Theoretical Computer Science 38 (1985) 35-53
North-Holland

THEORY OF REPRESENTATIONS

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Communicated by E. Engeler
Received January 1984
Revised October 1984

Abstract. An approach for a simple, general, and unified theory of effectivity on sets with cardinality not greater than that of the continuum is presented. A standard theory of effectivity on $\mathbb{F} = \{f : \mathbb{N} \to \mathbb{N}\}$ has been developed in a previous paper. By representations $\delta : \mathbb{F} \dashrightarrow M$ this theory is extended to other sets M. Topological and recursion theoretical properties of representations are studied, where the final topology of a representation plays an essential role. It is shown that for any separable T_0-space an (up to equivalence) unique admissible representation can be defined which reflects the topological properties correctly.

1. Introduction

Definitions of Type 2 computability, i.e., computability on sets with cardinality not greater than that of the continuum, have been given in several ways (see, e.g., [4, 11, 10]). Most of these definitions are equivalent or at least dependent from each other but there is no generally accepted approach as in the case of computability on denumerable sets.

This paper presents the concept of representations as a foundation for a unified Type 2 computability theory. Its basic idea is that real world computers cannot operate on abstract elements of a set M but only on names. We have chosen the set $\mathbb{F}$ of sequences of natural numbers as a standard set of names and have defined computability on $\mathbb{F}$ explicitly (see [12]). Computability on other sets M can then be derived from computability on $\mathbb{F}$ by means of representations, i.e., (partial) mappings from $\mathbb{F}$ onto M. The same computability theory could be obtained by using sets like P_ω as standard sets but considering the applications of our theory $\mathbb{F}$ seems to be the better one. For example, infinite objects are often defined by sequences of finite objects (e.g., Cauchy sequences, chains etc.) and not by sets of finite objects. Furthermore, the computation model for functions on $\mathbb{F}$ is easy to understand and allows studying computational complexity.

Computable functions turn out to be continuous in general, and in most cases functions which are not computable are not even continuous. Hence, topological considerations are fundamental for Type 2 theory, and continuity w.r.t. representations will also be studied. Therefore, two versions of Type 2 theory are developed simultaneously, a topological (t-) and a computable (c-) one.

0304-3975/85/$3.30

It is assumed that the reader is familiar with ordinary recursion theory and some basic concepts of recursion theory on $\mathbb{F}$. Our terminology and notation will follow Rogers [10] and Weihrauch [12].

By $\mathbb{N}$ we denote the set of all natural numbers and by $W(\mathbb{N})$ the set of all finite words over $\mathbb{N}$. ε is the empty word and $\lg(w)$ is the length of the word w. If $w \in W(\mathbb{N})$ and $w = x_0x_1 \dots x_n$ (where $x_i \in \mathbb{N}$), then we define $w(i) := x_i$ for $0 \leq i \leq n$. By $f: A \dashrightarrow B$ (with dashed arrow) we denote a partial function from A to B, where 'partial' means $\operatorname{dom} f \subseteq A$. As usual, we write $\langle i_1, \dots, i_n\rangle$ instead of $\pi^{(n)}(i_1, \dots, i_n)$ where $\pi^{(n)}: \mathbb{N}^n \to \mathbb{N}$ is Cantor's bijection. φ denotes the standard numbering of the unary partial recursive functions.

Define $\mathbb{F} := \{p: \mathbb{N} \to \mathbb{N}\}$ and $\mathbb{B} := W(\mathbb{N}) \cup \mathbb{F}$. For $a, b \in \mathbb{B}$ define $a \sqsubseteq b :\Leftrightarrow a$ is a prefix of b. For $p \in \mathbb{F}$ and $i \in \mathbb{N}$ let $p^{[i]} := p(0) \dots p(i-1) \in W(\mathbb{N})$ and conversely for $v \in W(\mathbb{N})$ let $[v] := \{p \in \mathbb{F} \mid v \sqsubseteq p\}$. On $\mathbb{B}$ we consider the topology defined by the basis $\{O_v \mid v \in W(\mathbb{N})\}$ where $O_v := \{b \in \mathbb{B} \mid v \sqsubseteq b\}$. The induced topology on $\mathbb{F}$ is the well-known Baire's topology. On $\mathbb{N}$ we consider the discrete topology. $[\mathbb{F} \to \mathbb{F}]$ ($[\mathbb{F} \to \mathbb{N}]$) denotes the set of all partial continuous functions from $\mathbb{F}$ to $\mathbb{F}$ (respectively $\mathbb{N}$) with G_δ-sets (respectively open sets) as domain. (A G_δ-set is a countable intersection of open sets.) $\tilde{\psi}(\chi)$ is a standard representation of $[\mathbb{F} \to \mathbb{F}]$ ($[\mathbb{F} \to \mathbb{N}]$) satisfying a utm- and an smn-theorem.

Some more details can be found in the authors' technical report [8] and Weihrauch's paper [12].

2. Representations: Continuity, computability, and reducibility

Let M be a set with cardinality not greater than that of the continuum. A representation of M is a (partial) surjective function $\delta: \mathbb{F} \dashrightarrow M$. We say $p \in \mathbb{F}$ is a name for $x \in M$ if $\delta(p) = x$. Clearly, every $x \in M$ must have a name but it is possible for x to have more than one name. Note that a sequence $p \in \mathbb{F}$ may not be any name.

The following examples for representations will be used throughout this paper.

2.1. Example. For $p \in \mathbb{F}$ let $\mathbb{M}_p := \{i \in \mathbb{N} \mid i+1 \in \operatorname{range} p\}$. Then $\mathbb{M}: \mathbb{F} \to P_\omega$ with $\mathbb{M}(p) := \mathbb{M}_p$ is the enumeration representation of P_ω.

2.2. Example. The representation δ_{cf} of P_ω by characteristic functions is defined by $\operatorname{dom} \delta_{\mathrm{cf}} := \{p \in \mathbb{F} \mid \operatorname{range} p \subseteq \{0, 1\}\}$ and $\delta_{\mathrm{cf}}(p) := p^{-1}\{0\}$ whenever $p \in \operatorname{dom} \delta_{\mathrm{cf}}$.

2.3. Example. Weihrauch [12] introduced the following standard representations:

$$\psi: \mathbb{F} \to [\mathbb{F} \to \mathbb{B}], \quad \text{where } [\mathbb{F} \to \mathbb{B}] := \{\Gamma: \mathbb{F} \to \mathbb{B} \mid \Gamma \text{ continuous}\},$$

$$\tilde{\psi}: \mathbb{F} \to [\mathbb{F} \to \mathbb{F}], \quad \text{and} \quad \chi: \mathbb{F} \to [\mathbb{F} \to \mathbb{N}],$$

$$\omega: \mathbb{F} \to O(\mathbb{F}), \quad \text{where } O(\mathbb{F}) := \{X \subseteq \mathbb{F} \mid X \text{ is open}\},$$

$$\xi: \mathbb{F} \to G_\delta(\mathbb{F}), \quad \text{where } G_\delta(\mathbb{F}) := \{X \subseteq \mathbb{F} \mid X \text{ is a } G_\delta\text{-set}\}.$$

2.4. Example. Homeomorphisms $\Pi^{(n)}:\mathbb{F}^n\to\mathbb{F}$ and $\Pi^{(\infty)}:\mathbb{F}^{\mathbb{N}}\to\mathbb{F}$ can be defined by

$$\Pi^{(1)}(p):=p,$$

$$\Pi^{(n+1)}(p_1,\ldots,p_{n+1})(x):=\begin{cases}\Pi^{(n)}(p_1,\ldots,p_n)(y) & \text{if } x=2y,\\ p_{n+1}(y) & \text{if } x=2y+1,\end{cases}$$

and $\Pi^{(\infty)}(p_0,p_1,p_2,\ldots)\langle i,j\rangle:=p_i(j)$.

We shall use

$$\langle p_1,\ldots,p_n\rangle:=\Pi^{(n)}(p_1,\ldots,p_n)\quad\text{and}\quad\langle p_i\rangle_i:=\Pi^{(\infty)}(p_0,p_1,\ldots).$$

$\Pi_n:=(\Pi^{(n)})^{-1}$ and $\Pi_\infty:=(\Pi^{(\infty)})^{-1}$ are the standard representations of $\mathbb{F}^n$ and $\mathbb{F}^{\mathbb{N}}$.

2.5. Example. For $i\in\mathbb{N}$, $p\in\mathbb{F}$ let $\langle i,p\rangle(0):=i$, $\langle i,p\rangle(n+1):=p(n)$. Then the function $\Pi:\mathbb{N}\times\mathbb{F}\to\mathbb{F}$ defined by $\Pi(i,p):=\langle i,p\rangle$ is a homeomorphism. Π^{-1} is the standard representation of $\mathbb{N}\times\mathbb{F}$.

