Theoretical Computer Science 38 (1985) 83-98
North-Holland

PERMUTATION REPRESENTATION OF k-ARY TREES*

Prakash V. RAMANAN** and C.L. LIU

Department of Computer Science, University of Illinois at Urbana-Champaign, 1304 W. Springfield Avenue, Urbana, IL 61801, U.S.A.

Communicated by M.S. Paterson
Received October 1982
Revised September 1984

Abstract. In this paper, we study the permutation representation of k-ary trees. First, we extend the notion of traversals, from binary trees to k-ary trees. A pair of k-ary tree traversals can be used to assign a permutation of the integers $1, 2, \ldots$ to each k-ary tree T. A pair of traversals is a valid k-ary tree Representation Scheme (k-RS) if it does not assign the same permutation to two distinct k-ary trees. We characterize such pairs of traversals. We also characterize those k-RS which assign permutations that have their lexicographic order being consistent with some well-known 'natural' order defined on the trees they represent.

1. Introduction

The problem of representing k-ary trees by integer sequences has been studied extensively. The main motivation is to find efficient algorithms to generate, rank, and unrank k-ary trees with respect to some order defined either on the set of all k-ary trees (with n internal nodes), or on the set of integer sequences chosen to represent them. Ruskey and Hu [8] and Ruskey [9] studied a representation using the sequence of level numbers of the leaves of the tree. Zaks [11] studied a representation using balloting sequences on $\{0, 1\}$. Knott [1], Rotem and Varol [6], and Trojanowski [10] studied a representation of binary trees using permutations of the integers $1, 2, \ldots$. They label the nodes of a binary tree with the integers $1, 2, \ldots$ in the order they are visited according to one traversal, and read off these labels in the order the nodes are visited according to another traversal. This procedure assigns a permutation of the integers $1, 2, \ldots$ to each binary tree. The two traversals are so chosen that no two trees will be assigned the same permutation. Knott [1] and Rotem and Varol [6] label the nodes according to inorder, and read off the labels according to preorder. Knott [1] and Rotem [5, 7] also observed that the set of all permutations assigned to binary trees is the set of stack sortable permutations. Rotem and Varol [6] represent these permutations by balloting sequences, as described in [5]. Trojanowski [10] labels the nodes according to preorder, and reads off the labels according to inorder. Liu [3] extended the idea of using a pair of

* This research was supported in part by the Office of Naval Research under Grant N00014-79-C-0775.
** Present affiliation: Department of Computer Science, University of California, Santa Barbara, CA 93106, U.S.A.

traversals to assign permutations to binary trees, to k-ary trees. However, his representation is based on permutations of a multiset of integers, which, as we shall see later, is a special case of the representation studied here.

The lexicographic order of the integer sequences, or the permutations, induces an order on the trees they represent. The order induced by the permutations of Knott [1] and Rotem and Varol [6] coincides with a well-known order on trees, defined below as A-order. Zaks [11] showed that the order induced by the integer sequences of Ruskey and Hu [8] and Ruskey [9] coincides with another well-known order on trees, defined below as B-order. He also showed that the order induced by the permutations of Trojanowski [10], and the balloting sequences of Zaks [11] coincides with B-order. The order induced by the permutations of Liu [3] also coincides with B-order.

In this paper we study the permutation representation of k-ary trees. We first generalize the notion of traversals (e.g., preorder, inorder, and postorder traversals of binary trees) to k-ary trees. Then we show how a pair of k-ary tree traversals can be used to assign a permutation of the integers $1, 2, \ldots$ to each k-ary tree T, by extending the idea used by Knott [1], Liu [3], Rotem and Varol [6], and Trojanowski [10]. A pair of k-ary tree traversals will be called a Representation Scheme if it does not assign the same permutation to two distinct k-ary trees.

Our main results are to characterize: (1) those pairs of traversals that are valid Representation Schemes (Section 2), and (2) those Representation Schemes which are such that, the order induced on trees by the lexicographic order of the permutations they assign, coincides with some of the well-known 'natural' orders on k-ary trees (Section 3).

2. Characterization of Representation Schemes

The following definition of k-ary trees is well known.

Definition 2.1. A *k-ary tree T* is defined recursively as being either a leaf node, or an internal node r called the root of T, together with an ordered sequence $(T_1, T_2, \ldots, T_k)$ of k-ary trees. T_i is referred to as the ith subtree of T.

Hereafter, tree will mean a k-ary tree as defined above. It is clear that if a k-ary tree T has N leaves, then $N = n(k-1)+1$, where n is the number of internal nodes in T. We let $|T|$ denote the number of internal nodes in T.

The following definition extends the notion of traversals from binary trees to k-ary trees.

Definition 2.2. A *k-ary tree traversal ϕ* is a permutation of the symbols $T_1, T_2, \ldots, T_k$ and M $(M>0)$ copies of the symbol r, where $T_1, T_2, \ldots, T_k$ appear in that order in ϕ. r denotes the root.

A traversal specifies the order in which we recursively visit the internal nodes of a k-ary tree T. Note that each internal node of T will be visited M times when we traverse T in the order specified by ϕ. We let $|\phi|$ denote M. We let r^i denote i consecutive occurrences of r in ϕ. For example, $\phi = rrT_1T_2rT_3T_4$ will be written as $\phi = r^2T_1T_2rT_3T_4$. This means that we visit the root twice, then recursively visit the first subtree, followed by the second subtree, and again visit the root, and then recursively visit the third subtree, and then the fourth subtree. In this example, $k=4$ and $|\phi| = 3$. Note also that the preorder, inorder, and postorder traversals of a binary tree are rT_1T_2, T_1rT_2, and T_1T_2r, respectively.

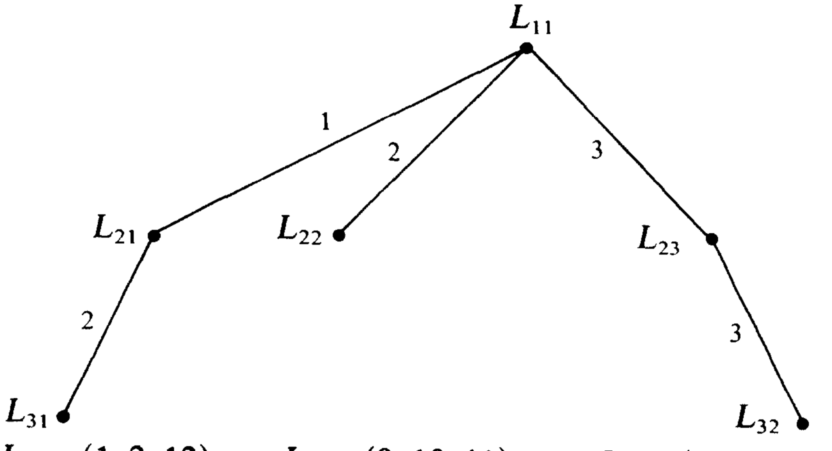
Definition 2.3. A *compatible pair of k-ary tree traversals* (ϕ_1, ϕ_2) is a pair of k-ary tree traversals ϕ_1 and ϕ_2, with $|\phi_1| = |\phi_2|$.

