mathrm{pr}(\langle i-1, i\rangle) \cdot \mathrm{pr}(\langle i, i+1\rangle) \cdot \mathrm{pr}(\langle i, i+2\rangle) \cdot \ldots \cdot \mathrm{pr}(\langle i, m\rangle)$,

(vii) $f = g$, *hence* $\Pi_f = \Pi_g$,

(viii) $\pi_{(i,j)} = \pi/\mathrm{pr}(\langle i,j\rangle)$, *and* $q_{ij} = \mathrm{pr}(\langle i,j\rangle)$.

Proof. (i) Follows from Theorem 6.11.

(ii) and (iii) Since $\mathrm{lcm}\{q_{ij}f \mid i<j\} \mid \pi$ as in the proof of Theorem 6.5, and $q_{ij} = \pi/\pi_{(i,j)}$, we have $f \cdot \mathrm{lcm}\{\pi/\pi_{(i,j)} \mid i<j\} \leq {}_mC_2 \cdot \mathrm{PP}({}_mC_2)$ by (i). But $f \geq {}_mC_2$ by Corollary 5.8 and $\mathrm{lcm}\{\pi/\pi_{(i,j)} \mid i<j\} \geq \mathrm{PP}({}_mC_2)$ by Proposition 2.4(ii). Hence $f = {}_mC_2$ and $\mathrm{lcm}\{\pi/\pi_{(i,j)} \mid i<j\} = \mathrm{PP}({}_mC_2)$.

(iv) From the definition, $F'_{ij} \subseteq B_i \cup B_j$ and $F'_{ij} \not\subseteq B_i$ nor $F'_{ij} \not\subseteq B_j$, and, by Proposition 4.4, such F'_{ij} exists for each i and j, $i \neq j$. Hence the total number of F'_{ij} is ${}_mC_2$, that is also the total number of f-channels $\mathbb{Z}_u$ in Π_f, by (ii). Now, Proposition 5.5 states that each F'_{ij} is partitioned into some elements of Π_f as $F'_{ij} = \mathbb{Z}_{k_1} \cup \mathbb{Z}_{k_2} \cup \cdots \cup \mathbb{Z}_{k_q}$. These results mean that the set of F'_{ij} and the set of f-channels $\mathbb{Z}_u$ are identical. Furthermore, for each B_i and B_j, $i<j$, there is a $\mathbb{Z}_u \in \Pi_f$ such that $\mathbb{Z}_u \cap B_i \neq \emptyset \neq \mathbb{Z}_u \cap B_j$, hence call such $\mathbb{Z}_u$ by the name $\mathbb{Z}_{\langle i,j\rangle}$. Then $F'_{ij} = \mathbb{Z}_{\langle i,j\rangle}$.

(v) $f_{ij} = \pi(F'_{ij}) = \pi(\mathbb{Z}_{\langle i,j\rangle})$ by (iv), hence $f_{ij} = f$.

(vi) For each $\mathbb{Z}_{\langle i,j\rangle}$ of a minimal prime pattern, let $\mathbb{Z}^i_{\langle i,j\rangle} = \mathbb{Z}_{\langle i,j\rangle} \cap B_i$ and $\mathbb{Z}^j_{\langle i,j\rangle} = \mathbb{Z}_{\langle i,j\rangle} \cap B_j$. Then both $h_{\langle i,j\rangle}(\mathbb{Z}^i_{\langle i,j\rangle})$ and $h_{\langle i,j\rangle}(\mathbb{Z}^j_{\langle i,j\rangle})$ have the primitive period $\mathrm{pr}(\langle i,j\rangle)$ in the compressed channel $\mathbb{Z}'_{\langle i,j\rangle}$, where $\mathrm{pr}(\langle i,j\rangle)$ is the $\langle i,j\rangle$th prime number. From the definition,