For formulating effectivity properties of theorems, functions, predicates, etc., we introduce the concept of correspondences (or multivalued functions).

A correspondence is a triple $f=(M,M',P)$ where $P\subseteq M\times M'$.

Define

$$\operatorname{dom} f:=\{x\in M\mid(\exists y\in M')\,(x,y)\in P\},$$

$$\operatorname{range} f:=\{y\in M'\mid(\exists x\in M)\,(x,y)\in P\}.$$

2.6. Definition. Let δ, δ' be representations of M respectively M' and let $f=(M,M',P)$ be a correspondence. Then f is called *weakly* (δ,δ')-t- (c-) *effective* iff there is some (computable) $\Gamma\in[\mathbb{F}\to\mathbb{F}]$ such that

$$(\delta q,\delta'\Gamma q)\in P\quad\text{for all } q\in\delta^{-1}\operatorname{dom} f.$$

The correspondence f is called (δ,δ')-t (c-) *effective* iff, in addition,

$$\Gamma(q)\text{ is undefined}\quad\text{for all } q\in\delta^{-1}(M\backslash\operatorname{dom} f).$$

A correspondence $f=(M,M',P)$ with $((x,y')\in P\wedge(x,z')\in P)\Rightarrow y'=z'$ is called a partial function and is denoted by $f:M \dashrightarrow M'$. Therefore, Definition 2.6 is applicable to partial functions.

A function $f:M\dashrightarrow M'$ is weakly (δ,δ')-effective if the following diagram commutes for some $\Gamma\in[\mathbb{F}\to\mathbb{F}]$:

$$\begin{tikzcd}
\mathbb{F} \arrow[r, dashed, "\Gamma"] \arrow[d, dashed, "\delta"'] & \mathbb{F} \arrow[d, dashed, "\delta'"] \\
M \arrow[r, dashed, "f"'] & M'
\end{tikzcd}$$

For (strong) (δ, δ')-effectively, Γ must also respect dom f. Therefore (δ, δ')-effective correspondences have natural domains.

(δ, ν)-effectivity of a correspondence $f = (M, S, P)$ where ν is a numbering of a set S is defined accordingly using $[\mathbb{F} \to \mathbb{N}]$ instead of $[\mathbb{F} \to \mathbb{F}]$. It is easy to see that our definition of effective correspondences generalizes Ershov's [5] definition of computability on numbered sets. For convenience we shall say 'continuous' instead of 't-effective' and 'computable' instead of 'c-effective'.

In recursion theory, the r.e. sets are the domains of computable functions and the recursive sets are defined by computable characteristic functions. The corresponding definitions in Type 2 theory are as follows.

2.7. Definition. Let δ be a representation of M. For any $A \subseteq M$ define

$$d_A := (M, \mathbb{N}, A \times \mathbb{N}),$$

$$c_A := (M, \mathbb{N}, \{(x, 0) \mid x \in A\} \cup \{(y, 1) \mid y \in M \setminus A\}).$$

(1) A is δ- (c-) *open* iff d_A is $(\delta, \mathrm{id}_{\mathbb{N}})$-t- (c-) effective.

(2) A is δ- (c-) *clopen* iff c_A is $(\delta, \mathrm{id}_{\mathbb{N}})$-t- (c-) effective.

Usually we shall say 'provable' instead of 'c-open' and 'decidable' instead of 'c-clopen'. Note that a set $A \subseteq M$ is δ-open (clopen) iff $\delta^{-1}A$ is open (clopen) in dom δ.

Representation-effective functions are closed under composition.

2.8. Lemma. *Let the δ_i's be representations of M_i $(i = 1, 2, 3)$. Let $f : M_1 \dashrightarrow M_2$ be (δ_1, δ_2)-computable and $g : M_2 \dashrightarrow M_3$ be (δ_2, δ_3)-computable.*

Then $g \circ f : M_1 \dashrightarrow M_3$ is (δ_1, δ_3)-computable.

The proof immediately follows by composition of the operators computing f and g. A corresponding version can be proved for 'continuous', 'weakly-computable', and 'weakly-continuous' instead of 'computable'. Note that, for g strongly (δ_2, ν)-effective, a similar lemma does not hold.

From given representations certain new representations can be constructed. We shall introduce some of these.

2.9. Definition. Let δ_i be representations of M_i $(i \in \mathbb{N})$ and let ν be a numbering of S.

(1) The representation $[\delta_i]_i$ of the set of sequences $M_0 \times M_1 \times \cdots$ is defined by

$$\langle p_i \rangle_i \in \mathrm{dom}[\delta_i]_i \; :\Leftrightarrow \; (\forall i) p_i \in \mathrm{dom}\, \delta_i,$$

$$[\delta_i]_i \langle p_i \rangle_i := (\delta_0(p_0), \delta_1(p_1), \ldots).$$

If $(\forall i)$ $\delta_i = \delta$, we write δ^∞ instead of $[\delta_i]_i$.

(2) The representation $[\delta_1, \ldots, \delta_n]$ of the finite product $M_1 \times \cdots \times M_n$ (respectively $\delta^n : \mathbb{F} \dashrightarrow \times_{i=1}^n M_i$) can be defined accordingly.

(3) The representation $[\nu, \delta]$ of $S \times M$ is defined by

$$\langle i, p\rangle \in \mathrm{dom}[\nu, \delta] \;:\Leftrightarrow\; i \in \mathrm{dom}\,\nu \text{ and } p \in \mathrm{dom}\,\delta,$$

$$[\nu, \delta]\langle i, p\rangle := (\nu(i), \delta(p)).$$

If $M = \{x\}$ and $(\forall p \in \mathbb{F})\ \delta(p) = x$, we write δ_ν instead of $[\nu, \delta]$ since $S \times M$ is isomorphic with S.

(4) The representation $[\delta_1 \to \delta_2]$ of all the (δ_1, δ_2)-continuous functions is defined by

$$p \in \mathrm{dom}[\delta_1 \to \delta_2]$$

$$:\Leftrightarrow \begin{cases} \tilde{\psi}_p(\mathrm{dom}\,\delta_1) \subseteq \mathrm{dom}\,\delta_2 \text{ and} \\ (\forall q, q' \in \mathrm{dom}\,\delta_1)(\delta_1 q = \delta_1 q' \Rightarrow \delta_2\tilde{\psi}_p(q) = \delta_2\tilde{\psi}_p(q')), \end{cases}$$

$$[\delta_1 \to \delta_2](p)(x) := \delta_2\tilde{\psi}_p(q) \quad \text{for some arbitrary } q \in \delta_1^{-1}\{x\}.$$

(5) The representation ω_δ of all the δ-open subsets of M is defined by

$$p \in \mathrm{dom}\,\omega_\delta \;:\Leftrightarrow\; (\forall q, q' \in \mathrm{dom}\,\delta)\ (\delta q = \delta q' \Rightarrow \chi_p(q) = \chi_p(q')),$$

$$\omega_\delta(p) := \delta(\mathrm{dom}\,\chi_p) \quad \text{whenever } p \in \mathrm{dom}\,\omega_\delta.$$

(6) A representation ξ_δ of all the δ-clopen subsets of M can be defined similarly.

The following examples show that different representations of a set may imply different kinds of continuity and computability.

(1) The function Union: $(P_\omega)^{\mathbb{N}} \to P_\omega$ with $\mathrm{Union}(A_0, A_1, \ldots) := \bigcup_i A_i$ is $(\mathbb{M}^\infty, \mathbb{M})$-computable but not even weakly $(\delta_{\mathrm{cf}}^\infty, \delta_{\mathrm{cf}})$-continuous.

(2) The function Complement: $P_\omega \to P_\omega$ with $\mathrm{Complement}(A) := \mathbb{N}\setminus A$ is $(\delta_{\mathrm{cf}}, \delta_{\mathrm{cf}})$-computable but not even weakly $(\mathbb{M}, \mathbb{M})$-continuous.

The proof is easy; in case of negative results one can never decide in finitely many steps whether a natural number n is not in $\mathrm{Union}(\delta_{\mathrm{cf}}(p_0), \delta_{\mathrm{cf}}(p_1), \ldots)$ or whether n will not appear in the range of a function $p \in \mathbb{F}$.

A representation may be changed in a certain way without changing the kind of effectivity defined by it.

2.10. Definition (Reducibility and equivalence of representations). For any two representations δ, δ' of M (respectively M') we define

$$\delta \leqslant_{\mathrm{t}} \delta' \;:\Leftrightarrow\; M \subseteq M' \quad \text{and} \quad \mathrm{id}_{M,M'} \text{ is } (\delta, \delta')\text{-t-effective},$$

$$\delta \equiv_{\mathrm{t}} \delta' \;:\Leftrightarrow\; \delta \leqslant_{\mathrm{t}} \delta' \quad \text{and} \quad \delta' \leqslant_{\mathrm{t}} \delta.$$

c-reducibility ($\leqslant_{\mathrm{c}}$) and c-equivalence ($\equiv_{\mathrm{c}}$) is defined accordingly.