Let (ϕ_1, ϕ_2) be a compatible pair of k-ary tree traversals, and let T be a k-ary tree. We assign a permutation of the integers $1, 2, \ldots, |\phi_1| \times |T|$ to T as follows: Label the internal nodes of T with the integers $1, 2, \ldots$ in the order the nodes are visited according to ϕ_1. Each internal node will be visited $|\phi_1|$ times and hence assigned as many labels. Labels at a node are ordered in the order they are assigned, namely in increasing order. Now traverse the tree in the order specified by ϕ_2. Each time an internal node is visited during this traversal, read off the smallest label of the node and remove it from the set of labels assigned to that node.

To keep the figures legible, we will not be showing the leaves of the trees. We indicate that an internal node in the tree is the ith son of its father, by placing the integer i adjacent to the edge joining the two nodes.

We shall now illustrate the above definitions and notations by an example.

Example 2.4. Let T be the 4-ary tree shown in Fig. 1. Let (ϕ_1, ϕ_2) be a pair of traversals, where $\phi_1 = r^2T_1T_2rT_3T_4$ and $\phi_2 = T_1rT_2T_3rT_4r$. The labels assigned to the internal nodes according to the traversal ϕ_1 are also shown in the figure. Now, when



$L_{11} = (1, 2, 12)$ $L_{22} = (9, 10, 11)$ $L_{31} = (5, 6, 7)$

$L_{21} = (3, 4, 8)$ $L_{23} = (13, 14, 15)$ $L_{32} = (16, 17, 18)$

Fig. 1.

we read off the labels of the nodes in the order they are visited by ϕ_2, we obtain the permutation

$$\pi(T) = 3\ 5\ 6\ 7\ 4\ 8\ 1\ 9\ 10\ 11\ 13\ 16\ 17\ 18\ 14\ 15\ 2\ 12.$$

Clearly, if ϕ_1 and ϕ_2 were not chosen carefully, then two distinct k-ary trees with n internal nodes might be assigned the same permutation. This leads us to the following definition.

Definition 2.5. A compatible pair of k-ary tree traversals (ϕ_1, ϕ_2) is called a k-ary tree *Representation Scheme* (k-RS), if it does not assign the same permutation to two distinct k-ary trees.

Let (ϕ_1, ϕ_2) be a compatible pair of k-ary tree traversals, with $|\phi_1| = |\phi_2| = M$. Let

$$\phi_1 = r^{a_1} T_1 r^{a_2} T_2 \ldots r^{a_k} T_k r^{a_{k+1}} \quad \text{and} \quad \phi_2 = r^{b_1} T_1 r^{b_2} T_2 \ldots r^{b_k} T_k r^{b_{k+1}}.$$

We let A_i denote $\sum_{j=1}^{i} a_j$, and B_i denote $\sum_{j=1}^{i} b_j$. Note that $A_{k+1} = B_{k+1} = M$. We also let $\pi(T)$ denote the permutation assigned to a k-ary tree T by a specified compatible pair (ϕ_1, ϕ_2) of k-ary tree traversals, and $\pi(r^{b_j})$ denote the b_j labels in $\pi(T)$ corresponding to the term r^{b_j} in ϕ_2. Now, we present the main theorem of this section.

Theorem 2.6. *A compatible pair of k-ary tree traversals (ϕ_1, ϕ_2) is a k-*RS *if and only if the following conditions hold*:

(i) *There exists no i, $1 \leq i < k$, such that $A_i = A_{i+1}$, and $B_i = B_{i+1}$.*

(ii) *There exist no i, j, $1 \leq i < j \leq k$, such that $A_i = B_i$, and $A_j = B_j$.*

Proof. First let us prove the ***necessity*** of the two conditions. Let (ϕ_1, ϕ_2) be a compatible pair of k-ary tree traversals for which condition (i) fails to hold. So there exists an i, $1 \leq i < k$, such that $A_i = A_{i+1}$, and $B_i = B_{i+1}$. Let T and T' be two k-ary trees with two internal nodes each, such that T_i and T'_{i+1} are the only nonempty subtrees of T and T', respectively. Considering, separately, each of the three cases $A_i < B_i$, $A_i = B_i$, and $A_i > B_i$, it is easy to see that T and T' are assigned the same permutation.

Now let (ϕ_1, ϕ_2) be a compatible pair of k-ary tree traversals for which condition (ii) fails to hold. So there exist i, j, $1 \leq i < j \leq k$, such that $A_i = B_i$, and $A_j = B_j$. Let T and T' be two k-ary trees with two internal nodes each, where T_i and T'_j are the only nonempty subtrees of T and T', respectively. It is easily seen that both T and T' are assigned the same permutation, namely the trivial permutation $1\ 2 \ldots (2M)$.

Now let us prove the ***sufficiency*** of the conditions. Let (ϕ_1, ϕ_2) be a compatible pair of k-ary tree traversals that satisfies conditions (i) and (ii). Let T and T' be two distinct k-ary trees which are assigned permutations $\pi(T)$ and $\pi(T')$, respectively. Note that $|\pi(T)| = |T| \times |\phi_1|$, and $|\pi(T')| = |T'| \times |\phi_1|$. So if $|T| \neq |T'|$, then $\pi(T) \neq \pi(T')$. Hence, we only need to consider the case $|T| = |T'|$.

We claim that if T and T' are two distinct k-ary trees with $|T| = |T'|$, then $\pi(T) \neq \pi(T')$. The proof proceeds by induction on $|T|$. If $|T| = 1$, then our claim is vacuously true, since there is only one k-ary tree with one internal node. Let our claim be true for all $|T| < n$. Now let $|T| = |T'| = n$. Let their subtrees be denoted by T_i and T'_i respectively, for $1 \leq i \leq k$. Let $T_i = T'_i$ for all i, $1 \leq i < h$, and $T_h \neq T'_h$ for some h, $1 \leq h \leq k$. So we have

$$\pi(T) = \underline{\pi(r^{b_1})\pi(T_1)\pi(r^{b_2})\pi(T_2)\ldots\pi(r^{b_h})}\pi(T_h)\ldots\pi(r^{b_k})\pi(T_k)\pi(r^{b_{k+1}}),$$

$$\pi(T') = \underline{\pi(r'^{b_1})\pi(T'_1)\pi(r'^{b_2})\pi(T'_2)\ldots\pi(r'^{b_h})}\pi(T'_h)\ldots\pi(r'^{b_k})\pi(T'_k)\pi(r'^{b_{k+1}}).$$

Since $T_i = T'_i$ for all i, $1 \leq i < h$, the underlined parts of $\pi(T)$ and $\pi(T')$ have the same length. If they are not identical, we are done. So, suppose they are identical. If $|T_h| = |T'_h|$, then $\pi(T_h) \neq \pi(T'_h)$ by induction hypothesis, implying that $\pi(T) \neq \pi(T')$. So, without loss of generality, let $|T_h| < |T'_h|$. Note that this implies $h < k$. Now there are three different cases to consider.