$$B_i = (\mathbb{Z}^i_{\langle 1,i\rangle} \cup \cdots \cup \mathbb{Z}^i_{\langle i-1,i\rangle}) \cup (\mathbb{Z}^i_{\langle i,i+1\rangle} \cup \cdots \cup \mathbb{Z}^i_{\langle i,m\rangle}),$$

and each $\mathbb{Z}^i_{\langle u,v\rangle}$ in it has a unique primitive period $\mathrm{pr}(\langle u, v\rangle)$ in $h_{\langle u,v\rangle}(\mathbb{Z}_{\langle u,v\rangle})$. Also $h_{\langle u,v\rangle}$ reduces the period in $\mathbb{Z}$ by $f = {}_mC_2$, by (ii). Hence each $\mathbb{Z}^i_{\langle u,v\rangle}$ has the primitive period of $f \cdot \mathrm{pr}(\langle u, v\rangle)$, or $f \cdot \mathrm{pr}(\langle u, v\rangle) \mid \pi_i$. Hence, if we let

$$r = f \cdot \mathrm{pr}(\langle 1, i\rangle) \cdot \ldots \cdot \mathrm{pr}(\langle i-1, i\rangle) \cdot \mathrm{pr}(\langle i, i+1\rangle) \cdot \ldots \cdot \mathrm{pr}(\langle i, m\rangle),$$

then $r \mid \pi_i$. On the other hand, $f_{ij} \mid \pi_i$ from the definition, hence $f \mid \pi_i$ by (v). Then for each $\mathbb{Z}^i_{\langle u,v\rangle}$ in B_i, $t_{\pi_i}(\mathbb{Z}^i_{\langle u,v\rangle}) = \mathbb{Z}^i_{\langle u,v\rangle}$, hence $\pi_i \mid r$.

(vii) $g = (\pi_1, \pi_2, \ldots, \pi_m) = f$ by (vi).

(viii) Let B_i and B_j, $i < j$, be combined into a same block, then

$$\begin{aligned} B_i \cup B_j = {} & (\mathbb{Z}^i_{\langle 1,i\rangle} \cup \cdots \cup \mathbb{Z}^i_{\langle i-1,i\rangle}) \cup (\mathbb{Z}^i_{\langle i,i+1\rangle} \cup \cdots \cup \hat{\mathbb{Z}}^i_{\langle i,j\rangle} \cup \cdots \cup \mathbb{Z}^i_{\langle i,m\rangle}) \\ & \cup (\mathbb{Z}^j_{\langle 1,j\rangle} \cup \cdots \cup \hat{\mathbb{Z}}^j_{\langle i,j\rangle} \cup \cdots \cup \mathbb{Z}^j_{\langle j-1,j\rangle}) \cup (\mathbb{Z}^j_{\langle j,j+1\rangle} \cup \cdots \cup \mathbb{Z}^j_{\langle j,m\rangle}) \\ & \cup \mathbb{Z}_{\langle i,j\rangle}. \end{aligned}$$

Here, each $\mathbb{Z}^i_{\langle u,v\rangle}$ or $\mathbb{Z}^j_{\langle u,v\rangle}$ has retained a unique primitive period $\mathrm{pr}(\langle u, v\rangle)$ in $h_{\langle u,v\rangle}(\mathbb{Z}^i_{\langle u,v\rangle})$ or $h_{\langle u,v\rangle}(\mathbb{Z}^j_{\langle u,v\rangle})$. But $\mathbb{Z}_{\langle i,j\rangle} = \mathbb{Z}^i_{\langle i,j\rangle} \cup \mathbb{Z}^j_{\langle i,j\rangle}$ are now in identical letters, and $h_{\langle i,j\rangle}(\mathbb{Z}_{\langle i,j\rangle}) = \mathbb{Z}$. $f_{ij} \mid \pi_{ij}$ from the definition, then $f \mid \pi_{ij}$ by (v). Hence through a similar argument as that of (vi), we see

$$\pi_{ij} = \pi(B_i \cup B_j) = [\pi_i, \pi_j]/\mathrm{pr}(\langle i, j\rangle).$$