2.11. Lemma. *Let δ, δ' be representations of M. Then the following properties are equivalent*:

(1) $\delta \leqslant_{\mathrm{t}} \delta'$.

(2) *For any representation* $\delta_1 : \mathbb{F} \dashrightarrow M_1$ *and any* $f : M_1 \dashrightarrow M$: *f is (weakly)* (δ_1, δ)-t-*effective* $\Rightarrow f$ *is (weakly)* (δ_1, δ')-t-*effective.*

(3) *For any representation* $\delta_2 : \mathbb{F} \dashrightarrow M_2$ *and any* $g : M \dashrightarrow M_2$: *g is (weakly)* (δ', δ_2)-t-*effective* $\Rightarrow g$ *is (weakly)* (δ, δ_2)-t-*effective.*

The proof immediately follows from Lemma 2.8.

Since Lemma 2.11 also holds for the computable (c-) case it is easy to see that two representations are t- (c-) equivalent if and only if they induce the same continuity (computability) theory. Especially t- (c-) equivalent representations define the same continuous (computable) functions and the same (c-) open and (c-) clopen sets. Furthermore, equivalence can be transferred to the derived representations (in the sense of Definition 2.9).

2.12. Lemma. *Let* δ_i (δ_i') *be representations of* M_i (M_i') $(i \in \mathbb{N})$ *and let* ν (ν') *be a numbering of* S (S').

$$(1) \quad (\forall i \leq n)\ \delta_i \leq_t \delta_i' \Rightarrow [\delta_0, \ldots, \delta_n] \leq_t [\delta_0', \ldots, \delta_n'],$$

$$(2) \quad (\forall i)\ \delta_i \leq_t \delta_i' \Rightarrow [\delta_i]_i \leq_t [\delta_i'],$$

$$(3) \quad (\delta_1 \leq_t \delta_1' \wedge \nu \leq \nu') \Rightarrow [\nu, \delta_1] \leq_t [\nu', \delta_1'],$$

$$(4) \quad (\delta_1' \leq_t \delta_1 \wedge \delta_2 \leq_t \delta_2') \Rightarrow [\delta_1 \to \delta_2] \leq_t [\delta_1' \to \delta_2'],$$

$$(5) \quad \delta_1 \leq_t \delta_1 \wedge M_1 = M_1' \Rightarrow (\omega_{\delta_1'} \leq_t \omega_{\delta_1} \text{ and } \xi_{\delta_1'} \leq_t \xi_{\delta_1}).$$

The properties (1), (3), (4), and (5) hold correspondingly for the computable case but a computable version of (2) would require reducibility uniform in *i*. Therefore, we can only formulate the simple version,

$$(2') \quad \delta_1 \leq_c \delta_1' \Rightarrow \delta_1^\infty \leq_c \delta_1'^\infty.$$

Proof. (1)–(3) Let $\Gamma_i \in [\mathbb{F} \to \mathbb{F}]$ such that $\delta_i(p) = \delta_i' \Gamma_i(p)$ whenever $p \in \operatorname{dom} \delta_i$ and let f be partial recursive with $\nu(i) = \nu' f(i)$ for $i \in \operatorname{dom} \nu$.

Define, for (1), $\Gamma\langle p_0, \ldots, p_n\rangle := \langle \Gamma_0(p_0), \ldots, \Gamma_n(p_n)\rangle$, for (2), $\Gamma\langle p_i\rangle_i := \langle \Gamma_i(p_i)\rangle_i$, and, for (3), $\Gamma\langle i, p\rangle := \langle f(i), \Gamma_1(p)\rangle$.

(4) Suppose $\delta_1' = \delta_1 \Gamma$ and $\delta_2 = \delta_2' \Delta$. By the utm- and smn-theorem there is some total $\Sigma \in [\mathbb{F} \to \mathbb{F}]$ with

$$\tilde{\psi}_{\Sigma(p)}(q) = \Delta \tilde{\psi}_p \Gamma(q) \quad \text{for every } p, q \in \mathbb{F}.$$

Hence, for every $p \in \operatorname{dom}[\delta_1 \to \delta_2]$, $x \in M_1$,

$$[\delta_1 \to \delta_2](p)(x) = [\delta_1' \to \delta_2']\Sigma(p)(x).$$

(5) Suppose $\delta_1 = \delta_1' \Gamma$. There is some total $\Sigma \in [\mathbb{F} \to \mathbb{F}]$ with

$$\chi_{\Sigma(p)}(q) = \chi_p \Gamma(q) \quad \text{for every } p, q \in \operatorname{dom} \Gamma.$$

It follows that $\omega_{\delta'_1}(p) = \delta_1 \Gamma^{-1} \operatorname{dom} \chi_p = \omega_{\delta_1}\Sigma(p)$ for $p \in \operatorname{dom} \omega_{\delta_1}$ and $\xi_{\delta'_1}(q) = \xi_{\delta_1}\Sigma(q)$ whenever $q \in \operatorname{dom} \xi_{\delta'_1}$. $\square$

The class of representations with the relation $\leqslant_t$ ($\leqslant_c$) is a preorder and therefore for any set Y of representations the following sets are well-defined:

$$\mathrm{Sup}_t\ Y := \{\delta \mid \delta \text{ is a least upper bound of } Y \text{ w.r.t.} \leqslant_t\},$$

$$\mathrm{Inf}_t\ Y := \{\delta \mid \delta \text{ is a greatest lower bound of } Y \text{ w.r.t.} \leqslant_t\},$$

and accordingly $\mathrm{Sup}_c\ Y$ and $\mathrm{Inf}_c\ Y$.

Clearly Sup Y and Inf Y are either empty or consist exactly of a single equivalence class.

2.13. Theorem. *Let δ_1 and δ_2 be representations. Define $\underset{\sim}{\delta}$ and $\tilde{\delta}$ by*

$$\operatorname{dom} \underset{\sim}{\delta} := \{\langle p_1, p_2\rangle \mid p_i \in \operatorname{dom} \delta_i\ (i = 1, 2) \textit{ and } \delta_1 p_1 = \delta_2 p_2\},$$

$$\underset{\sim}{\delta}\langle p_1, p_2\rangle := \delta_1(p_1) \quad \textit{whenever } \langle p_1, p_2\rangle \in \operatorname{dom} \underset{\sim}{\delta},$$

and

$$\tilde{\delta}(p) := \begin{cases} \delta_1 q & \textit{if } p = 2q \textit{ and } q \in \operatorname{dom} \delta_1, \\ \delta_2 q & \textit{if } p = 2q+1 \textit{ and } q \in \operatorname{dom} \delta_2, \\ \operatorname{div} & \textit{otherwise} \end{cases}$$

(where $(2q)(n) = 2 \cdot q(n)$, etc.). Then

(1) $\underset{\sim}{\delta} \in \mathrm{Inf}_c\{\delta_1, \delta_2\} \subseteq \mathrm{Inf}_t\{\delta_1, \delta_2\}$,

(2) $\tilde{\delta} \in \mathrm{Sup}_c\{\delta_1, \delta_2\} \subseteq \mathrm{Sup}_t\{\delta_1, \delta_2\}$.

The proof is immediate, and therefore omitted.

Let M_1 and M_2 be the sets represented by δ_1 and δ_2. Then $\underset{\sim}{\delta}$ is a representation of $M_1 \cap M_2$ and $\tilde{\delta}$ represents $M_1 \cup M_2$. Therefore, we shall use the notations $\delta_1 \cap \delta_2 := \underset{\sim}{\delta}$ and $\delta_1 \cup \delta_2 := \tilde{\delta}$.

The following example explains the relation between the representations $\mathbb{M}$ and δ_{cf} of P_ω.

2.14. Example. Let $\mathbb{M}$ be the enumeration representation of P_ω and δ_{cf} be the representation of P_ω by characteristic functions. Then $\delta_{cf} \in \mathrm{Inf}_c\{\mathbb{M}, \mathbb{M}^c\}$ (where $\mathbb{M}^c(p) := \mathbb{N} \setminus \mathbb{M}_p$).

The proof is similar to the proof of "A set is recursive iff it is r.e. and its complement is r.e.".

3. Recursion-theoretical properties of representations

In this section, precompleteness (see [5]) is studied for representations. Most of the interesting representations are precomplete. The recursion theorem and Rice's theorem are consequences of precompleteness. Any representation δ of a set M induces a canonical numbering ν_δ of the computable elements of M. The relation between δ and ν_δ is studied.

We shall start with precompleteness and give some examples.