Case 1. $A_h < B_h$. Let m be the smallest integer, $h < m \leq k$, such that

$$|T_h| + |T_{h+1}| + \cdots + |T_m| = |T'_h| + |T'_{h+1}| + \cdots + |T'_m|.$$

Since $|T| = |T'|$, such an m exists. Since $A_h < B_h$, the underlined parts of $\pi(T)$ and $\pi(T')$ contain labels assigned to the roots of T and T' respectively, after the traversal of their hth subtrees. If these labels were assigned to their roots before the traversal of their mth subtrees, then the underlined parts cannot be identical, contradicting our assumption. So, we have $a_i = 0$ for all i, $h < i \leq m$. But then by condition (i) of the theorem, we have $b_i > 0$ for all i, $h < i \leq m$. In particular, $b_{h+1} > 0$. So the first element of $\pi(r^{b_{h+1}})$, which is greater than $A_m + \sum_{j=1}^{m} |T_j| \times M$, is greater than every element of $\pi(T'_h)$. Since $|\pi(T_h)| < |\pi(T'_h)|$, this implies that $\pi(T) \neq \pi(T')$.

Case 2. $A_h = B_h$. Note that the labels in $\pi(T'_h)$ are consecutive integers. If $\pi(T) = \pi(T')$, then we must have one of the following two subcases.

Subcase 2.1. $\pi(T'_h)$ is a prefix of $\pi(T_h)\pi(r^{b_{h+1}})\pi(T_{h+1})\pi(r^{b_{h+2}})\ldots\pi(T_{j-1})\pi(r^{b_j})$, but not a prefix of $\pi(T_h)\pi(r^{b_{h+1}})\pi(T_{h+1})\pi(r^{b_{h+2}})\ldots\pi(T_{j-1})$, for some j, $h < j \leq k$. Clearly, this implies that $b_j > 0$. Since $|\pi(T'_h)|$ is a multiple of M, we have that $B_j = M$, $A_h = B_h = 0$, and $\pi(T'_h) = \pi(T_h)\pi(r^{b_{h+1}})\pi(T_{h+1})\pi(r^{b_{h+2}})\ldots\pi(T_{j-1})\pi(r^{b_j})$. Let m be the smallest index, $j \leq m \leq k$, such that T_m is not empty. Since $|T| = |T'|$, such an m exists. We have $B_m = B_j = M$. If $A_m < M$, then $\pi(T_h)\pi(r^{b_{h+1}})\pi(T_{h+1})\pi(r^{b_{h+2}})\ldots\pi(T_{j-1})\pi(r^{b_j})$ contains a label that was assigned to the root of T after the traversal of T_m, and hence is larger than any label in $\pi(T_m)$. Since $\pi(T'_h)$ does not contain any element of $\pi(T_m)$, such a label would also be larger than any label in $\pi(T'_h)$, implying that $\pi(T'_h) \neq \pi(T_h)\pi(r^{b_{h+1}})\pi(T_{h+1})\pi(r^{b_{h+2}})\ldots\pi(T_{j-1})\pi(r^{b_j})$, a contradiction. Hence $A_m = B_m = M$. This, together with $A_h = B_h = 0$, violates condition (ii) of the theorem. So we conclude that $\pi(T) \neq \pi(T')$.

Subcase 2.2. $\pi(T'_h)$ is a prefix of $\pi(T_h)\pi(r^{b_{h+1}})\pi(T_{h+1})\pi(r^{b_{h+2}})\ldots\pi(r^{b_j})\pi(T_j)$, but not a prefix of $\pi(T_h)\pi(r^{b_{h+1}})\pi(T_{h+1})\pi(r^{b_{h+2}})\ldots\pi(r^{b_j})$, for some j, $h < j \leq k$. If

$B_j < A_j$, then there is a label assigned to the root of T after the traversal of T_h, but before the traversal of T_j, that will appear in $\pi(T'_h)$, but not in $\pi(T_h)\pi(r^{b_{h+1}})\pi(T_{h+1})\pi(r^{b_{h+2}})\ldots\pi(r^{b_j})\pi(T_j)$, a contradiction. So we have $B_j \geqslant A_j$. This also implies that $B_j > A_j$. Otherwise, we have $A_j = B_j$, and $A_h = B_h$, violating condition (ii) of the theorem.

Suppose $\pi(T'_h)$ is a proper prefix of $\pi(T_h)\pi(r^{b_{h+1}})\pi(T_{h+1})\pi(r^{b_{h+2}})\ldots\pi(r^{b_j})\pi(T_j)$. Since $B_j > A_j$, there is a label assigned to the root of T after the traversal of T_j, that will appear in $\pi(T_h)\pi(r^{b_{h+1}})\pi(T_{h+1})\pi(r^{b_{h+2}})\ldots\pi(r^{b_j})$, but not in $\pi(T'_h)$, a contradiction.

So let $\pi(T'_h) = \pi(T_h)\pi(r^{b_{h+1}})\pi(T_{h+1})\pi(r^{b_{h+2}})\ldots\pi(r^{b_j})\pi(T_j)$. This implies $A_h = B_h = 0$ and $B_j = M$. Let m be the smallest index, $j < m \leqslant k$, such that T_m is not empty. Since $|T| = |T'|$, such an m exists. If $B_j > A_m$, then we have $\pi(T'_h) \neq \pi(T_h)\pi(r^{b_{h+1}})\pi(T_{h+1})\pi(r^{b_{h+2}})\ldots\pi(r^{b_j})\pi(T_j)$, a contradiction. So we have $A_m = B_m = M$. This, together with $A_h = B_h = 0$, violates condition (ii) of the theorem. So we conclude that $\pi(T) \neq \pi(T')$.

Case 3. $A_h > B_h$. Let us assume that $\pi(T) = \pi(T') = p_1 p_2 \ldots p_{nM}$. Label the internal nodes of T and T' with the sequence of labels $p_1, p_2, \ldots, p_{nM}$, in the order the nodes are visited according to ϕ_2. Read off the labels of the nodes in the order the nodes are visited according to ϕ_1. Now we will obtain the same permutation $1\,2\ldots(nM)$ for both T and T'. But in Case 1 we proved that this is not possible.