Hence,

$$\begin{aligned} \pi_{(i,j)} &= [\pi_{ij}, \pi_1, \ldots, \hat{\pi}_i, \ldots, \hat{\pi}_j, \ldots, \pi_m] \\ &= [[\pi_i, \pi_j]/\mathrm{pr}(\langle i, j\rangle), \pi_1, \ldots, \hat{\pi}_i, \ldots, \hat{\pi}_j, \ldots, \pi_m]. \end{aligned}$$

But each π_w here consists of the product of ${}_mC_2$ and a mutually disjoint set of $m-1$ prime numbers as was shown in (vi). The fact that $\pi_{(i,j)} < \pi$ also shows that $\Pi^{(m)}$ is a prime pattern as claimed. That $q_{ij} = \mathrm{pr}(\langle i, j\rangle)$ is obvious. □

Note that, from (ii), Π_f are also the compressed channels which we used in constructing the minimal prime pattern.

8. An example of minimal prime patterns

We first give an example of minimal prime patterns. As it is given by Theorem 7.1(i), an m component minimal prime pattern has the primitive period of $\pi^{(m)} = {}_mC_2 \cdot \mathrm{PP}({}_mC_2)$, which is computed for some values of m, as shown in Table 1, together with the number of f-channels, $f^{(m)} = {}_mC_2$. These values clearly indicate that the only practically explicitly presentable example is for the case $m = 3$. The example given below is a variant of what was first constructed by Hayes [1] which

Table 1

m	$f^{(m)}$	$\pi^{(m)}$
3	3	90
4	6	180 180
5	10	64 696 932 300
6	15	9.22×10^{18}
7	21	8.55×10^{30}

for the first time established by demonstration the existence of a prime pattern for $m = 3$. Considering the enormous sizes of the primitive periods for $m > 3$, it is unlikely to hit upon a prime pattern by cut and try approaches, and Hayes' example had a great influence on our subsequent research.

Example 8.1 (*Minimal prime pattern for* $m = 3$). Since $f^{(3)} = {}_3C_2 = 3$, we choose the index coding of $1 = \langle 1, 2\rangle$, $2 = \langle 1, 3\rangle$ and $3 = \langle 2, 3\rangle$. Then, as the f-channels of $\mathbb{Z}$, we have

$$\mathbb{Z}_1 = \mathbb{Z}_{\langle 1,2\rangle} = \{z \mid z = 0 \bmod 3\},$$

$$\mathbb{Z}_2 = \mathbb{Z}_{\langle 1,3\rangle} = \{z \mid z = 1 \bmod 3\},$$

$$\mathbb{Z}_3 = \mathbb{Z}_{\langle 2,3\rangle} = \{z \mid z = 2 \bmod 3\},$$

and clearly each of these has the primitive period of 3. Since $\mathrm{pr}(1) = 2$, $\mathrm{pr}(2) = 3$, and $\mathrm{pr}(3) = 5$, we use the types of partitions of $\mathbb{Z}_u$ suggested previously when we discussed d-diluted R-patterns and obtain,

$$\mathbb{Z}^1_{\langle 1,2\rangle} = \{z \mid z = 0 \bmod 6\},$$

$$\mathbb{Z}^2_{\langle 1,2\rangle} = \mathbb{Z}_{\langle 1,2\rangle} - \mathbb{Z}^1_{\langle 1,2\rangle},$$

corresponding to a generator sequence of type *ab*,

$$\mathbb{Z}^1_{\langle 1,3\rangle} = \{z \mid z = 1 \bmod 9\},$$

$$\mathbb{Z}^3_{\langle 1,3\rangle} = \mathbb{Z}_{\langle 1,3\rangle} - \mathbb{Z}^1_{\langle 1,3\rangle},$$

corresponding to a generator sequence of type *acc*, and

$$\mathbb{Z}^2_{\langle 2,3\rangle} = \{z \mid z = 2 \bmod 15\},$$

$$\mathbb{Z}^3_{\langle 2,3\rangle} = \mathbb{Z}_{\langle 2,3\rangle} - \mathbb{Z}^2_{\langle 2,3\rangle},$$

corresponding to a generator sequence of type *bcccc*.

