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TYPE 2 RECURSION THEORY

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Abstract. An attempt is made to lay a basis for a general, unified, concise, and simple theory of computable and continuous functions from $\mathbb{F}$ to $\mathbb{F}$ or $\mathbb{N}$, where $\mathbb{F}=\{f:\mathbb{N}\to\mathbb{N}\}$. The theory is formally very similar to ordinary recursion theory. It splits into a purely topological version and more special theory of computability. The basic definitions are given and fundamental properties are proved. As an example it is shown how the theory of recursively enumerable subsets of $\mathbb{N}$ can be transferred to a theory of open and a theory of computably open subsets of $\mathbb{F}$.

1. Introduction

Ordinary recursion theory or Type 1 recursion theory, i.e., the theory of computability on denumerable sets, is well established. In this theory, first computability of functions on some standard set, usually $\mathbb{N}$, is defined explicitly. Then via numberings the concepts are transferred to functions on sets different from this standard set. There are many good presentations of ordinary recursion theory in textbooks. One of the best references is still the book by Rogers [11]. The best reference to the theory of numberings is Ershov's paper [5].

The situation is different for Type 2 computability, i.e., computability on sets with cardinality not greater than that of the continuum. Typical sets of this kind are $2^{\mathbb{N}}$, $\Sigma^{\mathbb{N}}$ for a finite set Σ, $\mathbb{F}:=\mathbb{N}^{\mathbb{N}}:=\{f:\mathbb{N}\to\mathbb{N}\}$, $\mathbb{P}:=\{f:\mathbb{N}\dashrightarrow\mathbb{N}\}$ (a dashed arrow indicates partial functions), general effective cpo's [4], $\mathbb{O}:=$ the set of countable ordinals; $\mathbb{R}:=$ set of real numbers, $O(\mathbb{R}):=$ set of open subsets of $\mathbb{R}$, etc. There are several explicit definitions of computable operators on such sets which are equivalent or at least dependent from each other (see, e.g., [4, 11, 12]) and much is known about such operators but seemingly there is no generally accepted approach as in the case of ordinary recursion theory. This paper attempts to lay a basis for a unified and concise Type 2 recursion theory. We have chosen the set $\mathbb{F}$ of sequences of natural numbers as the standard set and define computability of functions on $\mathbb{F}$ explicitly. Computability on other sets S can then be derived from this Type 2 recursion theory on $\mathbb{F}$ via representations $\delta:\mathbb{F}\to S$. The theory of representations will be developed in a second paper. Equivalent Type 2 theories can be obtained by starting with sets like $2^{\mathbb{N}}$ with Scott's topology, $\{0,1\}^{\mathbb{N}}$ with Cantor's topology or $\mathbb{P}$ with the usual cpo-topology. The choice of $\mathbb{F}$ is a compromise with regard to simplicity, generality, and concreteness. This paper and a forthcoming one on representations shall demonstrate that the formalism based on $\mathbb{F}$ is simple and sufficiently general. The theory with $2^{\mathbb{N}}$ might be simpler in some cases [12]. But, in practice, infinite objects are usually defined as limits of sequences of finite objects and not by sets of finite objects. For this reason, $\mathbb{F}$ is more adequate for representations and a concrete representation for computers, e.g., by $\Sigma^{\mathbb{N}}$, can be easily derived. The approach to computability on $\mathbb{F}$ is a special case of more general concepts (e.g., Ershov's effective and complete fo-spaces [6] or effectively given domains [4]). It is the spirit of this approach first to develop a very simple (but sufficiently general) theory of continuity and computability on a standard Type 2 set (namely $\mathbb{F}$). Then, by means of representations, the results can be used as a basis for other theories, e.g., computable analysis, cpo-theory [4], higher type theory [6, 7, 10].