Thus we conclude that (ϕ_1, ϕ_2) is a k-RS. □

In [2, Section 2.3.1, Example 7], it is shown that $(rT_1T_2,\ T_1rT_2)$ is a 2-RS. It is also well known that $(T_1rT_2,\ T_1T_2r)$ is a 2-RS, whereas $(rT_1T_2,\ T_1T_2r)$ is not. Theorem 2.6 generalizes these results to k-ary trees.

Corollary 2.6.1. *(ϕ_1, ϕ_2) is a k-RS if and only if (ϕ_2, ϕ_1) is a k-RS.*

Corollary 2.6.2. *For any k-RS, $M \geqslant \lceil \frac{1}{2}k \rceil$.[1] Moreover, for any k there exists a k-RS with $M = \lceil \frac{1}{2}k \rceil$.*

Proof. By condition (i) of Theorem 2.6, for any k-RS $M \geqslant \lceil \frac{1}{2}(k-1) \rceil$.

When k is even, the pair of traversals (ϕ_1, ϕ_2), where

$$\phi_1 = rT_1rT_2\ldots rT_{\frac{1}{2}k}T_{\frac{1}{2}k+1}\ldots T_k \quad \text{and} \quad \phi_2 = T_1T_2\ldots T_{\frac{1}{2}k}rT_{\frac{1}{2}k+1}r\ldots T_k,$$

is a k-RS with $M = \frac{1}{2}k$.

However, when k is odd, any pair of k-ary tree traversals (ϕ_1, ϕ_2) with $|\phi_1| = |\phi_2| = \frac{1}{2}(k-1)$, that satisfies condition (i) of Theorem 2.6, will have $A_1 = B_1 = 0$, and $A_k = B_k = \frac{1}{2}(k-1)$. This violates condition (ii) of the theorem. But it is easily seen

[1] $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

that the pair of k-ary tree traversals (ϕ_1, ϕ_2), where

$$\phi_1 = rT_1rT_2 \ldots rT_{\frac{1}{2}(k+1)}T_{\frac{1}{2}(k+3)} \ldots T_k$$

and

$$\phi_2 = T_1T_2 \ldots T_{\frac{1}{2}(k+1)}rT_{\frac{1}{2}(k+3)}r \ldots T_kr,$$

is a k-RS with $M = \frac{1}{2}(k+1) = \lceil \frac{1}{2}k \rceil$. □

3. A-order and B-order Representation Schemes

In this section, we characterize those k-RS with $|\phi_1| = |\phi_2| \leq k-1$ which are such that the order induced on k-ary trees by the lexicographic order of the permutations they assign coincides with some of the well-known 'natural' orders on k-ary trees.

First, we present the definition of two well-known natural orders on k-ary trees. The order given in [2] for binary trees, and in [10] for k-ary trees is as follows.

Definition 3.1. Given two k-ary trees T and T', we say that $T < T'$ in *A-order* if
 (i) $|T| < |T'|$, or
 (ii) $|T| = |T'|$, and for some i, $1 \leq i \leq k$, we have
  (a) $T_j = T'_j$ for $j = 1, 2, \ldots, i-1$, and
  (b) $T_i < T'_i$.

The order used in [11] is as follows.

Definition 3.2. Given two k-ary trees T and T', we say that $T < T'$ in *B-order* if
 (i) T is a single leaf node, and T' has at least one internal node, or
 (ii) T has at least one internal node, and, for some i, $1 \leq i \leq k$, we have
  (a) $T_j = T'_j$ for $j = 1, 2, \ldots, i-1$, and
  (b) $T_i < T'_i$.

Now we shall define the order induced on k-ary trees by a k-RS.

Definition 3.3. Given two k-ary trees T and T' such that $|T| = |T'|$, we say that $T < T'$ in (ϕ_1, ϕ_2)*-order* if $\pi(T) < \pi(T')$ in the usual lexicographic order.

Note that (ϕ_1, ϕ_2)-order is defined only on trees that have the same number of internal nodes. Let τ_n denote the set of all k-ary trees with n internal nodes. We have the following definition.

Definition 3.4. A k-RS (ϕ_1, ϕ_2) is an *A-order k-ary tree Representation Scheme* (k-ARS) if (ϕ_1, ϕ_2)-order and A-order coincide on τ_n for all n.

A k-BRS is defined analogously.

Now, we shall present the theorems characterizing k-ARS and k-BRS, when $|\phi_1| = |\phi_2| \leq k-1$.

Theorem 3.5. *Let* (ϕ_1, ϕ_2) *be a* k-RS *with* $|\phi_1| = |\phi_2| \leq k-1$. *Then* (ϕ_1, ϕ_2) *is a* k-ARS *if and only if* $\phi_1 = T_1 r T_2 r \ldots T_{k-1} r T_k$ *and* $\phi_2 = r T_1 r T_2 \ldots r T_{k-1} T_k$.

Proof. Let (ϕ_1, ϕ_2) be a k-ARS with $|\phi_1| = |\phi_2| \leq k-1$. We shall prove that $\phi_1 = T_1 r T_2 r \ldots T_{k-1} r T_k$ and $\phi_2 = r T_1 r T_2 \ldots r T_{k-1} T_k$ in a sequence of steps.

Step 1. We claim that $A_i \leq B_i$ for all i, $1 \leq i < k$. If not, let j be the smallest index, $1 \leq j < k$, such that $A_j > B_j$. Consider the trees T and T', shown in Fig. 2, where the labels assigned to the nodes according to ϕ_1 are also shown. Note that $T < T'$ in A-order.

The permutation assigned to T is of the form

$$\pi(T) = 1\,2 \ldots B_j(A_j+1)(A_j+2) \ldots (A_j+B_j)(2A_j+1) \ldots.$$

If $B_k > B_j$, then the permutation assigned to T' is of the form

$$\pi(T') = 1\,2 \ldots B_j(A_j+1)(A_j+2) \ldots (A_j+B_j)(A_j+B_j+1) \ldots.$$

Since $A_j > B_j$, we have $\pi(T) > \pi(T')$, a contradiction. So, we must have $B_j = B_k$. But then by the uniqueness condition (i) of Theorem 2.6, we have $A_j < A_k$.

Now consider the trees $\hat{T}$ and $\hat{T}'$, $\hat{T} < \hat{T}'$, as shown in Fig. 3. The permutations assigned to $\hat{T}$ and $\hat{T}'$ are of the form $\pi(\hat{T}) = 1\,2 \ldots B_j(A_k+1) \ldots$ and $\pi(\hat{T}') = 1\,2 \ldots B_j(A_j+1) \ldots$. Since $A_j < A_k$, we have $\pi(\hat{T}) > \pi(\hat{T}')$, a contradiction.