3.1. Definition. A representation $\delta:\mathbb{F} \dashrightarrow M$ is called t- (c-) *precomplete* iff for every (computable) $\Gamma \in [\mathbb{F} \to \mathbb{F}]$ there is some (computable) total $\Delta \in [\mathbb{F} \to \mathbb{F}]$ such that

$$\delta\Gamma(p) = \delta\Delta(p) \quad \text{whenever } p \in \operatorname{dom} \Gamma.$$

3.2. Examples. (1) *The enumeration representation* $\mathbb{M}:\mathbb{F} \to P_\omega$ *is* c-*precomplete.*

(2) *The representations* ψ, $\tilde{\psi}$ *and* χ *are* c-*precomplete.*

Proof. (1) Let $\Gamma:\mathbb{F} \dashrightarrow \mathbb{F}$ be computable and let $\bar{\Gamma}$ be an oracle-Turing-machine computing Γ (see [12]). For every input $p \in \mathbb{F}$ let $\bar{\Gamma}(p)(i) \in \mathbb{N} \cup \{\varepsilon\}$ be the information written onto the output tape by $\bar{\Gamma}$ at step i.

Define $\Gamma:\mathbb{F} \to \mathbb{F}$ by

$$\Delta(p)(i) := \begin{cases} 0 & \text{if } \bar{\Gamma}(p)(i) = \varepsilon, \\ \bar{\Gamma}(p)(i) & \text{otherwise.} \end{cases}$$

Then Δ is computable and $\mathbb{M}_{\Delta(p)} = \mathbb{M}_{\Gamma(p)}$ holds for every $p \in \operatorname{dom} \Gamma$. Note that $\mathbb{M}_{\Delta(p)}$ is finite if $p \notin \operatorname{dom} \Gamma$.

(2) See [4]. □

Note that if $\delta:\mathbb{F} \dashrightarrow M$ is precomplete and $\delta' = H \circ \delta$ for some $H: M \dashrightarrow M'$, then also $\delta':\mathbb{F} \dashrightarrow M'$ is precomplete. Therefore, any representation $\delta = H \circ \mathbb{M}$ where $H: P_\omega \dashrightarrow M$ is precomplete. Also, $[\delta \to \delta']$, ω_δ, and ξ_δ are precomplete for arbitrary representations δ and δ'.

3.3. Definition. Let $\delta:\mathbb{F} \dashrightarrow M$ be a representation.

δ *satisfies the* t- (c-) *recursion theorem* iff there is some total (computable) $\Omega \in [\mathbb{F} \to \mathbb{F}]$ such that $\delta\Omega(p) = \delta\tilde{\psi}_p\Omega(p)$ holds for every $p \in \mathbb{F}$ with $\tilde{\psi}_p$ total.

The precomplete representations are exactly those which satisfy the recursion theorem.

3.4. Theorem. *A representation is* t- (c-) *precomplete iff it satisfies the* t- (c-) *recursion theorem.*

Proof. ($\Rightarrow$): Let $\delta:\mathbb{F} \dashrightarrow M$ be c-precomplete. Since $\Gamma:\mathbb{F} \dashrightarrow \mathbb{F}$ with $\Gamma(p) := \tilde{\psi}_p(p)$ is computable, there is some computable $\Delta: \mathbb{F} \to \mathbb{F}$ such that

$$\delta\Delta(p) = \delta\tilde{\psi}_p(p) \quad \text{whenever } p \in \text{dom } \tilde{\psi}_p.$$

By the translation lemma there is some computable total $\Sigma: \mathbb{F} \to \mathbb{F}$ such that $(\forall p)$ $\tilde{\psi}_{\Sigma(p)} = \tilde{\psi}_p \circ \Delta$. Define $\Omega := \Delta \circ \Sigma$. Then Ω is computable and, for every $p \in \mathbb{F}$ with $\tilde{\psi}_p$ total,

$$\delta\tilde{\psi}_p\Omega(p) = \delta\tilde{\psi}_p\Delta\Sigma(p) = \delta\tilde{\psi}_{\Sigma(p)}\Sigma(p) = \delta\Delta\Sigma(p) = \delta\Omega(p).$$

$(\Leftarrow)$: Let $\delta: \mathbb{F} \dashrightarrow M$ satisfy the c-recursion theorem by some computable $\Omega: \mathbb{F} \to \mathbb{F}$. Let $\Gamma: \mathbb{F} \dashrightarrow \mathbb{F}$ be computable. Then there is some computable $\Sigma: \mathbb{F} \to \mathbb{F}$ such that $(\forall p, q)$ $\tilde{\psi}_{\Sigma(p)}(q) = \Gamma(p)$. Furthermore, $\tilde{\psi}_{\Sigma(p)}$ is total whenever $p \in \text{dom } \Gamma$ and therefore

$$\delta\Omega\Sigma(p) = \delta\tilde{\psi}_{\Sigma(p)}\Omega\Sigma(p) = \delta\Gamma(p).$$

Hence $\Delta := \Omega \circ \Sigma: \mathbb{F} \to \mathbb{F}$ has the desired property. $\square$

As in the case of numberings for precomplete representations the equivalence classes of names are inseparable.

3.5. Theorem. *Let $\delta: \mathbb{F} \dashrightarrow M$ be a* t- (c-) *precomplete representation and let $x, y \in M$, $x \neq y$.*
Then $\delta^{-1}\{x\}$ and $\delta^{-1}\{y\}$ are t- (c-) *effectively inseparable.*

Proof. Let $q \in \delta^{-1}\{x\}$, $q' \in \delta^{-1}\{y\}$. There is some computable $\Gamma: \mathbb{F} \dashrightarrow \mathbb{F}$ such that

$$\Gamma(p) = \begin{cases} q & \text{if } \chi_p(p) = 0, \\ q' & \text{if } p \in \text{dom } \chi_p \text{ and } \chi_p(p) \neq 0, \\ \text{div} & \text{otherwise.} \end{cases}$$

From [12] we know that $A_0 := \{p \mid \chi_p(p) = 0\}$ and $A_1 := \{p \mid \chi_p(p) = 1\}$ are c-effectively inseparable. Clearly, $\Gamma(A_0) \subseteq \delta^{-1}\{x\}$ and $\Gamma(A_1) \subseteq \delta^{-1}\{y\}$. Since δ is t-precomplete, there is some continuous total $\Delta: \mathbb{F} \to \mathbb{F}$ such that $\delta\Delta(p) = \delta\Gamma(p)$, whenever $p \in \text{dom } \Gamma$. Hence, $\Delta(A_0) \subseteq \delta^{-1}\{x\}$ and $\Delta(A_1) \subseteq \delta^{-1}\{y\}$ and therefore [12, Theorem 4.7] $\delta^{-1}\{x\}$ and $\delta^{-1}\{y\}$ are t-effectively inseparable. $\square$

Rice's theorem is a consequence. We only formulate the topological version since it is stronger than the computable one.

3.6. Corollary (Rice's theorem). *Let $\delta: \mathbb{F} \dashrightarrow M$ be* t-*precomplete and let $\emptyset \neq A \subsetneq M$.*
Then A is not δ-clopen.

Proof. Assume A is δ-clopen. Then there is some total $\Delta \in [\mathbb{F} \to \mathbb{F}]$ such that

$$\delta^{-1}(A) = \Delta^{-1}\{0\} \cap \text{dom } \delta \quad \text{and} \quad \delta^{-1}(M \setminus A) = \Delta^{-1}(\mathbb{N} \setminus \{0\}) \cap \text{dom } \delta.$$

Since $\Delta^{-1}\{0\}$ and $\Delta^{-1}(\mathbb{N} \setminus \{0\})$ are open and therefore not effectively inseparable, there is a contradiction (cf. [12, Theorem 4.7]). $\square$

3.7. Theorem. *Let* $\delta:\mathbb{F} \dashrightarrow M$ *be a* t- (c-) *precomplete representation and let* $A \subseteq M$. *Then* $\delta^{-1}(A) \not\leq_t \overline{\delta^{-1}(A)}$.

Proof. Assume $\delta^{-1}(A) \leq_t \overline{\delta^{-1}(A)}$, i.e., there is some total continuous $\Gamma:\mathbb{F} \dashrightarrow \mathbb{F}$ such that $(\forall p)\ \delta(p) \in A \Leftrightarrow \delta\Gamma(p) \notin A$.

Let $\Gamma = \tilde{\psi}_q$. Since δ satisfies the t-recursion theorem, there is some total continuous $\Omega:\mathbb{F} \to \mathbb{F}$ with

$$\delta\Omega(q) = \delta\tilde{\psi}_q\Omega(q) = \delta\Gamma\Omega(q).$$

Since $\delta\Omega(q) \in A \Leftrightarrow \delta\Gamma\Omega(q) \notin A$, there is a contradiction. □

Elements with computable names play a fundamental role for computability theory. For any representation δ there is a canonical numbering ν_δ of the computable elements.