Type 2 recursion theory on $\mathbb{F}$ turns out to be formally similar to ordinary recursion theory. Remarkably, there is a slightly more general topological version of this theory. Since topological considerations are fundamental for Type 2 theory, we develop the two versions, the topological and the computational one, simultaneously. Since an exhaustive development of the theory is out of the scope of a simple publication like this and since many interesting questions are not yet answered, in this paper the basic definitions are given, several fundamental properties are proved, and it is shown by examples how ordinary recursion theory can be transferred to two versions of Type 2 theory, a topological and a recursive one. For some of the proofs only an outline is given. More detailed proofs have been elaborated by Dettki and Schuster [3].

It is assumed that the reader is familiar with ordinary recursion theory. As a main reference, the book by Rogers [11] is suggested. But also other books on recursion theory are suitable. Some notations will be used throughout this paper. By $f: A \dashrightarrow B$ (with dashed arrow) a partial function from A to B is denoted, where 'partial' means $\mathrm{dom}(f) \subseteq A$ and not necessarily $\mathrm{dom}(f) = A$. By φ the standard numbering of $P^{(1)}$, the unary partial recursive functions, is denoted. As usual we write $\langle i_1, \ldots, i_n\rangle$ instead of $\pi^{(n)}(i_1, \ldots, i_n)$ where $\pi^{(n)}: \mathbb{N}^n \to \mathbb{N}$ is Cantor's n-tuple bijection. If X is a set, $W(X)$ denotes the set of all words over X. The empty word is denoted by ε, and $\lg(w)$ is the length of the word w. If $w \in W(X)$ and if $w = x_0x_1 \ldots x_n$ (where $x_i \in X$), then we define $w(i) := x_i$ for any i, $0 \leq i \leq n$.

By $\mathbb{N}$ we denote the set of natural numbers $\{0, 1, 2, \ldots\}$. Define $\mathbb{F} := \mathbb{N}^{\mathbb{N}}$ and $\mathbb{B} := W(\mathbb{N}) \cup \mathbb{F}$. Thus, $\mathbb{B}$ is the set of all finite and infinite (ω-)words over $\mathbb{N}$. On $\mathbb{B}$ a (partial) order is defined by $b \sqsubseteq c :\Leftrightarrow b$ is a prefix of c. (*Remark*: $(\mathbb{B}, \sqsubseteq, \varepsilon)$ is a cpo, see, e.g., Egli and Constable [4].) For any $p \in \mathbb{F}$ and $i \in \mathbb{N}$ define $p^{[i]} := p(0) \ldots p(i-1) \in W(\mathbb{N})$. For any $v \in W(\mathbb{N})$ define $[v] = \{p \in \mathbb{F} \mid v \sqsubseteq p\}$. A function $\gamma: W(\mathbb{N}) \to W(\mathbb{N})$ is isotone if $v \sqsubseteq w \Rightarrow \gamma(v) \sqsubseteq \gamma(w)$. On the set $\mathbb{B}$, a standard topology is defined by the basis $\{O_v \mid v \in W(\mathbb{N})\}$ where $O_v = \{b \in \mathbb{B} \mid v \sqsubseteq b\}$. On $\mathbb{F}$ we consider the induced topology, i.e., $\{[v] \mid v \in W(\mathbb{N})\}$ is a basis of it. This is the well-known Baire's topology. On $\mathbb{N}$ we consider the discrete topology.

2. The standard representation of $[\mathbb{F}\to\mathbb{B}]$

The existence of an 'effective' numbering $\varphi:\mathbb{N}\to P^{(1)}$ of the unary partial recursive functions is one of the most fundamental properties in ordinary recursion theory. Here, we shall define an 'effective' representation $\psi:\mathbb{F}\to[\mathbb{F}\to\mathbb{B}]$ of the set of all continuous functions from $\mathbb{F}$ to $\mathbb{B}$, which plays a similar role. Here, the empty word $\varepsilon\in\mathbb{B}$ corresponds to 'undefined' in Type 1 recursion theory. The theory of cpo's [4] provides a general background for both theories. The following lemma (which can be generalized to appropriate cpo's) is the key to the definition of ψ.

