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Theoretical Computer Science 313 (2004) 417–425

Theoretical Computer Science

www.elsevier.com/locate/tcs

Periodicity in one-dimensional peg duotaire

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Received 5 July 2002; accepted 28 November 2002

Abstract

We consider the single-hop version of one-dimensional peg duotaire, a two player version of peg solitaire in which players move alternatively and the last player to move wins. We determine the nim-values of all positions consisting of two sets of consecutive pegs separated by a hole. We show that two classes of positions produce periodic sequences of nim-values, and we conjecture that two other classes exhibit similar periodicity.

Keywords: Combinatorial game theory; Grundy values; Peg solitaire

1. Introduction

One-dimensional peg duotaire is a two-player version of a one-dimensional version of peg solitaire. The game is played on a row of holes that can hold pegs. Each move consists of jumping one peg over another adjacent peg and landing in an empty hole; the peg that was jumped over is removed. In the solitaire version, the goal is to find a sequence of moves which reduces the initial position to a single peg. In Ref. [2] it was shown that the set of solvable positions is a regular language. In the two-player version (peg duotaire), originally proposed in Ref. [3], the winner is the last player to move. In [2] two variations of duotaire were considered: multi-hop duotaire, in which multiple jumps can be made with the same peg on a given turn, and single-hop duotaire, in which a turn consists of exactly one jump. In this paper we focus exclusively on the single-hop version of the game.

Since peg duotaire is an impartial game, its positions are completely characterized by their Grundy values. The Grundy value (or nim-value) $G(X)$ of a position X is the smallest non-negative integer not appearing in the nim-values of its options [1]. Fig. 1 gives the nim-values of some simple positions.

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0304-3975/$ - see front matter
doi:10.1016/j.tcs.2002.11.003

$$G(\circ) = G(\bullet) = 0 \qquad G(\bullet\bullet) = G(\bullet\bullet\bullet) = 1$$
$$G(\bullet\circ\bullet\bullet) = 2 \qquad G(\bullet^n) = \lfloor n/2 \rfloor \pmod 2$$

Fig. 1. Some easy-to-compute nim-values in peg duotaire ($\bullet$ = peg, $\circ$ = hole).

In [2] it is conjectured that there are peg duotaire positions with arbitrarily large nim-values. To investigate this claim, we considered sequences of positions of the form XP^n, where X and P are duotaire positions, hoping to find an increasing sequence of nim-values. Instead, for many choices of X and P that we considered we found the sequence $G(XP^n)$ to be eventually periodic. In this paper we present some of our results concerning this periodicity. In Section 2, we begin by determining the value $G(\bullet^m \circ \bullet^n)$ of for arbitrary m, n. In Section 3 we show that for two simple positions P, $G(XP^n)$ is eventually periodic for any X. We conjecture that the same is true for two other positions P; our conjectures have been verified by computer for all positions X of length at most 8.

2. $G(\bullet^m \circ \bullet^n)$

We begin with two simple lemmas.

Lemma 1. *In the position $X \circ\circ$, no sequence of moves can place a peg in the second hole.*

Proof. The proof is by induction on the number of pegs in X; the base cases with zero pegs or one pegs are trivial. The only way to place a peg in the second hole is to jump over a peg in the first hole, and the only way to place a peg in the first hole is jump there from X (not necessarily on the first move). After this jump the position will be

$$Y \circ\circ\bullet\circ$$

for some position Y with fewer pegs than X. By induction no sequence of moves can place a peg in the second hole to the right of Y, hence we can never jump to the rightmost hole. □

Lemma 2. *$G(X \circ\circ\circ Y) = G(X) \oplus G(Y)$ for any positions X, Y, where $\oplus$ denotes nim-addition.*

Proof. By Lemma 1, neither X nor Y can place a peg in the middle hole. It follows that this hole will always be empty, so the sub-games X and Y will never interact. □

A slightly stronger version of these lemmas appears in [2], but the current formulation is sufficient for our purposes. The following theorem provides a stepping stone to our main result. In the statement of the theorem and all that follows, we write $X \equiv Y$ to denote $G(X) = G(Y)$ for positions X, Y.