3.8. Definition. Let δ be a representation of M and let $x \in M$.

x is called *δ-computable* if $x = \delta(p)$ for some recursive p. The induced numbering ν_δ of $M_c := \{x \in M \mid x \text{ is } \delta\text{-computable}\}$ is defined by $\nu_\delta(i) := \delta\varphi_i$.

3.9. Examples. (1) Let $\mathbb{M}$, δ_{cf} be the representations of P_ω as defined above. Then, for $X \subseteq \mathbb{N}$,

$$X \text{ is } \mathbb{M}\text{-computable} \Leftrightarrow X \text{ is r.e.},$$

$$X \text{ is } \delta_{cf}\text{-computable} \Leftrightarrow X \text{ is recursive}.$$

Furthermore, $\nu_{\mathbb{M}}$ is recursively isomorphic to the standard numbering W with $W_i = \mathrm{dom}\ \varphi_i$ of the r.e. sets (see [10]).

(2) Let $\delta_{\mathbb{P}}:\mathbb{F} \dashrightarrow \mathbb{P} := \{f:\mathbb{N} \dashrightarrow \mathbb{N}\}$ be defined by

$$\mathrm{dom}\ \delta_{\mathbb{P}} := \{p \in \mathbb{F} \mid (\exists f \in \mathbb{P})\ \mathbb{M}_p = \mathrm{graph}(f) = \{\langle i, j\rangle \mid j = f(i)\}\},$$

$$\delta_{\mathbb{P}}(p) := \mathrm{graph}^{-1}\mathbb{M}_p \quad \text{if } p \in \mathrm{dom}\ \delta_{\mathbb{P}}.$$

Then $f \in \mathbb{P}$ is $\delta_{\mathbb{P}}$-computable iff f is computable and $\nu_{\delta_{\mathbb{P}}}$ is recursively isomorphic to φ.

(3) Let δ, δ' be arbitrary representations of M (respectively M'). Let $f: M \dashrightarrow M'$ and let $A \subseteq M$. Then

$$f \text{ is } [\delta \to \delta']\text{-computable} \Leftrightarrow f \text{ is } (\delta, \delta')\text{-computable},$$

$$A \text{ is } \omega_\delta\text{-computable} \Leftrightarrow A \text{ is } \delta\text{-provable},$$

$$A \text{ is } \xi_\delta\text{-computable} \Leftrightarrow A \text{ is } \delta\text{-decidable}.$$

Computability w.r.t. representations forces computability w.r.t. the induced numbering of the computable elements.

3.10. Lemma. *Let δ, δ' be representations of M (respectively M'), let $A \subseteq M$ and let $f: M \dashrightarrow M'$.*

(1) *f (δ, δ')-computable $\Rightarrow$ the restriction of f to M_c is $(\nu_\delta, \nu_{\delta'})$-computable.*

(2) *A δ-provable (decidable) $\Rightarrow A \cap M_c$ is ν_δ-provable (decidable).*

Proof. (1) Using oracle-Turing-machines as computability model for computable operators $\Gamma: \mathbb{F} \dashrightarrow \mathbb{F}$, it is easy to show that for every computable $\Gamma: \mathbb{F} \dashrightarrow \mathbb{F}$ there is some recursive g such that $\Gamma\varphi_i = \varphi_{g(i)}$ for every i. Therefore, if $f\delta = \delta'\Gamma$, then $f\nu_\delta = f\delta\varphi = \delta'\Gamma\varphi = \nu_{\delta'}g$.

(2) Similar to (1). □

The converse of Lemma 3.10 does not hold in general but it follows from the Myhill-Shepherdson theorem [9] that for certain representations (e.g., $\mathbb{M}: \mathbb{F} \to P_\omega$, $\delta_{\mathbb{P}}: \mathbb{F} \to \mathbb{P}$) and total functions computability w.r.t. the induced numberings of the computable elements forces representation computability.

An immediate consequence of Lemma 3.10 is that '$\delta \leq_c \delta' \Rightarrow \nu_\delta \leq_m \nu_{\delta'}$' holds for arbitrary representations. Furthermore, ν_δ is precomplete for every representation δ because φ is precomplete. Since precomplete m-equivalent numberings are recursively isomorphic (see [5]) we get the following.

3.11. Corollary. $\delta \equiv_c \delta' \Rightarrow \nu_\delta$ *and* $\nu_{\delta'}$ *are recursively isomorphic.*

4. Representations of topological spaces

Let $\delta: \mathbb{F} \dashrightarrow M$ be a representation. Since $(\mathbb{F}, \tau)$, where τ is the set of all open subsets of $\mathbb{F}$, is a topological space, δ induces a topology τ_δ on M by

$$X \in \tau_\delta \ :\Leftrightarrow\ \delta^{-1}X = A \cap \operatorname{dom} \delta \quad \text{for some } A \in \tau.$$

τ_δ is called the final topology of δ. It is easy to see that τ_δ is the set of all δ-open subsets of M.

If on M a topology τ is already defined, then $\tau = \tau_\delta$ should hold for any 'reasonable' representation δ of M.

We give some examples of final topologies.

4.1. Example. *Let $\mathbb{M}$ be the enumeration representation of P_ω. Then the set $\{O_e \mid e \subseteq \mathbb{N}$, e finite$\}$, where $O_e := \{X \subseteq \mathbb{N} \mid e \subseteq X\}$ is a basis of $\tau_{\mathbb{M}}$. Furthermore, $\mathbb{M}$ is an open mapping w.r.t. $\tau_{\mathbb{M}}$.*

Proof. Obviously, $\{O_e \mid e \subseteq \mathbb{N}$, e finite$\}$ is a basis of a topology τ on P_ω. For $e \subseteq \mathbb{N}$ finite we have $\mathbb{M}^{-1}O_e = \bigcup\{[w] \mid (\forall i \in e)\ (\exists j)\ w(j) = i+1\}$ is open in $\mathbb{F}$.

On the other hand, for $w \in W(\mathbb{N})$, $M([w]) = O_e$ with $e = \{i \mid (\exists j)\ w(j) = i+1\}$. Therefore, $\mathbb{M}$ is continuous and open w.r.t. τ. □

4.2. Example. The final topology of $\delta_{\mathrm{cf}}:\mathbb{F} \dashrightarrow P_\omega$ can be characterized by the basis $\{O_{d,e} \mid d, e \subseteq \mathbb{N}, \text{ finite}\}$ where $O_{d,e} = \{X \subseteq \mathbb{N} \mid d \subseteq X \subseteq \mathbb{N}\setminus e\}$.

4.3. Example. The final topologies $\Pi_n:\mathbb{F} \to \mathbb{F}^{\mathbb{N}}$ and $\Pi_\infty:\mathbb{F} \to \mathbb{F}^{\mathbb{N}}$ are the product topologies of $\mathbb{F}$. Π_n and Π_∞ are homeomorphisms. The same holds for $\Pi^{-1}:\mathbb{F} \to \mathbb{N} \times \mathbb{F}$.

For convenience we shall use the following notations: If (M_i, τ_i) are topological spaces and M is an arbitrary set, then

- $\tau_1|_M := \{X \cap M \mid X \in \tau_1\}$ is the topology on M induced by τ_1,
- $\inf(\tau_1, \tau_2)$ is the topology on $M_1 \cap M_2$ with basis $\{X_1 \cap X_2 \mid X_i \in \tau_i \ (i = 1, 2)\}$,
- $\sup(\tau_1, \tau_2) := \{X \subseteq M_1 \cup M_2 \mid X \cap M_i \in \tau_i \ (i = 1, 2)\}$,
- $\bigotimes_i \tau_i$ is the product topology on $\times_i M_i$.

The next lemma describes the behaviour of final topologies w.r.t. reduction and product.

4.4. Lemma. *Let the δ_i's be representations of M_i with final topologies τ_i. Then*

(1) $\delta_1 \leq_{\mathrm{t}} \delta_2 \Rightarrow \tau_2|_{M_1} \subseteq \tau_1$ *especially* $\delta_1 \equiv_{\mathrm{t}} \delta_2 \Rightarrow \tau_1 = \tau_2$,

(2) $\sup(\tau_1, \tau_2) = \tau_{(\delta_1 \cup \delta_2)}$, $\inf(\tau_1, \tau_2) \subseteq \tau_{(\delta_1 \cap \delta_2)}$,

(3) $\tau_1 \otimes \cdots \otimes \tau_n \subseteq \tau[\delta_1, \ldots, \delta_n]$, $\bigotimes_i \tau_i \subseteq \tau_{[\delta_i]_i}$.