2.1. Lemma. (1) *Let $\gamma: W(\mathbb{N})\to W(\mathbb{N})$ be an isotone function. Then the function $\bar{\gamma}:\mathbb{F}\to\mathbb{B}$, defined by $\bar{\gamma}(p):=\sup\{\gamma(w)\mid w\sqsubseteq p\}$, is continuous.*

(2) *Let $\Gamma:\mathbb{F}\to\mathbb{B}$ be a continuous function. Then there is some isotone $\gamma: W(\mathbb{N})\to W(\mathbb{N})$ with $\Gamma=\bar{\gamma}$.*

Proof. (1) Suppose, $p\in\mathbb{F}$, $q\in\mathbb{B}$ with $\bar{\gamma}(p)=q$. It is sufficient to show that for any $v\in W(\mathbb{N})$ with $q\in O_v$ there is some $x\in\mathbb{N}$ with $p\in[x]$ such that $\bar{\gamma}[x]\subseteq O_v$. Suppose, $v\in W(\mathbb{N})$ with $q\in O_v$. This implies $v\sqsubseteq q=\bar{\gamma}(p)=\sup\{\gamma(w)\mid w\sqsubseteq p\}$. Therefore, there is some $x\in W(\mathbb{N})$ with $v\sqsubseteq\gamma(x)$ and $p\in[x]$. Suppose, $p'\in[x]$. From $\bar{\gamma}(p')=\sup\{\gamma(w)\mid w\sqsubseteq p'\}$ and $x\sqsubseteq p'$ we conclude $v\sqsubseteq\gamma(x)\sqsubseteq\bar{\gamma}(p')$, hence $\bar{\gamma}(p')\in O_v$. This implies $\bar{\gamma}[x]\subseteq O_v$. Therefore, $\bar{\gamma}$ is continuous.

(2) For any $w\in W(\mathbb{N})$ define

$$M_w:=\{y\in W(\mathbb{N})\mid \lg(y)\leq\lg(w)\wedge\Gamma[w]\subseteq O_y\}.$$

Since M_w is finite, $\varepsilon\in M_w$, and ($y\sqsubseteq y'$ or $y'\sqsubseteq y$) if $O_y\cap O_{y'}\neq\emptyset$, $\max(M_w)$ exists. Define $\gamma(w):=\max(M_w)$. Obviously, γ is isotone. Suppose $p\in\mathbb{F}$. We shall prove: $(\forall v\in W(\mathbb{N}))(v\sqsubseteq\bar{\gamma}(p)\Leftrightarrow v\sqsubseteq\Gamma(p))$. This implies $\bar{\gamma}(p)=\Gamma(p)$.

Suppose $v\in W(\mathbb{N})$. Then

$$\begin{aligned}
&v\sqsubseteq\bar{\gamma}(p)\\
&\quad\Rightarrow(\exists w)(w\sqsubseteq p\wedge v\sqsubseteq\gamma(w)) &&\text{(by definition of }\bar{\gamma})\\
&\quad\Rightarrow(\exists w)(p\in[w]\wedge\Gamma[w]\subseteq O_v) &&\text{(by definition of }\gamma)\\
&\quad\Rightarrow\Gamma(p)\in O_v\\
&\quad\Rightarrow v\sqsubseteq\Gamma(p)
\end{aligned}$$

and

$$\begin{aligned}
&v\sqsubseteq\Gamma(p)\\
&\quad\Rightarrow\Gamma(p)\in O_v\\
&\quad\Rightarrow(\exists w)(p\in[w]\wedge\Gamma[w]\subseteq O_v) &&\text{(by continuity of }\Gamma)\\
&\quad\Rightarrow(\exists w)(\lg(v)\leq\lg(w)\wedge w\sqsubseteq p\wedge\Gamma[w]\subseteq O_v)
\end{aligned}$$

$$\Rightarrow (\exists w)(w \sqsubseteq p \wedge v \in M_w) \qquad \text{(by definition of } M_w)$$

$$\Rightarrow (\exists w)(w \sqsubseteq p \wedge v \sqsubseteq \gamma(w)) \qquad \text{(by definition of } \gamma)$$

$$\Rightarrow v \sqsubseteq \bar{\gamma}(w) \qquad \text{(by definition of } \bar{\gamma}).$$