Theorem 1. *For $a, b, c, d \geqslant 0$, let $P_{a,b,c,d} = \bullet^{2a} \circ (\circ\bullet)^b (\bullet\circ)^c \circ \bullet^{2d}$. Then for $a \geqslant 2$, $P_{a+2,b,c,d} \equiv P_{a,b,c,d}$, and similarly for b, c, d.*

Proof. Our proof is by simultaneous induction in all four indices on the number of pegs. The base case is $a, b, c, d \leqslant 4$, which we have verified by computer. From the position $P_{a,b,c,d}$ there are at most six moves: jump left/right from the first group of $2a$ stones, jump left/right from the last group of $2d$ stones, or jump left/right from the two stones in the center. Thus, $P_{a,b,c,d}$ has the following options:

$$\begin{aligned} &P_{a-1,b,c,d},\ P_{a-1,b+1,c,d},\ P_{a,b,c,d-1},\ P_{a,b,c+1,d-1}, \\ &\quad P_{a,b-1,1,0} \oplus P_{0,0,c-1,d},\ P_{a,b-1,0,0} \oplus P_{0,1,c-1,d}, \end{aligned} \tag{1}$$

where an option is valid only if its indices are non-negative. For the last two terms we have used Lemma 2, which tells us that a jump in either direction from the center splits the position into the disjunctive sum of two sub-positions. Now suppose that $a \geqslant 2$ and consider $P_{a+2,b,c,d}$. Replacing a with $a+2$ in (1) gives us its options

$$\begin{aligned} &P_{a+1,b,c,d},\ P_{a+1,b+1,c,d},\ P_{a+2,b,c,d-1},\ P_{a+2,b,c+1,d-1},\ P_{a+2,b-1,1,0} \oplus P_{0,0,c-1,d}, \\ &\quad P_{a+2,b-1,0,0} \oplus P_{0,1,c-1,d}. \end{aligned} \tag{2}$$

The base case is $a, b, c, d \leqslant 4$, so we can assume that one of a, b, c, d is greater than 4. Suppose first that $a > 4$. Then by induction each term of (2) is equal to the corresponding term of (1), so $P_{a+2,b,c,d}$ and $P_{a,b,c,d}$ have the same options and hence the same nim-values. Next suppose that $b > 4$. Then $b - 3 \geqslant 2$, so once again we can apply induction to the terms of (2), this time in the second index, and we find that $P_{a+2,b,c,d} \equiv P_{a+2,b-2,c,d}$. But $P_{a+2,b-2,c,d}$ has fewer pegs than $P_{a+2,b,c,d}$, so by induction $P_{a+2,b-2,c,d} \equiv P_{a,b-2,c,d} \equiv P_{a,b,c,d}$, hence $P_{a+2,b,c,d} \equiv P_{a,b,c,d}$. The cases $c > 4$ and $d > 4$ are identical, so in all cases $P_{a+2,b,c,d} \equiv P_{a,b,c,d}$. Finally, the arguments for the other three indices are the same as the argument for a, so our induction is complete. $\square$

From Theorem 1 it follows that to find the nim-value of $P_{a,b,c,d}$ for arbitrary a, b, c, d it suffices to generate the values for $a, b, c, d \leqslant 3$. Note that an easy induction on the number of pegs gives us

$$P_{a,b,c,d} \equiv \bullet P_{a,b,c,d} \equiv P_{a,b,c,d} \bullet \equiv \bullet P_{a,b,c,d} \bullet . \tag{3}$$

We are now ready to state our main result

Theorem 2. *For $m, n \geqslant 0$, let $A_{m,n} = \bullet^m \circ \bullet^n$. Then for $m \geqslant 7$, $A_{m+4,n} \equiv A_{m,n}$.*