Step 2. We claim that $A_i < B_i$ for all i, $1 \leq i < k$. If not, let j be the smallest index, $1 \leq j < k$, such that $A_j = B_j$. Since (ϕ_1, ϕ_2) is a k-RS, $A_h \neq B_h$ for any $h, j < h \leq k$. Let T and T' be two k-ary trees with two internal nodes each, where T_k and T'_j are the only nonempty subtrees of T and T', respectively. Clearly, $T < T'$. Since $A_j = B_j$, $\pi(T') = 1\,2 \ldots (2M)$. Hence, $\pi(T) > \pi(T')$, a contradiction.

Step 3. We claim that $A_i < A_{i+1}$ for all $i, 1 \leq i < k$. If not, let j be the smallest index, $1 \leq j < k$, such that $A_j = A_{j+1}$. Since (ϕ_1, ϕ_2) is a k-RS, we have $B_j < B_{j+1}$. From Step 2 we have $A_j < B_j$. So, $A_j = A_{j+1} < B_j < B_{j+1}$.

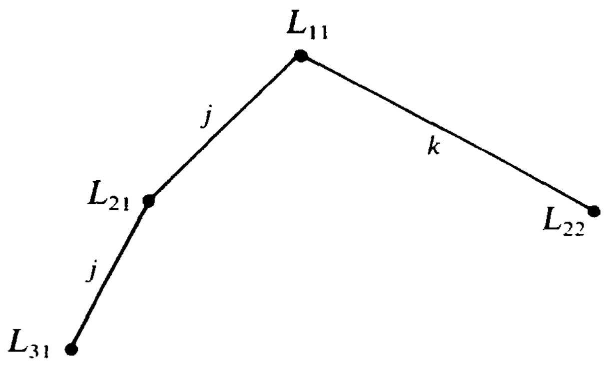
Let T and T' be two k-ary trees, $T < T'$, as shown in Fig. 4. Labels assigned to the nodes are also shown in this figure. The permutations assigned to T and T' are of the form

$$\pi(T) = 1\,2 \ldots A_j(A_j+M+1)(A_j+M+2) \ldots (B_j+M)$$
$$(B_j+M+1) \ldots (B_{j+1}+M) \ldots,$$
$$\pi(T') = 1\,2 \ldots A_j(A_j+M+1)(A_j+M+2) \ldots (B_j+M)(A_j+1) \ldots.$$

Since $A_j < B_j$, we have $\pi(T) > \pi(T')$, a contradiction.

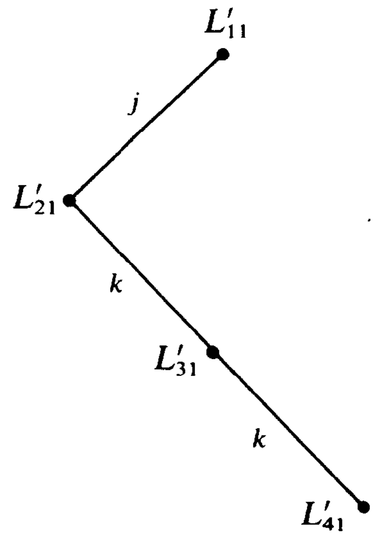
Step 4. Let M be no larger than $k-1$. Then, from Step 3, we have that $\phi_1 = T_1 r T_2 r \ldots T_{k-1} r T_k$.

Step 5. Again, let M be no larger than $k-1$. Then we claim that $\phi_2 = r T_1 r T_2 \ldots r T_{k-1} T_k$. From Steps 2 and 4 it is clear that $B_i \geq i$ for all i, $1 \leq i < k$. So $b_k = 0$. Let j be the smallest index, $1 \leq j < k$, such that $b_j > 1$.



$$L_{11} = (1, 2, \ldots, A_j, A_j+2M+1, A_j+2M+2, \ldots, A_k+2M, A_k+3M+1,$$
$$A_k+3M+2, \ldots, 4M)$$
$$L_{21} = (A_j+1,\ A_j+2, \ldots, 2A_j,\ 2A_j+M+1,\ 2A_j+M+2, \ldots, A_j+2M)$$
$$L_{22} = (A_k+2M+1, A_k+2M+2, \ldots, A_k+3M)$$
$$L_{31} = (2A_j+1,\ 2A_j+2, \ldots, 2A_j+M)$$

(a)



$$L'_{11} = (1, 2, \ldots, A_j, A_j+3M+1, A_j+3M+2, \ldots, 4M)$$
$$L'_{21} = (A_j+1, A_j+2, \ldots, A_j+A_k, \ldots)$$
$$L'_{31} = (A_j+A_k+1, A_j+A_k+2, \ldots, A_j+2A_k, \ldots)$$
$$L'_{41} = (A_j+2A_k+1, A_j+2A_k+2, \ldots, A_j+2A_k+M)$$

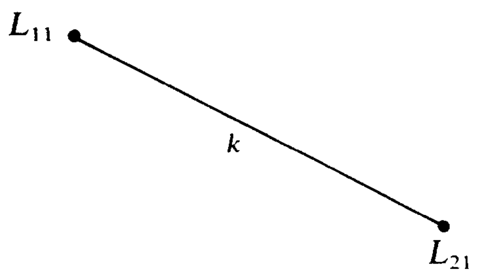
(b)

Fig. 2. (a) T. (b) T'.

Let T and T' be two k-ary trees as shown in Fig. 5. Labels assigned to the roots of T and T' are also shown. The permutations assigned to T and T' are of the form

$$\pi(T) = 1\,2\ldots(j-1)(j+2M)(j+3M+1)\ldots,$$
$$\pi(T') = 1\,2\ldots(j-1)(j+2M)(j+2M+1)\ldots.$$

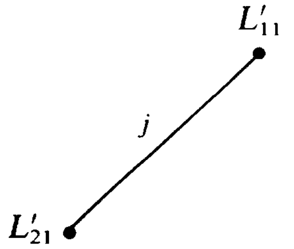
So $\pi(T) > \pi(T')$, a contradiction.



$$L_{11} = (1, 2, \ldots, A_k, A_k + M + 1, A_k + M + 2, \ldots, 2M)$$

$$L_{21} = (A_k + 1, A_k + 2, \ldots, A_k + M)$$

(a)



$$L'_{11} = (1, 2, \ldots, A_j, A_j + M + 1, A_j + M + 2, \ldots, 2M)$$

$$L'_{21} = (A_j + 1, A_j + 2, \ldots, A_j + M)$$

(b)

Fig. 3. (a) $\hat{T}$. (b) $\hat{T}'$.

From Steps 4 and 5 we finally conclude that if (ϕ_1, ϕ_2) is a k-ARS, then $\phi_1 = T_1 r T_2 r \ldots T_{k-1} r T_k$ and $\phi_2 = r T_1 r T_2 \ldots r T_{k-1} T_k$.