This implies $\bar{\gamma} = \Gamma$. □

Suppose, γ: $W(\mathbb{N}) \to W(\mathbb{N})$ is isotone, $\Gamma: \mathbb{F} \to \mathbb{B}$ is continuous, and $\bar{\gamma} = \Gamma$. By the definition of $\bar{\gamma}$, for any $p \in \mathbb{F}$ and $v \in W(\mathbb{N})$, if v is a prefix of p, then $\gamma(v)$ is a prefix of $\Gamma(p)$, and $\Gamma(p)$ can be approximated arbitrarily precisely by prefixes $\gamma(w)$ with $w \sqsubseteq p$. By Lemma 2.1 the mapping $\gamma \to \bar{\gamma}$ is a surjective mapping from $\{\gamma \mid \gamma \text{ isotone}\}$ to $[\mathbb{F} \to \mathbb{B}]$. We shall modify this mapping into a representation $\psi: \mathbb{F} \to [\mathbb{F} \to \mathbb{B}]$. For this purpose, any γ: $W(\mathbb{N}) \to W(\mathbb{N})$ will be represented by some $p \in \mathbb{F}$, and the resulting partial representation will be extended into a total one. First, we define the computable functions from $\mathbb{F}$ to $\mathbb{B}$ and characterize them by oracle Turing machines.

2.2. Definition. Let $\nu_N: \mathbb{N} \to W(\mathbb{N})$ be the bijective standard numbering of $W(\mathbb{N})$ defined by $\nu_N(\varepsilon) = 0$, $\nu_N(\langle x_0, x_1, \ldots, x_n, n\rangle + 1) = x_0 x_1 \ldots x_n$ (where $\langle\ \rangle$ is Cantor's tupling function). A function γ: $W(\mathbb{N}) \to W(\mathbb{N})$ is called computable iff it is (ν_N, ν_N)-computable, i.e., iff $\nu_N^{-1} \gamma \nu_N$ is total recursive.

Lemma 2.1 immediately leads to the definition of the computable functions from $\mathbb{F}$ to $\mathbb{B}$.

2.3. Definition. A continuous function $\Gamma: \mathbb{F} \to \mathbb{B}$ is computable iff $\Gamma = \bar{\gamma}$ for some computable function γ.

The computable functions $\mathbb{F} \to \mathbb{B}$ can be characterized by oracle Turing machines. This characterization admits informal but reliable specifications and proofs for computable operators.

2.4. Definition. An oracle Turing machine is a Turing machine T of the following type:
- T has a one-sided infinite read-only input tape on which the values $p(0)$, $p(1)$, $p(2), \ldots$ of the input $p \in \mathbb{F}$ are written (in binary notation).
- T has work tapes.
- T has a one-sided infinite write-only output tape onto which from time to time the values $q(0)$, $q(1), \ldots$ (in this order) of the output $q \in \mathbb{B}$ are written.

The machine is started with $p \in \mathbb{F}$ on the input tape, with empty output tape and the read and the write heads at position O. The machine may compute forever. The result $f_T(p)$ is the (finite or infinite) sequence of numbers it writes onto the output tape.

The idea of piecewise approximating the output from piecewise approximations of the input is another way of expressing continuity. Usually, [11] an oracle machine is defined in such a way that with (oracle) input $p \in \mathbb{F}$ and $n \in \mathbb{N}$ it yields a number $f(p, n)$ (or it diverges). If we consider only functions such that $f(p, n)$ exists if $n < m$ and $f(p, m)$ exists and if we interpret $f(p, n)$ as the nth number on the output tape, we obtain our concept of oracle machines as a special case.