Proof. Once again our proof is by induction on the number of pegs. This time the base case is $m, n \leqslant 12$, which we have again verified by computer. The options of

Table 1
Nim-values of $\bullet^m \circ \bullet^n$

$A_{m;n}$	0	1	2	3	4	5	6	7	8	9	10
0	0	0	1	1	0	0	1	1	0	0	1
1	0	0	2	2	1	1	0	0	1	1	0
2	1	2	0	3	1	2	0	3	1	2	0
3	1	2	3	1	4	0	2	1	3	0	2
4	0	1	1	4	0	0	1	1	0	0	1
5	0	1	2	0	0	1	4	0	1	1	0
6	1	0	0	2	1	4	0	2	1	3	0
7	1	0	3	1	1	0	2	1	3	0	2
8	0	1	1	3	0	1	1	3	0	2	1
9	0	1	2	0	0	1	3	0	2	1	3
10	1	0	0	2	1	0	0	2	1	3	0

The periodic region is highlighted.

$A_{m,n}$ are

$$A_{m-2,n}, A_{m,n-2}, P_{\lfloor (m+1)/2 \rfloor 0,0,\lfloor (n-2)/2 \rfloor}, P_{\lfloor (m-2)/2 \rfloor 0,0,\lfloor (n+1)/2 \rfloor}, \tag{4}$$

where the floor functions of the third and fourth terms are justified by observation (3) above. Now suppose that $m \geqslant 7$ and consider $A_{m+4,n}$. Its options are

$$A_{m+2,n}, A_{m+4,n-2}, P_{\lfloor (m+1)/2 \rfloor +2,0,0,\lfloor (n-2)/2 \rfloor}, P_{\lfloor (m-2)/2 \rfloor +2,0,0,\lfloor (n+1)/2 \rfloor}. \tag{5}$$

The base case is $m, n \leqslant 12$, so we can assume that one of m, n is greater than 12. First suppose that $m > 12$. Then by induction the first two terms of (5) are equal to the first two terms of (4). Furthermore, since $\lfloor (m-2)/2 \rfloor \geqslant 5 > 2$, we can apply Theorem 1 to the third and fourth terms of (5) to show that they are equal to the corresponding terms of (4). Thus, $A_{m+4,n} \equiv A_{m,n}$. Next suppose that $n > 12$. Then $n - 6 \geqslant 7$ and $\lfloor (n-2)/2 \rfloor - 2 \geqslant 3 > 2$, so we can apply induction and Theorem 1 to the terms of (5) in the 'n' indices to find that $A_{m+4,n} \equiv A_{m+4,n-4}$. Finally, $A_{m+4,n-4}$ has fewer pegs, so by induction $A_{m+4,n-4} \equiv A_{m,n-4} \equiv A_{m,n}$. Thus, in both cases $A_{m+4,n} \equiv A_{m,n}$ so our induction is complete. □

Table 1 gives the nim-values of $A_{m,n}$ up to the first period in both m and n.

3. Periodic sequences of nim-values

Given two duotaire positions X and P, we can define a sequence of nim-values $g_n = G(XP^n)$. Our motivation for considering these sequences was to try to find an increasing sequence of nim-values; instead what we found was a collection of eventually periodic sequences of nim-values. In this section we consider four different positions P. For the first two, we show that g_n is eventually periodic for any position X. We conjecture that the same is true for the other two positions as well.