Their primitive periods are 6 for $\mathbb{Z}^1_{\langle 1,2\rangle}$ and $\mathbb{Z}^2_{\langle 1,2\rangle}$, 9 for $\mathbb{Z}^1_{\langle 1,3\rangle}$ and $\mathbb{Z}^3_{\langle 1,3\rangle}$, and 15 for $\mathbb{Z}^2_{\langle 2,3\rangle}$ and $\mathbb{Z}^3_{\langle 2,3\rangle}$.

Now, we let Π be constituted by

$$B_1 = \mathbb{Z}^1_{\langle 1,2\rangle} \cup \mathbb{Z}^1_{\langle 1,3\rangle} \quad \text{(i.e. } a\text{'s)},$$

$$B_2 = \mathbb{Z}^2_{\langle 1,2\rangle} \cup \mathbb{Z}^2_{\langle 2,3\rangle} \quad \text{(i.e. } b\text{'s)},$$

$$B_3 = \mathbb{Z}^3_{\langle 1,3\rangle} \cup \mathbb{Z}^3_{\langle 2,3\rangle} \quad \text{(i.e. } c\text{'s)}.$$

As in Section 6, we map each f-channel $\mathbb{Z}_{\langle i,j\rangle}$ onto $\mathbb{Z}$ by $h_{\langle i,j\rangle}$, then we have a two block partition of each compressed channel, consisting of

$$h_{\langle i,j\rangle}(\mathbb{Z}^{i}_{\langle i,j\rangle}) \quad \text{and} \quad h_{\langle i,j\rangle}(\mathbb{Z}^{j}_{\langle i,j\rangle}) = \mathbb{Z} - h_{\langle i,j\rangle}(\mathbb{Z}^{i}_{\langle i,j\rangle}),$$

for each $\langle i,j\rangle = \langle 1,2\rangle$, $\langle 1,3\rangle$ and $\langle 2,3\rangle$.

Next, if we use letters A, B and C for B_1, B_2 and B_3, respectively, and likewise for the results of appropriate $h_{\langle i,j\rangle}$ maps according to the f-channels in which the element are, then we have the following three compressed channels:

$$h_{\langle 1,2\rangle}(\mathbb{Z}_{\langle 1,2\rangle}): \quad \ldots \underline{AB}ABABABABABABABABABABABABABABA\ldots,$$

$$h_{\langle 1,3\rangle}(\mathbb{Z}_{\langle 1,3\rangle}): \quad \ldots \underline{ACC}ACCACCACCACCACCACCACCACCACCA\ldots,$$

$$h_{\langle 2,3\rangle}(\mathbb{Z}_{\langle 2,3\rangle}): \quad \ldots \underline{BCCCC}BCCCCBCCCCBCCCCBCCCCBCCCCB\ldots,$$

where the leftmost letters are at location 0. It is easy to see that the images of f-channels from the top down have the primitive periods of 2, 3 and 5, respectively, as shown by the underscored generator sequences. This is precisely what we planned to happen when we defined the minimal prime patterns. That is, in general we take ${}_mC_2$ copies of $\mathbb{Z}$, and partition each $\mathbb{Z}$ into a two block pattern having a unique primitive period of a prime number, using again a unique pair of letters out of m. Then we have a set of $h_{\langle i,j\rangle}(\mathbb{Z}_{\langle i,j\rangle})$.