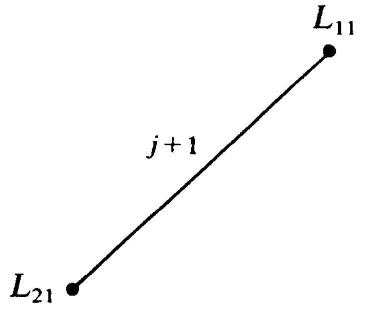
Now we shall prove that the pair of traversals (ϕ_1, ϕ_2), where $\phi_1 = T_1 r T_2 r \ldots r T_k$ and $\phi_2 = r T_1 r T_2 \ldots r T_{k-1} T_k$, is a k-ARS. We shall use subscripts to distinguish between the r's. So let $\phi_1 = T_1 r_1 T_2 r_2 \ldots r_{k-1} T_k$ and $\phi_2 = r_1 T_1 r_2 T_2 \ldots r_{k-1} T_{k-1} T_k$. We claim that if T and T' are two k-ary trees, $T < T'$, and $|T| = |T'| = p$, then $\pi(T) < \pi(T')$. The proof is by induction on p. For $p = 1$, the claim is vacuously true since there is only one k-ary tree with one internal node. Let the claim be true for all $p < n$. Let $|T| = |T'| = n$. Let j be the smallest index, $1 \leq j \leq k$, such that $T_i = T'_i$ for all i, $1 \leq i < j$, and $T_j < T'_j$.

The permutations assigned to T and T' are

$$\pi(T) = \underline{\pi(r_1)\pi(T_1)\pi(r_2)\pi(T_2) \ldots \pi(r_{j-1})\pi(T_{j-1})}$$
$$\pi(r_j)\pi(T_j) \ldots \pi(r_{k-1})\pi(T_{k-1})\pi(T_k),$$

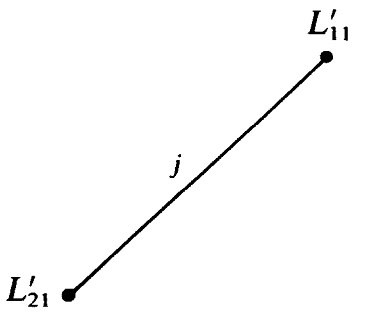
$$\pi(T') = \underline{\pi(r'_1)\pi(T'_1)\pi(r'_2)\pi(T'_2) \ldots \pi(r'_{j-1})\pi(T'_{j-1})}$$
$$\pi(r'_j)\pi(T'_j) \ldots \pi(r'_{k-1})\pi(T'_{k-1})\pi(T'_k),$$

respectively.



$$L_{11}=(1,2,\ldots,A_j,A_j+M+1,A_j+M+2,\ldots,2M)$$
$$L_{21}=(A_j+1,A_j+2,\ldots,A_j+M)$$

(a)



$$L'_{11}=(1,2,\ldots,A_j,A_j+M+1,A_j+M+2,\ldots,2M)$$
$$L'_{21}=(A_j+1,A_j+2,\ldots,A_j+M)$$

(b)

Fig. 4. (a) T. (b) T'.

Since $T_i=T'_i$ for all i, $1\leq i<j$, the underlined parts of $\pi(T)$ and $\pi(T')$ are identical. Now we need to consider two cases.

Case 1. $j=k$. This implies that $|T_k|=|T'_k|$. By induction hypothesis, $\pi(T_k)<\pi(T'_k)$, and hence $\pi(T)<\pi(T')$.

Case 2. $j<k$. In this case, there are two subcases to consider.

Subcase 2.1. $|T_j|=|T'_j|$. In this case, $\pi(r_j)=\pi(r'_j)$, and, by induction hypothesis, $\pi(T_j)<\pi(T'_j)$. So we have $\pi(T)<\pi(T')$.

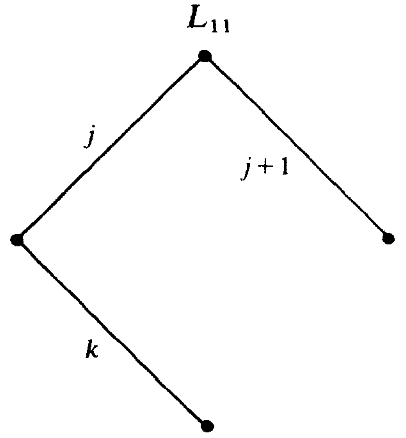
Subcase 2.2. $|T_j|<|T'_j|$. In this case, $\pi(r_j)<\pi(r'_j)$, and so $\pi(T)<\pi(T')$.

So we conclude that $(T_1rT_2r\ldots T_{k-1}rT_k, rT_1rT_2\ldots rT_{k-1}T_k)$ is the only k-ARS with $|\phi_1|=|\phi_2|\leq k-1$. $\square$

It is well known that the 2-RS (T_1rT_2, rT_1T_2) is a 2-ARS [1]. Theorem 3.5 generalizes this result to k-ary trees.

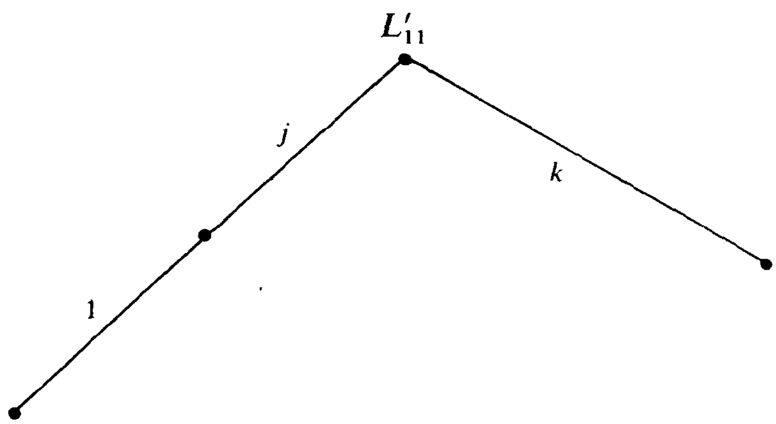
Now we shall present the theorem characterizing k-BRS with $|\phi_1|=|\phi_2|\leq k-1$.

Theorem 3.6. *Let (ϕ_1,ϕ_2) be a k-RS with $|\phi_1|=|\phi_2|\leq k-1$. Then (ϕ_1,ϕ_2) is a k-BRS if and only if the following conditions hold*:



$$L_{11} = (1, 2, \ldots, j-1, j+2M, j+3M+1, j+3M+2, \ldots, 4M)$$

(a)



$$L'_{11} = (1, 2, \ldots, j-1, j+2M, j+2M+1, \ldots, 3M)$$

(b)

Fig. 5. (a) T. (b) T'.