3.1. $P = \circ\bullet$

Theorem 3. *For any position X, we have $X\circ\bullet\circ\bullet \equiv X\circ\bullet\circ\bullet\circ\bullet\circ\bullet$.*

Proof. It suffices to show that the second player can win the in game $X\circ\bullet\circ\bullet \oplus X\circ\bullet\circ\bullet\circ\bullet\circ\bullet$. The winning strategy is to play symmetrically; when the first player moves in one copy of X, the second player makes the same move in the other copy. If the first player never jumps into the space to the right of X then the second player can always copy the first player's move and therefore wins. Otherwise, after this jump is made and mimicked the position will be

$$X'\circ\circ\bullet\bullet\circ\bullet \oplus X'\circ\circ\bullet\bullet\circ\bullet\circ\bullet\circ\bullet.$$

By Lemma 1, X' can never place a peg in the second hole, so we can rewrite this as

$$Y|\circ\bullet\bullet\circ\bullet \oplus Y|\circ\bullet\bullet\circ\bullet\circ\bullet\circ\bullet,$$

where we use the notation $Y|\circ$ to denote that the position Y can never place a peg in the hole to its right. The second player continues to copy the first player's moves, and at some point the first player must, in one of the sub-games, jump from the peg pair to the right of Y. If the jump is to the right then after the corresponding move in the other sub-game the position is

$$Z|\circ\circ\circ\bullet\bullet \oplus Z|\circ\circ\circ\bullet\bullet\circ\bullet\circ\bullet = Z \oplus \bullet\bullet \oplus Z \oplus \bullet\bullet\circ\bullet\circ\bullet \equiv \bullet\bullet \oplus \bullet\bullet\circ\bullet\circ\bullet.$$

Here we have used Lemma 2. But $G(\bullet\bullet) \equiv G(\bullet\bullet\circ\bullet\circ\bullet) = 1$, so $\bullet\bullet \oplus \bullet\bullet\circ\bullet\circ\bullet \equiv 0$ and the second player wins. On the other hand if, the jump is to the left, then after the corresponding move the position is

$$Z\bullet\circ\circ\circ\bullet \oplus Z\bullet\circ\circ\circ\bullet\circ\bullet\circ\bullet \equiv Z\bullet \oplus Z\bullet \equiv 0$$

so in all cases the second player wins. $\square$

From Theorem 3 it follows that $g_n = G(XP^n)$ is periodic with period 2.

3.2. $P = \bullet$

This position is trickier to work with than the previous one. We will make heavy use of the following:

Lemma 3. *Let $X_n, n \geqslant 0$, be a sequence of positions in an impartial game such that X_n is an option of X_{n+1} for all n. Let $\mathcal{G}_n = \{G(Y) \mid Y$ is an option of $X_n, Y \neq X_{n-1}\}$. If $\mathcal{G}_n$ is eventually periodic with period 2, then so is $G(X_n)$.*

Proof. Let $g_n = G(X_n)$ and suppose that $\mathcal{G}_n$ is eventually periodic with period 2. Then for large enough n we have $g_{2n} = \operatorname{mex}(S \cup \{g_{2n-1}\})$, where $S = \mathcal{G}_{2n}$ is a fixed set and $\operatorname{mex}(T)$ is the smallest non-negative integer not in the set T. It follows that if $g_{2n-1} \neq \operatorname{mex}(S)$ then $g_{2n} = \operatorname{mex}(S)$, otherwise $g_{2n} = \operatorname{mex}(S \cup \{\operatorname{mex}(S)\})$. Thus, for large n, the sequence $\{g_{2n}\}$ takes on at most two distinct values, so it either alternates between $\text{mex}(S)$ and $\text{mex}(S \cup \{\text{mex}(S)\})$ or contains two consecutive terms which are the same. The former case is impossible, for if $g_{2n} = \text{mex}(S)$ and $g_{2n+2} = \text{mex}(S \cup \{\text{mex}(S)\})$ then by the previous remarks $g_{2n+1} = \text{mex}(S) = g_{2n}$ which is impossible since X_{2n} is an option of X_{2n+1}. Thus $g_{2n} = g_{2n+2}$ for some n large enough such that $\mathscr{G}_n$ is periodic. But then

$$g_{2n+3} = \text{mex}(\mathscr{G}_{2n+3} \cup \{g_{2n+2}\}) = \text{mex}(\mathscr{G}_{2n+1} \cup \{g_{2n}\}) = g_{2n+1}$$

and so inductively $g_{m+2} = g_m$ for all $m \geqslant 2n$. Thus, g_n is eventually periodic with period 2. $\square$

Theorem 4. *For any position X, $G(X\bullet^n)$ is eventually periodic with period* 4.