In order to construct a minimal prime pattern by multiplexing these compressed channel patterns, now take the inverse maps $h^{-1}_{\langle i,j\rangle}(\mathbb{Z}_{\langle i,j\rangle})$, keeping the letter assignment to each integer fixed during the mapping. In our example this amounts to picking, from top to bottom, a letter from each displayed line above such that columns are strung up one after another from left to right. Namely, we pick the leftmost letter A of $h_{\langle 1,2\rangle}(\mathbb{Z}_{\langle 1,2\rangle})$ above for location $0 \in \mathbb{Z}$, then the leftmost letter A of $h_{\langle 1,3\rangle}(\mathbb{Z}_{\langle 1,3\rangle})$ above for location $0 \in \mathbb{Z}$, and so on, obtaining AAB, $BCC, \ldots$. The full string thus we would obtain is

$$\ldots AAB\ BCC\ ACC\ BAC\ ACC\ BCB\ AAC\ BCC\ ACC\ BAC$$

$$ACB\ BCC\ AAC\ BCC\ ACC\ BAB\ ACC\ BCC\ AAC\ BCC$$

$$ACB\ BAC\ ACC\ BCC\ AAC\ BCB\ ACC\ BAC\ ACC\ BCC\ldots$$

(For visual ease, the sequence is blocked into a unit of three letters. Note that after the above, the string of these 90 letters repeats.)

From the way the above example is constructed, it is clear that

$$\pi(B_1) = \pi(\mathbb{Z}^{1}_{\langle 1,2\rangle} \cup \mathbb{Z}^{1}_{\langle 1,3\rangle}) = f \cdot [\mathrm{pr}(\langle 1,2\rangle), \mathrm{pr}(\langle 1,3\rangle)]$$

$$= 3 \cdot 2 \cdot 3 = 18,$$

which is expected from Theorem 7.1(iv), and similarly

$$\pi(B_2) = 30, \qquad \pi(B_3) = 45,$$

and $\pi(\Pi)=[\pi_1, \pi_2, \pi_3]=[18, 30, 45]=90$, by Proposition 2.2, which is equal to ${}_3C_2 \cdot \mathrm{PP}(3)=3\cdot 2\cdot 3\cdot 5=90$, given by Theorem 7.1(i). It is easily verified that the reduced primitive period, when B_i and B_j are made into one block, is given, as stated in Theorem 7.1(v), by

$$\pi_{(1,2)}=90/2=45, \quad \pi_{(1,3)}=90/3=30 \quad \text{and} \quad \pi_{(2,3)}=90/5=18,$$

which may easily be verified by actually performing the reductions on the example sequence above, and showing that Π is indeed a prime pattern. Also, since $q_{ij}=\pi/\pi_{(i,j)}$, the factor with which the primitive period is reduced in the above, we may compute

$$\operatorname{lcm}\{q_{ij} \mid i\neq j\}=\operatorname{lcm}\{2, 3, 5\}=\mathrm{PP}(3)=30,$$

which checks with result stated as Theorem 7.1(iii).

9. Examples of nonminimal prime patterns

Examples of nonminimal prime patterns may also be readily constructed, similarly using primitive generator sequences for a given $R(k)$, $k\geqslant {}_mC_2$, and d ($\geqslant 1$) dilution sequences. We give below an example of d-diluted R-patterns, with some variations. Such variations, sometimes but not always valid in the construction of prime patterns, are

(i) the use of sequences, in compressed channels, other than those in the form $a_i(a_j)^u$, as the generator sequences of the d-diluted R-patterns.

(ii) the use of dilution sequences other than solid single letter sequences in d-diluted R-patterns, such as (a) duplicates from the generator sequences, or (b) other periodic sequences having the same in-channel primitive periods, or fraction thereof, and having the same letters, as generator sequences.

Example 9.1 (*Nonminimal prime pattern*). The following set of sequences for 5 compressed channels, of which two are generalized dilution sequences, will produce an example of general prime patterns.

$\mathbb{Z}_1'$: $(A\,B)$ (diluent of $\mathbb{Z}_5'$),

$\mathbb{Z}_2'$: $(C\,C\,C\,B\,B)$ (nonstandard generator sequence),

$\mathbb{Z}_3'$: $(B\,C\,B\,C\,C)$ (diluent of $\mathbb{Z}_2'$),

$\mathbb{Z}_4'$: $(A\,C\,C)$ (standard generator sequence),

$\mathbb{Z}_5'$: $(A\,B\,B\,B)$ (generator sequence with nonprime period).