(i) $\phi_2 = T_1 r T_2 r \ldots T_{k-1} r T_k$,
(ii) $A_i > B_i$ for $i = 1, 2, \ldots, k-1$, *and*
(iii) $A_k = B_k = k-1$.

To prove the theorem, we need the following lemma.

Lemma 3.7. *Let* (ϕ_1, ϕ_2) *be a k*-RS *satisfying the conditions mentioned in the theorem. Let T and T′ be two k-ary trees such that* $T < T'$ *in B-order. Then*
(i) *if* $|T| = |T'|$, *then* $\pi(T) < \pi(T')$.
(ii) *if* $|T| < |T'|$, *then* $\pi(T) \leqslant$ *the prefix of* $\pi(T')$ *of length* $|\pi(T)|$.
(iii) *if* $|T| > |T'|$, *then* $\pi(T') >$ *the prefix of* $\pi(T)$ *of length* $|\pi(T')|$.

Proof. Let $\max\{|T|, |T'|\} = p$, for some $p > 0$. We shall prove that the result stated in the lemma holds, by induction on p. Clearly, for $p = 1$ the result holds. Let the result hold for all $p < n$. Now let $\max\{|T|, |T'|\} = n$. If $|T| = 0$, then the proof is trivial.

So assume that $|T| > 0$. Let j be the smallest index, $1 \leq j \leq k$, such that $T_i = T'_i$ for all i, $1 \leq i < j$, and $T_j < T'_j$. Clearly, $\max\{|T_j|, |T'_j|\} < n$.

Using the notations introduced earlier, we see that

$$\pi(T) = \underline{\pi(T_1)\pi(r_1)\pi(T_2)\pi(r_2)\ldots\pi(r_{j-1})}$$
$$\pi(T_j)\pi(r_j)\ldots\pi(T_{k-1})\pi(r_{k-1})\pi(T_k),$$
$$\pi(T') = \underline{\pi(T'_1)\pi(r'_1)\pi(T'_2)\pi(r'_2)\ldots\pi(r'_{j-1})}$$
$$\pi(T'_j)\pi(r'_j)\ldots\pi(T'_{k-1})\pi(r'_{k-1})\pi(T'_k).$$

Since $A_i \geq B_i$ for all i, $1 \leq i \leq k$, the underlined parts of the permutations are identical. We shall now consider two cases.

Case 1. $j = k$. If $|T| = |T'|$, then $|T_j| = |T'_j|$. By induction hypothesis, $\pi(T_j) < \pi(T'_j)$. Hence $\pi(T) < \pi(T')$.

If $|T| < |T'|$, then $|T_j| < |T'_j|$. By induction hypothesis, $\pi(T_j) \leq$ the prefix of $\pi(T'_j)$ of length $|\pi(T_j)|$. So $\pi(T) \leq$ the prefix of $\pi(T')$ of length $|\pi(T)|$.

If $|T| > |T'|$, then $|T_j| > |T'_j|$. By induction hypothesis, $\pi(T'_j) >$ the prefix of $\pi(T_j)$ of length $|\pi(T'_j)|$. So $\pi(T') >$ the prefix of $\pi(T)$ of length $|\pi(T')|$.

Case 2. $j < k$. If $|T_j| = |T'_j|$, then, by induction hypothesis, $\pi(T_j) < \pi(T'_j)$, and we are done.

If $|T_j| < |T'_j|$, then $\pi(T_j) \leq$ the prefix of $\pi(T'_j)$ of length $|\pi(T_j)|$. Since $A_j > B_j$, $\pi(r_j)$ is less than every element of $\pi(T'_j)$. Hence $\pi(T_j)\pi(r_j) <$ the prefix of $\pi(T'_j)$ of length $|\pi(T_j)| + 1$, and we are done.

If $|T_j| > |T'_j|$, then, by induction hypothesis, $\pi(T'_j) >$ the prefix of $\pi(T_j)$ of length $|\pi(T'_j)|$, and we are done. □

Proof of Theorem 3.6. Let (ϕ_1, ϕ_2) be a k-BRS with $|\phi_1| = |\phi_2| \leq k - 1$. We shall prove that it satisfies the conditions stated above, in a sequence of steps.

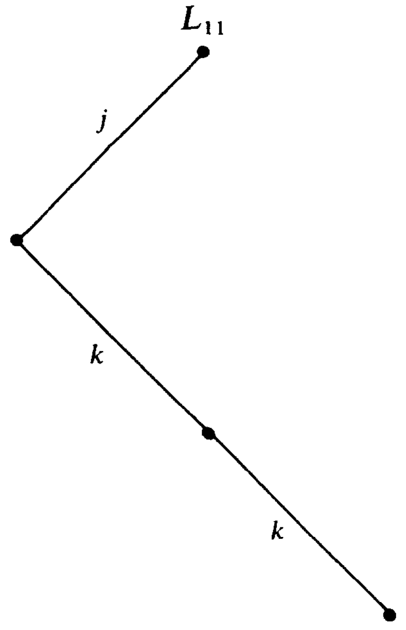
Step 1. First, we claim that $A_i \geq B_i$ for all i, $1 \leq i < k$. If not, let j be the smallest index, $1 \leq j < k$, such that $A_j < B_j$.

Let T and T' be two k-ary trees, $T < T'$, as shown in Fig. 6. Labels assigned to the roots of T and T' are also shown in this figure. The permutations assigned to T and T' are of the form $\pi(T) = 1\,2\ldots A_j(A_j + 3M + 1)\ldots$ and $\pi(T') = 1\,2\ldots A_j(A_j + 2M + 1)\ldots$. So we have $\pi(T) \geq \pi(T')$, a contradiction.

Step 2. We claim that $A_i > B_i$ for all i, $1 \leq i < k$. From Step 1 we have that $A_i \geq B_i$ for all i, $1 \leq i < k$. Let j be the smallest index, $1 \leq j < k$, such that $A_j = B_j$. Then, by Theorem 2.6, $A_h \neq B_h$ for any $h, j < h \leq k$. In particular, $A_k \neq B_k$.

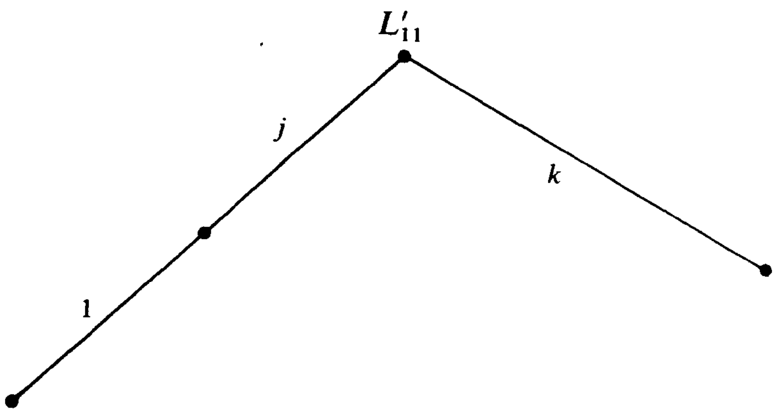
Let T and T' be two k-ary trees with two internal nodes each, where T_k and T'_j are the only nonempty subtrees of T and T', respectively. Clearly, $T < T'$. Since $A_j = B_j$, we have that $\pi(T') = 1\,2\ldots(2M)$. So $\pi(T) > \pi(T')$, a contradiction.