Proof. We use induction on the number of pegs in X; the nature of our argument does not require the base cases to be considered separately. The options of $X\bullet^n$ are $X'\bullet^n$ (where X' is an option of X), $X\bullet^{n-2}$, and if X is of the form $Y\circ$ then also $Y\bullet\circ\circ\bullet^{n-2}$. By induction, $G(X'\bullet^n)$ is eventually periodic for each option X' of X. If we can show that $G(Y\bullet\circ\circ\bullet^n)$ is eventually periodic with period 4, then we can apply Lemma 3 separately to the odd and even terms of $\{X\bullet^n\}$ to show that $G(X\bullet^n)$ is also eventually periodic. Thus, it suffices to show that $G(X\circ\circ\bullet^n)$ is eventually periodic with period 4 for any position X.

We again use induction on the number of pegs in X. The options of $X\circ\circ\bullet^n$ are $X'\circ\circ\bullet^n$, $X\circ\circ\bullet^{n-2}$, $X\circ\bullet\circ\circ\bullet^{n-2}$, and if X is of the form $Y\bullet\bullet$ then also $Y\circ\circ\bullet\circ\bullet^n$. Repeating the previous argument, it suffices to show that $G(X\circ\bullet\circ\circ\bullet^n)$ and $G(X\circ\circ\bullet\circ\bullet^n)$ are eventually periodic for any position X. Equivalently, using Lemma 1, it suffices to show that $G(X\circ\bullet\circ\circ\bullet^n)$ and $G(X|\circ\bullet\circ\bullet^n)$ are eventually periodic, where as before we use the notation $X|\circ$ to denote a position that can never place a peg in the hole to its right.

Once again we use induction. The options of $X\circ\bullet\circ\circ\bullet^n$ are $X'\circ\bullet\circ\circ\bullet^n$, $X\circ\bullet\circ\circ\bullet^{n-2}$, $X\circ\bullet\circ\bullet\circ\circ\bullet^{n-2}$, and, if X is of the form $Y\bullet\bullet$, then also $Y\circ\circ\bullet\bullet\circ\circ\bullet^n$. The options of $X|\circ\bullet\circ\bullet^n$ are $X'|\circ\bullet\circ\bullet^n$, $X|\circ\bullet\circ\bullet^{n-2}$, and $X|\circ\bullet\bullet\circ\circ\bullet^{n-2}$. Thus, again using Lemma 1, it suffices to show that $G(X\circ\bullet\circ\bullet\circ\circ\bullet^n)$ and $G(X|\circ\bullet\bullet\circ\circ\bullet^n)$ are eventually periodic.

We use induction yet again. The options of $X\circ\bullet\circ\bullet\circ\circ\bullet^n$ are $X'\circ\bullet\circ\bullet\circ\circ\bullet^n$, $X\circ\bullet\circ\bullet\circ\circ\bullet^{n-2}$, $X\circ\bullet\circ\bullet\circ\bullet\circ\circ\bullet^{n-2}$, and if X is of the form $Y\bullet\bullet$ then also $Y\circ\circ\bullet\bullet\circ\bullet\circ\circ\bullet^n$. The options of $X|\circ\bullet\bullet\circ\circ\bullet^n$ are $X'|\circ\bullet\bullet\circ\circ\bullet^n$, $X|\circ\bullet\bullet\circ\circ\bullet^{n-2}$, $X|\circ\bullet\bullet\circ\bullet\circ\circ\bullet^{n-2}$, $X\bullet\oplus\bullet^n$ (which is periodic), and $X\oplus\bullet\circ\bullet^n$ (which is also periodic since $\bullet\circ\bullet^n = A_{1,n}$). Thus it suffices to show that $G(X\circ\bullet\circ\bullet\circ\bullet\circ\circ\bullet^n)$ and $G(X|\circ\bullet\bullet\circ\bullet\circ\circ\bullet^n)$ are eventually periodic.