Proof. (1) Let $A_i := \operatorname{dom} \delta_i$ and let $\delta_i' := \delta_i|_{A_i}$. Suppose $\delta_1 \leq_{\mathrm{t}} \delta_2$. Then there is some continuous $\Sigma: A_1 \to A_2$ with $\delta_1' = \delta_2'\Sigma$. Therefore, for every $X \subseteq M_2$:

$$X \in \tau_2 \Rightarrow (\delta_2')^{-1}X \text{ is open} \Rightarrow \Sigma^{-1}(\delta_2')^{-1}X = (\delta_1')^{-1}(X \cap M_1) \text{ is open}$$
$$\Rightarrow X \cap M_1 \in \tau_2 \text{ holds for every } X \subseteq M_2.$$

(2) Suppose $X \in \sup(\tau_1, \tau_2)$. Then there are $V_i \subseteq W(\mathbb{N})$ with $\delta_i^{-1}(X \cap M_i) = \bigcup\{[w] \mid w \in V_i\}$ $(i = 1, 2)$.

Let $V := \{2w \mid w \in V_1\} \cup \{2w+1 \mid w \in V_2\}$. Then $(\delta_1 \cup \delta_2)^{-1}X = \bigcup\{[w] \mid w \in V\}$ and hence $X \in \tau_{(\delta_1 \cup \delta_2)}$.

Conversely, for $X \in \tau_{(\delta_1 \cup \delta_2)}$ follows $X \cap M_i \in \tau_i$ $(i = 1, 2)$ by (1), i.e., $X \in \sup(\tau_1, \tau_2)$.

Now let $X = X_1 \cap X_2$ where $X_i \in \tau_i$. Then, by (1), $\{X_1 \cap M_2, X_2 \cap M_1\} \subseteq \tau_{(\delta_1 \cup \delta_2)}$, hence $X = X_1 \cap M_2 \cap X_2 \cap M_1 \in \tau_{(\delta_1 \cap \delta_2)}$.

(3) Suppose $O_i \in \tau_i$ $(i \in \mathbb{N})$. Then $[\delta_i]_i(O_0 \times O_1 \times \cdots) = \pi^{(\infty)}(\delta_0^{-1}O_0 \times \delta_1^{-1}O_1 \times \cdots)$ is open in $\operatorname{dom}[\delta_i]_i$. Therefore, $O_0 \times O_1 \times \cdots \in \tau_{[\delta_i]_i}$.

The proof for $[\delta_1, \ldots, \delta_n]$ is similar. □

It should be noted that there are representations δ and δ' with $\tau_\delta = \tau_{\delta'}$ but not $\delta \equiv_{\mathrm{t}} \delta'$. Examples are the decimal representation and the standard representation of the real numbers.

In the examples above we characterized the final topologies for given representations. Now we shall define 'natural' representations for given topologies. The spaces we consider are separable T_0-spaces.

(A topological space is separable iff it has a countable basis. It is a T_0-space iff any two points can be distinguished by open sets.)

4.5. Definition. Let (M, τ) be a separable T_0-space and let U be a numbering of a basis of τ. For $x \in M$ let $\varepsilon_u(x) := \{i \in \mathbb{N} \mid x \in U_i\}$. A *standard representation* $\delta_u : \mathbb{F} \dashrightarrow M$ of (M, τ) is defined by $\operatorname{dom} \delta_u := \mathbb{M}^{-1}\varepsilon_u(M)$ and $\delta_u(p) := \varepsilon_u^{-1}\mathbb{M}_p$ whenever $p \in \operatorname{dom} \delta_u$.

Since τ is a T_0-space, $\varepsilon_u : M \to P_\omega$ is injective and therefore $\delta_u(p)$ is well-defined. A standard representation of a separable T_0-space has remarkable properties.

4.6. Theorem. *Let (M, τ), U, and δ_u be as above. Then*

(1) *δ_u is continuous and open, especially $\tau = \tau_{\delta_u}$.*

(2) *For any topological space (M', τ') and any $H : M \dashrightarrow M'$, $H\delta_u$ continuous $\Rightarrow H$ continuous.*

(3) *$\zeta \leqslant_t \delta_u$ for any continuous $\zeta : \mathbb{F} \dashrightarrow M$. (Note that ζ is a representation of* range $\zeta \subseteq M$.)

Proof. (1) It is easy to see that $\varepsilon_u : M \to P_\omega$ is continuous and open. Since the same holds for the representation $\mathbb{M} : \mathbb{F} \to P_\omega$, also δ_u is open and continuous.

(2) Immediate from (1).

(3) Let $\zeta : \mathbb{F} \dashrightarrow M$ be continuous. Then

$$(\forall n)\ (\forall p \in \operatorname{dom} \zeta)\ (\zeta(p) \in U_n \Leftrightarrow (\exists k)\ \zeta[p^{[k]}] \subseteq U_n).$$

There is some continuous $\Delta : \mathbb{F} \to \mathbb{F}$ such that

$$\mathbb{M}_{\Delta p} = \{n \mid (\exists k)\ \zeta[p^{[k]}] \subseteq U_n\}.$$

Therefore, $(\forall p \in \operatorname{dom} \zeta)\ \zeta(p) = \delta_u \Delta(p)$. $\square$

An immediate consequence of Theorem 4.6 is that all the standard representations of a separable T_0-space are topologically equivalent. Therefore, the equivalence class $\{\delta \mid \delta \equiv_t \delta_u\}$ is independent of the numbering U.

Since t-equivalent representations induce the same kind of continuity theory, the following definition is reasonable.

4.7. Definition. Let δ be a representation of a separable T_0-space (M, τ). δ is t-*effective* (admissible) w.r.t. τ iff $\delta \equiv_t \delta_u$ for some standard representation δ_u.

Clearly, the admissible representations of (M, τ) form exactly the equivalence class $\{\delta \mid \delta \equiv_t \delta_u\}$ for arbitrary U.

4.8. Corollary. *Let δ represent a separable* T_0*-space* (M, τ).
δ is admissible
$\Leftrightarrow$ *δ is continuous and $\zeta \leq_t \delta$ for any continuous $\zeta : \mathbb{F} \dashrightarrow M$*
$\Leftrightarrow$ *δ is continuous and $\delta_u \leq_t \delta$ for some numbering U of a basis of τ.*

Examples for admissible representations are the following:
(1) The enumeration representation $\mathbb{M}$ of P_ω.
(2) The representations $\Pi_n : \mathbb{F} \to \mathbb{F}^n$, $\Pi_\infty : \mathbb{F} \to \mathbb{F}^{\mathbb{N}}$, and $\Pi^{-1} : \mathbb{F} \to \mathbb{N} \times \mathbb{F}$.
(3) The representations of effective cpo's defined by Weihrauch and Schäfer [13].

Proof of Corollary 4.8. (1) By $U_i := \{A \subseteq \mathbb{N} \mid D_i \subseteq \mathbb{N}\}$ a numbering U of a basis of $\tau_{\mathbb{M}}$ can be defined. We prove $\delta_u \leq_c \mathbb{M}$.

By definition, $\delta_u(p) = \varepsilon_u^{-1}\mathbb{M}_p = \bigcup \{D_i \mid i \in \mathbb{M}_p\}$ if $p \in \mathrm{dom}\ \delta_u$. Let

$$\Gamma(p)\langle i, j, k\rangle := \begin{cases} k+1 & \text{if } p(j) = i+1 \text{ and } k \in D_j, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\Gamma : \mathbb{F} \to \mathbb{F}$ is computable and $\mathbb{M}\Gamma(p) = \delta_u(p)$ for every $p \in \mathrm{dom}\ \delta_u$.

(2) Since Π_n is a homeomorphism for any continuous $\zeta : \mathbb{F} \dashrightarrow \mathbb{F}^n$, $\zeta(p) = \Pi_n(\Pi^{(n)}\zeta)(p)$ if $p \in \mathrm{dom}\ \zeta$, and therefore $\zeta \leq_t \Pi_n$. The proofs for Π_∞ and Π^{-1} are similar. □

Note that a representation can be admissible only w.r.t. its final topology. Clearly, there are representations of separable T_0-spaces which are not admissible. The decimal representation of real numbers is an example.

Topological continuity and continuity w.r.t. admissible representations are closely related.

4.9. Theorem. *Let (M_i, τ_i) be separable* T_0*-spaces and let $\delta_i : \mathbb{F} \dashrightarrow M_i$ be admissible representations $(i = 1, 2)$. Let $\mathbb{F} = M_1 \dashrightarrow M_2$.*
(1) *F is (τ_1, τ_2)-continuous $\Leftrightarrow$ F is weakly (δ_1, δ_2)-continuous.*
(2) *F is (τ_1, τ_2)-continuous and* dom *$F \in G_\delta(\tau_1) \Rightarrow F$ is (δ_1, δ_2)-continuous.*

Proof. W.l.o.g. we may assume δ_1 and δ_2 to be standard representations.