2.5. Lemma. *A function $\Gamma: \mathbb{F} \to \mathbb{B}$ is computable iff it is computed by some oracle Turing machine.*

Proof. Suppose, Γ is computable. Then there is some computable isotone γ: $W(\mathbb{N}) \to W(\mathbb{N})$ with $\Gamma = \bar{\gamma}$. Let T be an oracle Turing machine which operates in stages $n = 0, 1, 2, \ldots$ as follows. Let p be the input.

Stage n: read $w = p(0) \ldots p(n)$; determine $\gamma(w)$; append $x \in W(\mathbb{N})$ to the output, where x is the single word with $vx = \gamma(w)$, $v :=$ inscription already on the output tape.

Indeed, there is an oracle machine which operates this way, and, obviously, $\bar{\gamma}(p) = \sup\{\gamma(w) \mid w \sqsubseteq p\}$ is the output for input p. On the other hand, suppose Γ is computed by some oracle Turing machine T. For $w \in W(\mathbb{N})$, let $\gamma(w)$ be the word on the output tape after $\lg(w)$ steps if w is written onto the first positions of the input tape. Note that in $\lg(w)$ steps T cannot require more input than given by the word w. Therefore, γ is well-defined. Also, γ is isotone and computable and $\bar{\gamma}$ is the function computed by T. $\square$

Below we shall introduce a representation $\psi: \mathbb{F} \to [\mathbb{F} \to \mathbb{B}]$ by its universal function Γ_u. Γ_u will be defined by an oracle Turing machine which is based on the definition of $\bar{\gamma}$ by γ. First we introduce the tupling functions.

2.6. Definition (Tupling functions)

(1) Define $\Pi: \mathbb{F}^2 \to \mathbb{F}$ by $\Pi(p, q)\ (2i) := p(i)$, $\Pi(p, q)\ (2i+1) := q(i)$.

(2) Define $\Pi^{(k)}: \mathbb{F}^k \to \mathbb{F}$ for $k \geq 1$ inductively by $\Pi^{(1)}(q) := q$, $\Pi^{(k+1)}(q_1, \ldots, q_{k+1}) := \Pi(\Pi^{(k)}(q_1, \ldots, q_k), q_{k+1})$. Usually, we write $\langle q_1, \ldots, q_k \rangle$ instead of $\Pi^{(k)}(q_1, \ldots, q_k)$.

(3) Define $\Pi^{(\infty)}: \mathbb{F}^{\mathbb{N}} \to \mathbb{F}$ by $\Pi^{(\infty)}(p_0, p_1, \ldots)\langle i, j \rangle := p_i(j)$.

The following lemma summarizes the most important properties of the tupling functions. On the sets $\mathbb{F}^k$ and $\mathbb{F}^{\mathbb{N}}$ we consider the product topologies.

2.7. Lemma (1) *$\Pi^{(k)}$ is a homeomorphism, and, for any i, $1 \leq i \leq k$, $\langle p_1, \ldots, p_k \rangle \mapsto p_i$ is computable.*

(2) *$\Pi^{(\infty)}$ is a homeomorphism, and for any i the function $\Pi_i^{(\infty)}$ with $\Pi_i^{(\infty)}: (p_0, p_1, \ldots) = p_i$ is computable.*

Proof. Showing continuity of $\Pi^{(k)}$ and $\Pi^{(\infty)}$ and their inverses is a simple exercise in topology. Computability of the 'projections' can easily be shown using oracle Turing machines. $\square$

Using $\Pi^{(k)}$, any continuous k-ary function $\Gamma:\mathbb{F}^k \dashrightarrow M$ can be uniquely represented by the continuous unary function $\Gamma' = \Gamma(\Pi^{(k)})^{-1}$. Therefore, up to the tupling functions only unary functions have to be considered. We now define a binary universal function by its unary equivalent.