The primitive period of the multiplexed pattern for this choice will be $(3+2)\cdot 3\cdot 4\cdot 5=300$, as can readily be verified.

Recall that, by Theorem 7.1(vii) and (viii), $\Pi_g = \Pi_f$ and $F_{ij} \in \Pi_g$ for minimal prime patterns. However, when a prime pattern is nonminimal, we have the following proposition.

Proposition 9.2. *There exists a (nonminimal) prime pattern* $\Pi^{(m)}$, $m \geqslant 3$, *such that* $\Pi_f = \Pi_g$, *and which possesses* F'_{ij} *such that* $F'_{ij} \notin \Pi_g$.

Proof. (i) For $m = 3$, take the above example. It is easy to see for this example that $\pi_{(A,B)} = \pi(B_C) = 75$, $\pi_{(B,C)} = \pi(B_A) = 60$ and $\pi_{(C,A)} = \pi(B_B) = 100$, and $g = 5$. From Proposition 5.4, $f \mid \pi_{(i,j)}$ for all of these. Hence $f = 5$, and $\Pi_f = \Pi_g$. Furthermore, from Proposition 5.5, F_{ij}'s may consist only of the elements of Π_f. There are two possible such aggregations, namely $\mathbb{Z}_{AB} = \mathbb{Z}'_1 \cup \mathbb{Z}'_5$ and $\mathbb{Z}_{BC} = \mathbb{Z}'_2 \cup \mathbb{Z}'_3$. It is readily checked that they indeed satisfy the requirement for frames, hence $F_{AB} = \mathbb{Z}_{AB}$, and $F_{BC} = \mathbb{Z}_{BC}$.

(ii) For $m > 3$, an example is similarly constructed by an appropriate dilution of a minimal prime pattern. Then a similar proof follows. □

Example 9.3. Take six compressed channels, $\mathbb{Z}'_1$ through $\mathbb{Z}'_6$, and assign them the following six generator sequences in that order; $G_1 = AB$, $G_2 = ACC$, $G_3 = BCCCC$, $G_4 = B$, $G_5 = C$ and $G_6 = C$. Then we have a 3-diluted $R(3)$-pattern.

It is easy to see that the resulting pattern has primitive block periods $\pi_A = 36$, $\pi_B = 60$, and $\pi_C = 90$, and $g = (\pi_A, \pi_B, \pi_C) = 6$, hence $\Pi_g = (\mathbb{Z}'_1, \ldots, \mathbb{Z}'_6)$. On the other hand, let $\mathbb{Z}''_1 = \mathbb{Z}'_1 \cup \mathbb{Z}'_4$, $\mathbb{Z}''_2 = \mathbb{Z}'_2 \cup \mathbb{Z}'_5$, $\mathbb{Z}''_3 = \mathbb{Z}'_3 \cup \mathbb{Z}'_6$, then $(\mathbb{Z}''_1, \mathbb{Z}''_2, \mathbb{Z}''_3)$ may be viewed as a basic pattern having generator sequences $G'_1 = AB^3$, $G'_2 = AC^5$ and $G'_3 = BC^9$, which no longer appears to be a d-diluted R-pattern, although it is still the same pattern. Now, it can be readily shown that $(\mathbb{Z}''_1, \mathbb{Z}''_2, \mathbb{Z}''_3)$ indeed constitutes Π_f hence $f = 3$, and $f < g$ and $f \mid g$.

Proposition 9.4. *With a nonminimal prime pattern, it is possible to have a case such that* $\Pi_f \neq \Pi_g$ *and* $F'_{ij} \in \Pi_f$.