(1) Let $F : M_1 \dashrightarrow M_2$ be (τ_1, τ_2)-continuous and let $\delta' := F \circ \delta_1$. Then $\delta' : F \dashrightarrow M$ is continuous and, by Theorem 4.6(3), $\delta' \leq_t \delta_2$. I.e., $F\delta_1(p) = \delta_2\Gamma(p)$ for all $p \in \mathrm{dom}\ F\delta_1$ with some continuous Γ. Conversely, let F be weakly (δ_1, δ_2)-continuous, i.e., $F\delta_1 = \delta_2\Gamma$ for some continuous Γ. Since δ_2 is continuous, the same holds for $F\delta_1$ and by Theorem 4.6(2) also F is continuous.

(2) Let F be continuous and $\mathrm{dom}\ F \in G_\delta(\tau_1)$, i.e., $\mathrm{dom}\ F = \bigcap_{i \in \mathbb{N}} O_i$, where $O_i \in \tau_1$ for $i \in \mathbb{N}$. Since δ_1 is continuous, there are sets O_i' open in $\mathbb{F}$ such that $\mathrm{dom}\ F\delta_1 = \delta_1^{-1}\ \mathrm{dom}\ F = \bigcap_{i \in \mathbb{N}} O_i' \cap \mathrm{dom}\ \delta_1$.

By (1) there is some $\Gamma \in [\mathbb{F} \to \mathbb{F}]$ with $F\delta_1(p) = \delta_2 \Gamma(p)$ for all $p \in \mathrm{dom}\ F\delta_1$. Now let Γ_1 be the restriction of Γ to the G_δ-set $\bigcap O_i$. Then, $\mathrm{dom}\ \Gamma_1 = \mathrm{dom}\ \Gamma \cap \bigcap O_i$ is a G_δ-set and hence $\Gamma_1 \in [\mathbb{F} \to \mathbb{F}]$, and, for every $p \in \mathrm{dom}\ \delta_1$,

$$p \in \mathrm{dom}\ F\delta_1 \Rightarrow \delta_2 \Gamma_1(p) = F\delta_1(p)$$

and

$$p \notin \mathrm{dom}\ F\delta_1 \Rightarrow p \notin \mathrm{dom}\ \Gamma_1.$$

This means that F is strongly (δ_1, δ_2)-continuous. □

For some special representations, the converse of (2) also holds. We shall introduce such a representation of the real numbers in a following paper.

For (strongly) (δ, δ')-continuous functions also an effective version of (τ, τ')-continuity can be shown. Let U (U') be a numbering of some basis of τ (τ') with $(\forall i, j)\ (\exists k)\ U_i \cap U_j = U_k$ and let $\omega_{\mathrm{u}}(p) := \bigcup\{U_i \mid i \in \mathbb{M}_p\}$ be the 'natural' representation of τ. A function $F: (M, \tau) \dashrightarrow (M', \tau')$ is called effectively (τ, τ')-continuous iff $F^{-1}: \tau' \to \tau|_{\mathrm{dom}\ F}$ is $(\omega_{\mathrm{u'}} \omega_{\mathrm{u}} | \mathrm{dom}\ F)$-continuous.

4.10. Lemma. *Let δ be an admissible representation of (M, τ) and let U be a numbering of a basis of τ with $(\forall i, j)\ (\exists k)\ U_i \cap U_j = U_k$. Then $\omega_{\mathrm{u}} \equiv_{\mathrm{t}} \omega_\delta$.*

Proof. W.l.o.g. assume $\delta = \delta_{\mathrm{u}}$. Then since δ is continuous and open,

$$\delta(p) \in \omega_{\mathrm{u}}(q) \Leftrightarrow (\exists j \in \mathbb{M}_q)\ (\exists k)\ \delta[q^{[k]}] \subseteq U_j$$

holds whenever $p \in \mathrm{dom}\ \delta$, $q \in \mathbb{F}$. Using the smn-theorem one can easily construct some total $\Sigma \in [\mathbb{F} \to \mathbb{F}]$ such that $\mathrm{dom}\ \chi_{\Sigma(q)} = \{p \mid \delta(p) \in \omega_{\mathrm{u}}(q)\}$, i.e., $\omega_{\mathrm{u}}(q) = \omega_\delta \Sigma(q)$ for all $q \in \mathbb{F}$. Conversely, for $p \in \mathrm{dom}\ \omega_\delta$, $\omega_\delta(p) = \bigcup\{\delta[w] \mid [w] \subseteq \mathrm{dom}\ \chi_p\}$ holds. Since for $w \in W(\mathbb{N})$ there is some $j \in \mathbb{N}$ with $\delta[w] = U_j$, there is some $\Gamma \in [\mathbb{F} \to \mathbb{F}]$ with $\mathbb{M}_{\Gamma(p)} = \{j \mid (\exists [w] \subseteq \mathrm{dom}\ \chi_p)\ \delta[w] = U_j\}$ and hence $\omega_\delta(p) = \omega_{\mathrm{u}} \Gamma(p)$ whenever $p \in \mathrm{dom}\ \omega_\delta$. □

An immediate consequence is that the notion of effective (τ, τ')-continuity is independent of the numbering U.

4.11. Theorem. *Let δ (δ') be admissible representations of (M, τ) (respectively (M', τ')).*

A function $F: M \dashrightarrow M'$ is (δ, δ')-continuous iff it is effectively (τ, τ')-continuous.

Proof. Assume $\delta = \delta_{\mathrm{u}}$ $(\delta' = \delta_{\mathrm{u'}})$. Then there are $\Delta, \Delta' \in [\mathbb{F} \to \mathbb{F}]$ with $\omega'_{\mathrm{u}}(p) = \omega_{\delta'} \Delta'(p)$, $\omega_\delta(q) = \omega_{\mathrm{u}} \Delta(q)$ whenever $p \in \mathbb{F}$, $q \in \mathrm{dom}\ \omega_\delta$. Let F be (δ, δ')-continuous by $\Gamma \in [\mathbb{F} \to \mathbb{F}]$. Then, by the smn-theorem there is some total $\Sigma \in [\mathbb{F} \to \mathbb{F}]$ with $\chi_{\Sigma(p)} = \chi_p \Gamma \Delta'$ and hence

$$F^{-1} \omega_{\mathrm{u'}}(p) = F^{-1} \omega_{\delta'} \Delta'(p) = \omega_\delta \Sigma(p) \cap \mathrm{dom}\ F = \omega_{\mathrm{u}} \Delta \Sigma(p) \cap \mathrm{dom}\ F$$

whenever $p \in \mathrm{dom}\ \omega_{\mathrm{u'}}$, i.e., F^{-1} is $(\omega_{\mathrm{u'}}, \omega_{\mathrm{u}} | \mathrm{dom}\ F)$-continuous.

Conversely, let F^{-1} be $(\omega_{u'}, \omega_u|\text{dom } F)$-continuous by Ω and let $r \in \mathbb{F}$ with $\mathbb{M}_{\varphi_{r(i)}} = \{i\}$. Then,

$$F^{-1}U_i = F^{-1}\omega_u(\varphi_{r(i)}) = \bigcup\{U_j \mid j \in \mathbb{M}_{\Omega\varphi_{r(i)}}\}$$

and therefore $F\delta(p) \in U'_i \Leftrightarrow (\exists j \in \mathbb{M}_{\Omega\varphi_{r(i)}})\ \delta(p) \in U_j$.

Choose $\Gamma \in [\mathbb{F} \to \mathbb{F}]$ such that

$$\mathbb{M}_{\Gamma(p)} = \{i \mid (\exists j \in \mathbb{M}_p)\ j \in \mathbb{M}_{\Omega\varphi_{r(i)}}\}, \qquad (\forall i \in \mathbb{N})\ (\forall p \in \mathbb{F})\ \Gamma(p)(i) \neq 0.$$

Then, for $p \in \text{dom } \delta$,

$$p \in \text{dom } F\delta \Rightarrow \delta'\Delta(p) = F\delta(p)$$

and

$$p \notin \text{dom } F\delta \Rightarrow \mathbb{M}_{\Delta(p)} = \emptyset \Rightarrow \Delta(p) = \text{div}.$$

Hence F is (δ, δ')-continuous. □

Also Theorem 4.11 shows that our approach is very consistent.

Separable complete metric spaces are important in analysis and functional analysis. The following example gives a direct construction of an admissible representation of a separable complete metric space for which a numbering of a dense subset is given.