2.8. Definition. Define a function $\Gamma_u:\mathbb{F}\to\mathbb{B}$ by an oracle Turing machine T as follows. T works in stages $n = 0, 1, 2, \ldots$. Let $\langle p, q\rangle$ be the input of T.

Stage n: Let $z \in W(\mathbb{N})$ be the word already written on the output tape.
If $(\forall i, j \leq n)\ (\nu_N(i) \sqsubseteq \nu_N(j) \Rightarrow \nu_N p(i) \sqsubseteq \nu_N p(j))$
then $y := \max\{\nu_N p(i) \mid i \leq n \wedge \nu_N(i) \sqsubseteq q\}$
else $y := z$;
write x, where x is the word determined by $zx = y$.

Let $\psi:\mathbb{F}\to[\mathbb{F}\to\mathbb{B}]$ be defined by

$$\psi_p(q) := \psi(p)(q) := \Gamma_u\langle p, q\rangle.$$

It remains to show that ψ is a well-defined surjection. The next lemma also shows the connection of ψ and the function $\gamma \mapsto \bar{\gamma}$ for isotone functions γ.

2.9. Lemma. (1) $\psi:\mathbb{F}\to[\mathbb{F}\to\mathbb{B}]$ *is a well-defined surjective function,*
(2) $\psi(\nu_N^{-1}\gamma\nu_N) = \bar{\gamma}$ *for any isotone* γ: $W(\mathbb{N}) \to W(\mathbb{N})$,
(3) $\Gamma:\mathbb{F}\to\mathbb{B}$ *is computable iff* $\Gamma = \psi(p)$ *for some computable function* $p \in \mathbb{F}$.

Proof. (1), (2) First we show that y and z exist for any stage n. If $(\forall i, j \leq n)(\ldots)$, then $\{\nu_N(i) \mid i \leq n \wedge \nu_N(i) \sqsubseteq q\}$ is linearly ordered by "$\sqsubseteq$". The condition $(\forall i, j)(\ldots)$ implies that also $\{\nu_N p(i) \mid i \leq n \wedge \nu_N(i) \sqsubseteq q\}$ is linearly ordered, therefore y exists. If $n = 0$, then $z = \varepsilon$ and $x = y$. Otherwise, $z = \max\{\nu_N p(i) \mid i \leq n-1 \wedge \nu_N(i) \sqsubseteq q\} \subseteq y$. Therefore, x exists. If not $(\forall i, j \leq n)(\ldots)$, then $y = z$ and $x = \varepsilon$. Therefore, ψ is well-defined. Now suppose, γ is isotone. Define $p := \nu_N^{-1}\gamma\nu_N$. Then $\nu_N(i) \sqsubseteq \nu_N(j) = \nu_N p(i) \sqsubseteq \nu_N p(j)$ for all $i, j \in \mathbb{N}$, and the output after Stage n is $\max\{\nu_N p(i) \mid i \leq n \wedge \nu_N(i) \sqsubseteq q\}$. Therefore,

$$\psi_p(q) = \sup\{\nu_N p(i) \mid \nu_N(i) \sqsubseteq q\} = \sup\{\gamma\nu_N(i) \mid \nu_N(i) \sqsubseteq q\} = \bar{\gamma}.$$

This proves (2), and surjectivity follows from Lemma 2.1 (2).