Proof, For $\Pi^{(3)}$, take the above example. Then from Corollary 5.8, $f \geqslant {}_3C_2 = 3$. Hence $(\mathbb{Z}''_1, \mathbb{Z}''_2, \mathbb{Z}''_3)$ is a candidate for Π_f. It can readily be checked that this set indeed satisfies the requirement for Π_f. For $\Pi^{(m)}$, $m > 3$, a similar example can be constructed, for which a similar proof procedure is applicable. □

Combining the features of the above two examples gives the following example.

Example 9.5. Let $G_1 = AB$, $G_2 = ACC$, $G_3 = BCCCC$, $G_4 = AB$, $G_5 = B$, $G_6 = C$, $G_7 = C$ and $G_8 = B$. Then we have a variant of 5-diluted $R(3)$-patterns where diluent G_4 is a duplicate of G_1.

This pattern is prime with the primitive period of $(3+5)\cdot 2\cdot 3\cdot 5=240$. Furthermore, $\pi_A=48$, $\pi_B=80$, $\pi_C=120$; $\pi_{(A,B)}=120$, $\pi_{(B,C)}=48$, $\pi_{(C,A)}=80$, and $g=8$. But $\mathbb{Z}_1''=\mathbb{Z}_1'\cup\mathbb{Z}_5'$, $\mathbb{Z}_2''=\mathbb{Z}_2'\cup\mathbb{Z}_6'$, $\mathbb{Z}_3''=\mathbb{Z}_3'\cup\mathbb{Z}_7'$, and $\mathbb{Z}_4''=\mathbb{Z}_4'\cup\mathbb{Z}_8'$ will give us $G_1'=AB^3$, $G_2'=AC^5$, $G_3'=BC^9$, $G_4'=AB^3$, and $\pi(\mathbb{Z}_1'')=16$, $\pi(\mathbb{Z}_2'')=24$, $\pi(\mathbb{Z}_3'')=40$, $\pi(\mathbb{Z}_4'')=16$.

Proposition 9.6. *With a nonminimal prime pattern, it is possible to have a case such that* $\Pi_f\neq\Pi_g$ *and* $F'_{ij}\notin\Pi_f$.

Proof. For $\Pi^{(3)}$, take the above example. Then, from Corollary 5.8, $f\geq 3$, and from Proposition 5.4, $f\mid\pi_{(i,j)}$, hence f is the smallest realizable one of either 4 or 8. But $(\mathbb{Z}_1'',\mathbb{Z}_2'',\mathbb{Z}_3'',\mathbb{Z}_4'')$ above satisfies the requirement for being Π_f, hence $f=4$. But it is easy to see that $F_{AB}=\mathbb{Z}_1''\cup\mathbb{Z}_4''\notin\Pi_f$. For $\Pi^{(m)}$, $m>3$, a similar proof is constructed with a similar and appropriate example. □

The following example is *not* a prime pattern.

Example 9.7. To six compressed channels $\mathbb{Z}_1'$ through $\mathbb{Z}_6'$, assign the following six sequences in that order; $G_1=AB$, $G_2=ACC$, $G_3=BCCCC$, $G_4=D$, $G_5=CAC$ and $G_6=CCBCC$, where each sequence contains at most two distinct letters.

The multiplexed pattern of this example is not a basic pattern, and it can be readily checked that it is not even a prime pattern. Also, $\pi_A=36$, $\pi_B=60$, $\pi_C=45$, $\pi_D=6$, hence $g=(36,60,45,6)=3$. It follows then that $\Pi_g=(\mathbb{Z}_1'',\mathbb{Z}_2'',\mathbb{Z}_3'')$ and a possible assignment of generator sequences for this Π_g is $G_1'=ADBD$, $G_2'=ACC$ and $G_3'=BCCCC$, hence $\mathbb{Z}_1''$ contains elements from three blocks of $\Pi^{(4)}$. Note that the generator sequences contain two pairs of the conjugate words in the example.

Now, Corollary 3.7 states that a g-channel of *any* 3-block pattern contains elements from at most two blocks of the pattern. Proposition 5.5(ii) states that an f-channel of an m block prime pattern contains elements also from at most two blocks of the pattern. With a proper extension of the f-channel concept to nonprime patterns, this may still hold. In the meantime, the following corollary holds.