Step 3. We claim that $B_i < B_{i+1}$ for all i, $1 \leq i < k$. If not, let j be the smallest index, $1 \leq j < k$, such that $B_j = B_{j+1}$. From Theorem 2.6 we have $A_j < A_{j+1}$. So, from Step 2 we have $B_j = B_{j+1} < A_j < A_{j+1}$.



$$L_{11} = (1, 2, \ldots, A_j, A_j + 3M + 1, A_j + 3M + 2, \ldots, 4M)$$

(a)



$$L'_{11} = (1,\ 2, \ldots, A_j,\ A_j + 2M + 1, A_j + 2M + 2, \ldots, A_k + 2M,\ A_k + 3M + 1,\\ A_k + 3M + 2, \ldots, 4M)$$

(b)

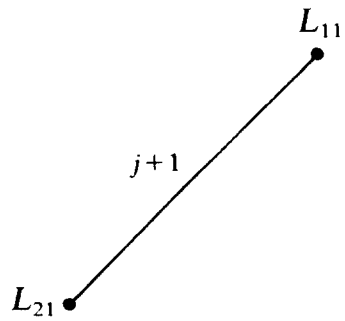
Fig. 6. (a) T. (b) T'.

Let T and T' be two k-ary trees, $T < T'$, as shown in Fig. 7. Labels assigned to the nodes are also shown in this figure. The permutations assigned to T and T' are of the form

$$\pi(T) = 1\ 2 \ldots B_j(A_{j+1} + 1)(A_{j+1} + 2) \ldots (A_{j+1} + M) \ldots,$$

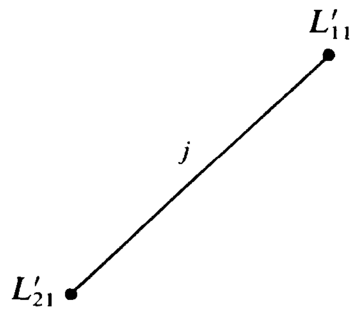
$$\pi(T') = 1\ 2 \ldots B_j(A_j + 1)(A_j + 2) \ldots (A_j + M) \ldots,$$

respectively. Since $A_{j+1} > A_j$, we have $\pi(T) > \pi(T')$, a contradiction.



$$L_{11} = (1, 2, \ldots, A_{j+1}, A_{j+1} + M + 1, A_{j+1} + M + 2, \ldots, 2M)$$
$$L_{21} = (A_{j+1} + 1, A_{j+1} + 2, \ldots, A_{j+1} + M)$$

(a)



$$L'_{11} = (1, 2, \ldots, A_j, A_j + M + 1, A_j + M + 2, \ldots, 2M)$$
$$L'_{21} = (A_j + 1, A_j + 2, \ldots, A_j + M)$$

(b)

Fig. 7. (a) T. (b) T'.

Step 4. If we restrict ourselves to the case $|\phi_1| = |\phi_2| \leq k-1$, then, from Step 3 we conclude that $\phi_2 = T_1 r T_2 r \ldots T_{k-1} r T_k$.

So we conclude that if (ϕ_1, ϕ_2) is a k-BRS, then it satisfies the conditions in the theorem.

To prove the sufficiency part, let (ϕ_1, ϕ_2) be a k-RS that satisfies the conditions of the theorem. Let T and T' be two k-ary trees, such that $|T| = |T'|$, and $T < T'$. From the preceding lemma it follows that $\pi(T) < \pi(T')$. So we conclude that (ϕ_1, ϕ_2) is a k-BRS. □

Trojanowski [10] studied the 2-RS (rT_1T_2, T_1rT_2). Zaks [11] has shown that this is a 2-BRS. Theorem 3.6 generalizes this result to k-ary trees.

4. Conclusions

In Sections 2 and 3 we studied some of the main properties of k-ary tree Representation Schemes that are of interest. It is clear that some Representation Schemes are more useful than others.

The Representation Scheme studied by Liu [3] can be thought of as the special case $\phi_1 = r^{k-1}T_1T_2 \ldots T_k$ and $\phi_2 = T_1rT_2r \ldots T_{k-1}rT_k$. If we label the nodes with the sequence of integers $1, 2, \ldots$ as described in Section 2, each node will be labeled with $k-1$ consecutive integers. Instead, Liu labels the nodes with the sequence of integers $1^{k-1}2^{k-1} \ldots |T|^{k-1}$ (i^j denotes j copies of the integer i). Consequently, each node is labeled with $k-1$ copies of the same integer, and each k-ary tree is assigned a permutation of the multiset $\{1, 2, \ldots, |T|\}^{k-1}$.

We can also characterize those Representation Schemes which assign permutations that are either all stack realizable, or all stack sortable. This is discussed in [4].

References

[1] G.D. Knott, A numbering system for binary trees, *Comm. Assoc. Comput. Mach.* **20** (1977) 113-115.
[2] D.E. Knuth, *The Art of Computer Programming, Vol. 1: Fundamental Algorithms* (Addison-Wesley, Reading, MA, 1968).
[3] C.L. Liu, Generation of k-ary trees, *Advances in Math.*, to appear.
[4] P. Ramanan, Topics in combinatorial algorithms, Ph.D. Thesis, Dept. of Computer Science, Univ. of Illinois, 1984.
[5] D. Rotem, On a correspondence between binary trees and a certain type of permutations, *Inform. Process. Lett.* **4** (1975) 58-61.
[6] D. Rotem and Y.L. Varol, Generation of binary trees from ballot sequences, *J. Assoc. Comput. Mach.* **25** (1978) 396-404.
[7] D. Rotem, Stack sortable permutations, *Discr. Math.* **33** (1981) 185-196.
[8] F. Ruskey and T.C. Hu, Generating binary trees lexicographically, *SIAM J. Comput.* **6** (1977) 745-758.
[9] F. Ruskey, Generating t-ary trees lexicographically, *SIAM J. Comput.* **7** (1978) 424-439.
[10] A.E. Trojanowski, Ranking and listing algorithms for k-ary trees, *SIAM J. Comput.* **7** (1978) 492-509.
[11] S. Zaks, Lexicographic generation of ordered trees, *Theoret. Comput. Sci.* **10** (1980) 63-82.