(3) Suppose $p \in \mathbb{F}$ is computable. Then there is an oracle machine T with $f_T(q) = \langle p, q\rangle$ for any $q \in \mathbb{F}$. Combining T with the oracle machine for Γ_u yields an oracle machine which transforms q into $\psi(p)(q)$ for any $q \in \mathbb{F}$. Therefore, $\psi(p):\mathbb{F}\to\mathbb{B}$ is computable. On the other hand, suppose Γ is computable. Then $\Gamma = \bar{\gamma}$ for some computable γ. By Definition 2.2, $p := \nu_N^{-1}\gamma\nu_N$ is computable and $\bar{\gamma} = \psi(p)$ by (2). $\square$

By Definition 2.8 and by Lemma 2.5, the universal function Γ_u of ψ is computable. This corresponds to the ordinary universal Turing machine theorem which says that the universal function of φ is computable. Also, for the second effectiveness property of φ, the 'smn-theorem' there is a corresponding property for ψ. We formulate it as a 'translation lemma'.

2.10. Theorem. (1) ('utm-*theorem*'): *There is a computable function* $\Gamma_u:\mathbb{F}\to\mathbb{B}$ *such that* $(\forall p, q\in\mathbb{F})\ \Gamma_u\langle p, q\rangle=\psi_p(q)$.

(2) (*'translation lemma'*): *For any computable function* $\Gamma:\mathbb{F}\to\mathbb{B}$ *there is a computable function* $\Sigma:\mathbb{F}\to\mathbb{B}$ *with* $\Sigma(\mathbb{F})\subseteq\mathbb{F}$ *such that* $(\forall p, q\in\mathbb{F})\ \psi_{\Sigma(p)}(q)=\Gamma\langle p, q\rangle$.

Proof. (1) follows immediately from Lemma 2.5.

(2) Suppose $\Gamma:\mathbb{F}\to\mathbb{B}$ is computable. By Definition 2.3 there is some computable $\gamma: W(\mathbb{N})\to W(\mathbb{N})$ with $\Gamma=\bar{\gamma}$. Let $p\in\mathbb{F}$ be any function on $\mathbb{N}$. Define $\gamma_p: W(\mathbb{N})\to W(\mathbb{N})$ by $\gamma_p(w):=\gamma\langle p^{[\lg w]}, w\rangle$, where $\langle i_1i_2\ldots i_n, j_1j_2\ldots j_n\rangle:=i_1j_1i_2j_2\ldots i_nj_n$. Obviously, γ_p is isotone.

2.10.1. Proposition. $(\forall q\in\mathbb{F})\ \bar{\gamma}_p(q)=\bar{\gamma}\langle p, q\rangle$.

Proof

$$\begin{aligned}\bar{\gamma}_p(q)&=\sup\{\gamma_p(w)\mid w\sqsubseteq q\}\\&=\sup\{\gamma\langle p^{[\lg w]}, w\rangle\mid w\sqsubseteq q\}\\&\sqsubseteq\sup\{\gamma(z)\mid z\sqsubseteq\langle p, q\rangle\}\\&=\bar{\gamma}\langle p, q\rangle.\end{aligned}$$

On the other hand, suppose $y\in W(\mathbb{N})$ with $y\sqsubseteq\bar{\gamma}\langle p, q\rangle$. Since $\bar{\gamma}\langle p, q\rangle=\sup\{\gamma(z)\mid z\sqsubseteq\langle p, q\rangle\}$, there are z_1, z_2 with $\lg(z_1)=\lg(z_2)$, $\langle z_1, z_2\rangle\sqsubseteq\langle p, q\rangle$, and $y\sqsubseteq\gamma\langle z_1, z_2\rangle$. Under these conditions,

$$\gamma\langle z_1, z_2\rangle\sqsubseteq\sup\{\gamma\langle p^{[\lg w]}, w\rangle\mid w\sqsubseteq p\}=\bar{\gamma}_p(q).$$

Therefore, $y\sqsubseteq\bar{\gamma}\langle p, q\rangle$ implies $y\sqsubseteq\bar{\gamma}_p(q)$ for all y, hence $\bar{\gamma}\langle p, q\rangle\sqsubseteq\bar{\gamma}_p(q)$. This proves the proposition. □