Corollary 9.8. *When* $m>3$, *a g-channel of an m block pattern* $\Pi^{(m)}$ *may contain elements from more than two blocks of* $\Pi^{(m)}$.

We close this section by giving an example from those prime patterns whose primitive period reduces when the f-channels are permuted according to Corollary 6.8.

Example 9.9. The pattern consists of 90 compressed channels, the first, the second and the third, 30 of which contain AB-generated, ACC-generated and $BCCCC$-generated sequences, respectively.

Clearly it is a prime pattern, having the primitive period of 900. Now, permute these channels such that $(3z+1)$st, $(3z+2)$nd, and $(3z+3)$rd compressed channels, $0 \le z \le 29$, have the AB-generated, ACC-generated and $BCCCC$-generated sequences, respectively. Then we will have the minimal prime pattern of Example 8.1, having the primitive period of 90.

Corollary 9.10. *When a prime pattern is not basic, the permutation of* Π_f *does not always preserve the primitive period.*

10. Concluding remarks

The mapping we have studied in this report is the transition behavior among the periodic configurations of one-dimensional cellular automata, when

(i) cellular automata are of the (degenerate) kind such that the next state of a cell is dependent only on the present state of the cell itself as its neighbourhood,

(ii) the transition we observe is of only one step, and

(iii) the number of distinct states appearing in a configuration is required to reduce by one.

When such is the case, we have found that, among all periodic configurations, we can always find those whose primitive periods would reduce after the transition, no matter which next state function for all cells are chosen. We also have indicated some properties of such period-reducing prime sequences.

As for the areas of the further research, even within the study of periodic sequences in conjunction with the cellular automata having a single neighbourhood as we have reported in the present article, there are still many detailed questions unanswered, such as the fine structures of periodic sequences and channels, one-step reductions into even smaller alphabet than A_{m-1}, etc. In addition, it is our opinion that the following areas of further research among others should prove to be of interest:

(i) the cellular automata with single neighbourhood cells treated in this report are of a very much restricted kind, and it is natural to ask the transition behavior of those cellular automata whose cell neighbourhood consists of more than one cell [5]. We shall also see there that the study reported herein is fundamental to such an investigation.

(ii) By removing the restriction of having only one-step operations, we may consider the possibility of first transforming a given periodic sequence to another of the same primitive period, still containing the same number of distinct letters of the alphabet, by the application of a sequence of appropriate maps, then reducing the number of letters in the resultant periodic sequence by a map while keeping the primitive period as we have done in the present report. This involves a neighbourhood which is of the size larger than 1, as well as a problem of the connectivity among the periodic configurations of cellular automata under the transformation by the next state functions.

(iii) The third possible research problem is a special and autonomous case of the second in that the maps successively applied are all identical and having the same codomain as the domain. The application of the map is terminated when the desired result is obtained.

Acknowledgment

The authors are greatly indebted to Mr. C. Choffrut and an anonymous referee, who carefully read our earlier manuscript and made extensive and valuable comments and suggestions, which are all incorporated into the present version.

References

[1] D.L. Hayes, A private communication, The Moore School of Electrical Engineering, University of Pennsylvania, 1971.
[2] J. Hunter, *Number Theory* (Oliver & Boyd, Edinburgh, 1964).
[3] M. Imori and H. Yamada, Transformations of periodic sequences by cellular automata, Tech. Rept. 74-02, Information Science Laboratory, Faculty of Science, University of Tokyo, 1974.
[4] A. Lentin and M.P. Schützenberger, A combinatorial problem in the theory of free monoids, in: R.C. Bose and T.A. Dowling, eds., *Combinatorial Mathematics and Applications* (Univ. of North Carolina Press, 1969).
[5] H. Yamada and M. Imori, One-step transformation of periodic sequences by cellular automata, *SIAM J. Comput.* **12** (1983) 539–510.