We begin with $X|\circ\bullet\bullet\circ\bullet\circ\circ\bullet^n$, using induction once again. The options of $X|\circ\bullet\bullet\circ\bullet\circ\circ\bullet^n$ are $X'|\circ\bullet\bullet\circ\bullet\circ\circ\bullet^n$, $X|\circ\bullet\bullet\circ\bullet\circ\circ\bullet^{n-2}$, $X|\circ\bullet\bullet\circ\bullet\circ\bullet\circ\circ\bullet^{n-2}$, $X\bullet\oplus\bullet^n$ (which is periodic), and $X\oplus\bullet\bullet\circ\circ\bullet^n$ (which is also periodic since $\bullet\bullet\circ\circ\bullet^n = P_{1,0,0,\lfloor n/2\rfloor}$). Thus, it suffices to show that $G(X|\circ\bullet\bullet\circ\bullet\circ\bullet\circ\circ\bullet^n)$ is eventually periodic. We can in fact prove directly and more generally that $X|\circ\bullet(\bullet\circ)^m\circ\bullet^{n+4} \equiv X|\circ\bullet(\bullet\circ)^m\circ\bullet^n$ for

$m \geqslant 3$ by showing that the second player has a winning strategy in the game

$$X|\circ\bullet(\bullet\circ)^m\circ\bullet^{n+4}\oplus X|\circ\bullet(\bullet\circ)^m\circ\bullet^n. \tag{6}$$

The strategy is to copy the first player's moves, which keeps the game in the same form, until one of two events occurs: either the first player makes a move that cannot be copied, or the first player jumps in one of the games from the peg pair to the right of X. If the first event occurs, then at that point the position has $n=0$ or 1 and the move is a jump from the group of $n+4$ pegs. In this case the second player jumps in the same direction from the remaining $n+2$ pegs. If the jump was to the right, then the resulting position is $X|\circ\bullet(\bullet\circ)^m\oplus X|\circ\bullet(\bullet\circ)^m\equiv 0$. If the jump was to the left, then the resulting position is $X|\circ\bullet(\bullet\circ)^{m+2}\oplus X|\circ\bullet(\bullet\circ)^m$ which, since $m\geqslant 3$, is also zero by Theorem 3, so in both cases the second player wins. If the second event (a jump from the peg pair to the right of X) occurs, then the second player copies the jump. If the jump was to the left then the resulting position is $X\bullet\oplus\bullet^{n+4}\oplus X\bullet\oplus\bullet^n\equiv 0$ (since $\bullet^{n+4}\equiv\bullet^n$), and if the jump was to the right then the resulting position is

$$X\oplus\bullet(\bullet\circ)^{m-1}\circ\bullet^{n+4}\oplus X\oplus\bullet(\bullet\circ)^{m-1}\circ\bullet\equiv P_{0,1,m-1,\lfloor n/2\rfloor+2}\oplus P_{0,1,m-1,\lfloor n/2\rfloor}.$$

But we can verify that $P_{0,1,c,d+2}\equiv P_{0,1,c,d}$ for $c=2,3$ and $d=0,1$. From Theorem 1 it follows that $P_{0,1,c,d+2}\equiv P_{0,1,c,d}$ for all d when $c=2$ or 3, and therefore (applying Theorem 1 again in the 'c' index) the same is true for all $c\geqslant 2$. Thus $P_{0,1,m-1,\lfloor n/2\rfloor+2}\oplus P_{0,1,m-1,\lfloor n/2\rfloor}\equiv 0$ (since $m\geqslant 3$), so in all cases the second player wins.