4.12. Example (Separable complete metric space). *Let (M, d) be a metric space and let β be a numbering of a dense subset $C \subseteq M$. Then all the elements of M can be represented by Cauchy-sequences on C. This idea induces a representation δ_c of M by*

$$\text{dom } \delta_c = \{p \mid (\beta p(i))_{i\in\mathbb{N}} \text{ is a Cauchy-sequence}\}$$

and

$$\delta_c(p) = \lim \beta p(i) \quad \text{for all } p \in \text{dom } \delta_c.$$

But δ_c is not admissible because the final topology of δ_c is trivial (i.e., $\tau_{\delta_c} = \{\emptyset, M\}$). A second condition on the domain of the representations forcing the speed of convergence gives a satisfactory result:

Define $\delta_{NC} : \mathbb{F} \dashrightarrow M$ by

$$\text{dom } \delta_{NC} = \{p \mid (\forall i)\ d(\beta p(i+1), \beta p(i)) \leq 2^{-i}\}$$

and

$$\delta_{NC}(p) = \delta_c(p) \quad \text{for } p \in \text{dom } \delta_{NC}.$$

Then δ_{NC} is admissible w.r.t. the topology τ_d induced on M by d.

Proof. (1) Suppose $X \neq \emptyset$ is δ_c-open, i.e., there is some $A \subseteq W(\mathbb{N})$ with $\delta_c^{-1}X = \bigcup\{[w] \mid w \in A\} \cap \text{dom } \delta_c$. Choose an arbitrary $y \in M$. Then for every $v \in A$ there is some $p \in [v]$ with $\delta_c(p) = y$. Hence, $y \in X$, i.e., $X = M$.

(2) Let $B_{\langle i,j\rangle} := \{x \in M \mid d(x, \beta_i) < 2^{-j}\}$. Then the set $\{B_{\langle i,j\rangle} \mid \langle i,j\rangle \in \mathbb{N}\}$ is a basis of τ_δ and the corresponding standard representation δ_B satisfies

$$\operatorname{dom} \delta_B = \{p \mid (\exists x \in M)\ \mathbb{M}_p = \{\langle i,j\rangle \mid d(x, \beta_i) < 2^{-j}\}\},$$

$$\{\delta_B(p)\} = \bigcap\{B_{\langle i,j\rangle} \mid \langle i,j\rangle \in \mathbb{M}_p\}.$$

We show $\delta_B \equiv_t \delta_{\mathrm{NC}}$.

Since

$$\delta_{\mathrm{NC}}(p) \in B_{\langle i,j\rangle} \Leftrightarrow (\exists k)\ d(\beta_{p(k)}, \beta_i) < 2^{-j} - 2^{-k},$$

it is easy to construct some continuous $\Gamma : \mathbb{F} \to \mathbb{F}$ with $\mathbb{M}_{\Gamma(p)} = \{\langle i,j\rangle \mid \delta_{\mathrm{NC}}(p) \in B_{\langle i,j\rangle}\}$ and hence $\delta_{\mathrm{NC}}(p) = \delta_B \Gamma(p)$ whenever $p \in \operatorname{dom} \delta_{\mathrm{NC}}$.

Conversely, there is some computable $\Delta : \mathbb{F} \dashrightarrow \mathbb{F}$ such that, for $p \in \mathbb{F}, j \in \mathbb{N}$, $\Delta(p)(j) = \mu_i[\langle i, j+1\rangle \in \mathbb{M}_p]$. Suppose $p \in \operatorname{dom} \delta_B$. Then $p \in \operatorname{dom} \Delta$ and

$$d(\beta_{\Delta(p)(j)}, \beta_{\Delta(p)(j+1)}) \leq 2^{-j} \quad \text{for every } j \in \mathbb{N}.$$

Therefore, $\delta_B(p) = \delta_{\mathrm{NC}} \Delta(p)$. □

The following theorem gives closure properties of admissible representations.

4.13. Theorem. *Let the δ_i's be admissible representations of (M_i, τ_i) $(i \in \mathbb{N})$.*

(1) *$[\delta_1, \ldots, \delta_n]$ is an admissible representation of $(M_1 \times \cdots \times M_n, \tau_1 \otimes \cdots \otimes \tau_n)$.*

(2) *$[\delta_i]_i$ is an admissible representation of $(\times_i M_i, \times_i \tau_i)$.*

(3) *Any $\delta \in \mathrm{Inf}_t(\delta_1, \delta_2)$ is admissible w.r.t. $(M_1 \cap M_2, \inf(\tau_1, \tau_2))$.*

Proof. W.l.o.g. let the δ_i's be standard representations.

(1) Similar to (2), see below.

(2) By Lemma 4.4, $\tau_{[\delta_i]_i} \supseteq \bigotimes_i \tau_i$, i.e., $[\delta_i]_i$ is continuous. Let $\zeta : \mathbb{F} \dashrightarrow \times_i M_i$ be continuous, let $\zeta_i := \mathrm{pr}_i \zeta : \mathbb{F} \dashrightarrow M_i$. Then, for any $i \in \mathbb{N}$, ζ_i is continuous and therefore $\zeta_i = \delta_i \Gamma_i$ for some continuous $\Gamma_i : \mathbb{F} \dashrightarrow \mathbb{F}$. Let $\Gamma(p) = \langle \Gamma_i(p)\rangle_i$. Then Γ is continuous and satisfies $\zeta(p) = [\delta_i]_i \Gamma(p)$. Hence, $\zeta \leq_t [\delta_i]_i$ and, by Corollary 4.8, $[\delta_i]_i$ is admissible.

(3) Choose $\delta = \delta_1 \cap \delta_2$. Then, by Lemma 4.4, δ is continuous. Then $\zeta(p) = \delta_i \Gamma_i(p)$ for some continuous $\Gamma_i : \mathbb{F} \dashrightarrow \mathbb{F}$. Let $\Gamma(p) := \langle \Gamma_1(p), \Gamma_2(p)\rangle$. Then, $\zeta(p) = \delta \Gamma(p)$ for all $p \in \operatorname{dom} \varphi$, i.e., $\zeta \leq_t \delta$. Hence, δ is admissible and the same holds for $\delta \in \mathrm{Inf}_t(\delta_1, \delta_2)$. □

Note that, by Lemma 4.4, $\sup(\tau_1, \tau_2)$ is the final topology of $\delta_1 \cup \delta_2$ but generally $\delta_1 \cup \delta_2$ is not admissible. For counter-examples, see [8] or the following paper.

We now give a final example: We have already proved that $\mathbb{M} : \mathbb{F} \to P_\omega$ is admissible and that the sets $O_e := \{X \subseteq \mathbb{N} \mid e \subseteq X\}$ where $e \subseteq \mathbb{N}$ finite form a basis of $\tau_{\mathbb{M}}$. Similarly it can be shown that $\mathbb{M}^c$ is admissible with basis sets $U_e := \{X \subseteq \mathbb{N} \mid X \subseteq \mathbb{N} \setminus e\}$.

Hence, the representation δ_{cf} of P_ω is also admissible and its final topology is generated by $O_{d,e} := \{X \subseteq \mathbb{N} \mid d \subseteq X \subseteq \mathbb{N} \setminus e\}$.

5. Conclusion

This paper presents basic definitions and properties of the theory of representations as a tool for further research. In Section 2, the effectivity of subsets and functions relative to given representations is studied. It is shown that a representation is defined uniquely up to t- (c-) equivalence by the topological (computational) properties induced by it. Several standard constructions of new representations from given ones are introduced and it is shown that these constructions respect reducibility. Finally, it is shown that for any two representations the supremum and infimum exist. In Section 3, some recursion-theoretical properties are investigated. The recursion theorem for precomplete representations is proved and different versions of Rice's theorem are derived. It is shown that the concepts of computability introduced so far are consistent, and finally the numbering derived from a representation is considered. In Section 4, topological properties are investigated. Every representation induces a topology on the represented set, the final topology. For any separable T_0-space there is a distinguished uniquely defined (up to t-equivalence) representation which is called admissible. Admissible representations have very satisfactory properties some of which are investigated.

Especially it is shown that continuity and continuity w.r.t. admissible representations are reasonably related. There is no doubt that the admissible representations are the most reasonable ones for separable T_0-spaces. In the case of the real numbers $\mathbb{R}$ with standard topology, several representations such as the decimal representation are not admissible. The concept of admissibility leads to standard representations of the L^p-spaces [14] and other separable T_0-spaces from functional analysis. Therefore, the theory of constructive functional analysis is well-defined. Representations are also useful in constructive analysis since without using intuitionistic logic it can be studied whether mathematical objects (sets, functions, predicates etc.) are constructive or not. A unified approach to constructive (and recursive) analysis may serve as intermediary between traditional 'idealistic' mathematics (not concerning with constructivity) and intuitionistic mathematics [1, 2], which does not accept nonconstructive objects and proofs. This area will be investigated in a forthcoming paper.

The study of computational complexity [6, 7] is a further application of representations. Even a canonical approach to constructive measure theory (on $\mathbb{R}$) is possible by using an appropriate separable metric space. It should be mentioned that computability properties of the (t-) admissible representation δ_U depend on the numbering U, which should be chosen as 'c-effective' as possible. A general rule does not seem to exist.

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