Finally, it remains to show that $G(X\circ\bullet\circ\bullet\circ\bullet\circ\circ\bullet^n)$ is eventually periodic. We prove directly and more generally that $X\circ(\bullet\circ)^m\circ\bullet^{n+4}\equiv X\circ(\bullet\circ)^m\circ\bullet^n$ for $m\geqslant 3$ by showing that the second player has a winning strategy in the game $X\circ(\bullet\circ)^m\circ\bullet^{n+4}\oplus X\circ(\bullet\circ)^m\circ\bullet^n$. Again, the strategy is to copy the first player's moves until one of two events occurs: either the first player makes a move that cannot be copied, or the first player jumps into the hole to the right of X. In the first case, the same argument used for game (6) shows that the second player wins. In the second case, after the second player copies the move the position is of the form (6) so again the second player wins. Thus in all cases the second player wins, and we have finally managed to tie up all the loose ends and complete the proof of Theorem 4. □

3.3. Conjectures

We conjecture, but are unable at this time to prove, that a statement similar to that of Theorem 4 can be made for $P=\circ\circ\bullet\bullet$ and $P=\circ\circ\bullet$. Specifically:

Conjecture 1. *For any position* X, $G(X(\circ\circ\bullet\bullet)^n)$ *is eventually periodic with period* 8.

Conjecture 2. *For any position* X, $G(X(\circ\bullet\bullet)^n)$ *is eventually periodic with period* 17. (!)

For specific positions X, we can verify by computer that the sequences are periodic as follows:

Lemma 4. *Let* $A_n=(\circ\bullet\bullet\circ)^n$, $B_n=A_n\bullet$, $C_n=\bullet A_n\bullet$. *Then* $A_n\equiv A_{n-8}$ *for* $n\geqslant 10$, *and similarly for* B_n, C_n.

Proof. The options of A_n are $A_k\oplus B_{n-k-1}$, the options of B_n are $B_k\oplus B_{n-k-1}$ and $A_k\oplus C_{n-k-1}$, and the options of C_n are $C_k\oplus B_{n-k-1}$ ($0\leqslant k\leqslant n-1$ in all cases). We can verify the claim by hand for $n<20$. If $n\geqslant 20$, then for $0\leqslant k\leqslant n-1$ one of $(k,n-k-1)$ must be $\geqslant 10$. We can therefore apply induction to either the left or right summand of each option of A_n to show that A_n and A_{n-8} have the same options and hence $A_n\equiv A_{n-8}$. The proofs for B_n, C_n are the same. □

Theorem 5. *Let* $X|$ *be a position from which we can never jump to the right past* $|$. *If* N *is a positive integer such that*

(1) $X'|A_n\equiv X'|A_{n-8}$ *and* $X'|B_n\equiv X'|B_{n-8}$ *for* $n\geqslant N$ *and for all options* X' *of* X,

(2) $X|A_n\equiv X|A_{n-8}$ *and* $X|B_n\equiv X|B_{n-8}$ *for* $N-10\leqslant n<N$

then $X|A_n\equiv X|A_{n-8}$ *and* $X|B_n\equiv X|B_{n-8}$ *for all* $n\geqslant N$.

Proof. Suppose that N is as described and $n\geqslant N$. The options of $X|A_n$ are $X'|A_n$, $X|A_k\oplus B_{n-k-1}$, $X|B_k\oplus A_{n-k-1}$ and the options of $X|B_n$ are $X'|B_n$, $X|A_k\oplus C_{n-k-1}$, $X|B_k\oplus B_{n-k-1}$. As k varies from 0 to $n-1$, we always have either $n-k-1\geqslant 10$ or $k\geqslant n-10\geqslant N-10$. We can therefore use induction for each option to show that the options of $X|A_n$ are the same as the options of $X|A_{n-8}$, hence $X|A_n\equiv X|A_{n-8}$, and similarly $X|B_n\equiv X|B_{n-8}$. □

Note that $X(\circ\circ\bullet\bullet)^n$ can be rewritten as $X|(\circ\bullet\bullet\circ)^n$, thus Theorem 5 allows us to inductively check the periodicity of $G(X(\circ\circ\bullet\bullet)^n)$ by computer. The corresponding lemma and theorems for $X(\circ\circ\bullet)^n$ are similar, and the proofs are virtually identical (though more involved) so we leave them as exercises for the reader.

Lemma 5. *Let* $P_n=(\circ\bullet\bullet)^n$, $Q_n=(\circ\bullet\bullet)^n\bullet$, $R_n=(\circ\bullet\bullet)^n\circ\bullet$, $S_n=\bullet(\circ\bullet\bullet)^n\bullet$, $T_n=\bullet(\circ\bullet\bullet)^n\circ\bullet$, $U_n=\bullet\bullet\bullet(\circ\bullet\bullet)^n\bullet$. *Then* $P_n\equiv P_{n-17}$ *for* $n\geqslant 69$, *and similarly for* Q_n, R_n, S_n, T_n, U_n.

Theorem 6. *Let* $X|$ *be a position from which we can never jump to the right past* $|$. *If* N *is a positive integer such that*

(1) $X'|P_n\equiv X'|P_{n-17}$, $X'|Q_n\equiv X'|Q_{n-17}$, *and* $X'|R_n\equiv X'|R_{n-17}$ *for* $n\geqslant N$ *and all options* X' *of* X,

(2) $X|P_n\equiv X|P_{n-17}$, $X|Q_n\equiv X|Q_{n-17}$, *and* $X|R_n\equiv X|R_{n-17}$ *for* $N-69\leqslant n<N$,

then $X|P_n\equiv X|P_{n-17}$, $X|Q_n\equiv X|Q_{n-17}$, *and* $X|R_n\equiv X|R_{n-17}$ *for all* $n\geqslant N$.

Theorem 7. *Let* X *be any position. If* N *is a positive integer such that*

(1) $X'\bullet\bullet P_n\equiv X'\bullet\bullet P_{n-17}$, $X'\bullet\bullet Q_n\equiv X'\bullet\bullet Q_{n-17}$, *and* $X'\bullet\bullet R_n\equiv X'\bullet\bullet R_{n-17}$ *for* $n\geqslant N$ *and all options* X' *of* X,

(2) $X\bullet\bullet P_n\equiv X\bullet\bullet P_{n-17}$, $X\bullet\bullet Q_n\equiv X\bullet\bullet Q_{n-17}$, *and* $X\bullet\bullet R_n\equiv X\bullet\bullet R_{n-17}$ *for* $N-69\leqslant n<N$

then $X\bullet\bullet P_n \equiv X\bullet\bullet P_{n-17}$, $X\bullet\bullet Q_n \equiv X\bullet\bullet Q_{n-17}$, *and* $X\bullet\bullet R_n \equiv X\bullet\bullet R_{n-17}$ *for all* $n \geqslant N$.

Using Theorems 5–7, we have verified the two conjectures by computer for all positions X of length (pegs+holes) at most 8. As a final note, the largest nim-value we have observed for any position that cannot be decomposed as in Lemma 2 is 193, which was calculated for the following position:

$$\bullet\bullet\circ(\circ\bullet\bullet\circ\bullet\circ\bullet\circ)^{20601}\bullet.$$

Acknowledgements

Many thanks to Richard Nowakowski and Michael Albert for patiently listening to proofs and pointing out errors. Without them, this paper would be largely incorrect. Thanks also to Richard Nowakowski and the anonymous reviewer for numerous helpful suggestions.

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