Define Σ by $\Sigma(p):=\nu_N^{-1}\gamma_p\nu_N$. Then $\Sigma(\mathbb{F})\subseteq\mathbb{F}$ and by Lemma 2.9(2), $\psi_{\Sigma(p)}(q)=\bar{\gamma}_p(q)=\bar{\gamma}\langle p, q\rangle=\Gamma\langle p, q\rangle$. It remains to show that $\Sigma:\mathbb{F}\to\mathbb{B}$ is computable. By the definitions,

$$\Sigma(p)(i)=\nu_N^{-1}\gamma_p\nu_N(i)=\nu_N^{-1}\gamma\langle p^{[\lg\nu_N(i)]}, \nu_N(i)\rangle$$

holds for any $p\in\mathbb{F}$ and $i\in\mathbb{N}$. Since γ is computable, there is some oracle Turing machine which computes Σ. □

Two other versions of the translation lemma, the uniform 'smn-theorem' and the continuous translation lemma can easily be derived.

2.11. Corollary. (1) *There is some computable* $\Sigma:\mathbb{F}\to\mathbb{B}$ *with* $\Sigma(\mathbb{F})\subseteq\mathbb{F}$ *such that* $(\forall p,q,r\in\mathbb{F})\ \psi_{\Sigma\langle p,q\rangle}(r)=\psi_p\langle q,r\rangle$.

(2) *For any continuous function* $\Gamma:\mathbb{F}\to\mathbb{B}$ *there is some continuous function* $\Delta:\mathbb{F}\to\mathbb{B}$ *with* $\Delta(\mathbb{F})\subseteq\mathbb{F}$ *such that* $(\forall p,q\in\mathbb{F})\ \psi_{\Delta(p)}(q)=\Gamma\langle p,q\rangle$.

Proof. (1) The function $\Omega:\mathbb{F}\to\mathbb{B}$ with $\Omega\langle p,\langle q,r\rangle\rangle=\langle\langle p,q\rangle,r\rangle$ is computable and also $\Gamma:=\Gamma_{\mathrm{u}}\Omega$ is computable (use Lemma 2.5). For Γ there is some computable Σ which satisfies the conditions of Theorem 2.10(2). Then, for all $p,q,r\in\mathbb{F}$:

$$\psi_p\langle p,r\rangle=\Gamma_{\mathrm{u}}\langle p,\langle q,r\rangle\rangle=\Gamma_{\mathrm{u}}\Omega\langle\langle p,q\rangle,r\rangle=\psi_{\Sigma\langle p,q\rangle}(r).$$

(2) Since Γ is continuous, $\Gamma=\psi_r$ for some r. By (1), $\Gamma\langle p,q\rangle=\psi_{\Sigma\langle r,p\rangle}(q)$ for some computable Σ. Then the function Δ with $\Delta(p):=\Sigma\langle r,p\rangle$ has the desired properties. □

While Corollary 2.11(1) is equivalent to Theorem 2.10(2) the continuous translation lemma Corollary 2.11(2) is only a consequence of Theorem 2.10(2). Similarly to ordinary recursion theory, the smn- and the utm-theorems characterize ψ uniquely up to equivalence. First we introduce topological and computable reducibility and equivalence for representations.

2.12. Definition. (1) A representation of a set M is a surjective function $\delta:\mathbb{F}\dashrightarrow M$.

(2) On the class of all representations the following relations are defined:

$$\delta_1\leqslant_{\mathrm{t}}\delta_2\ :\Leftrightarrow\ (\forall p\in\mathrm{dom}(\delta_1))\ \delta_1(p)=\delta_2\Gamma(p)\text{ for some continuous }\Gamma:\mathbb{F}\to\mathbb{B},$$

$$\delta_1\leqslant_{\mathrm{c}}\delta_2\ :\Leftrightarrow\ (\forall p\in\mathrm{dom}(\delta_1))\ \delta_1(p)=\delta_2\Gamma(p)\text{ for some computable }\Gamma:\mathbb{F}\to\mathbb{B},$$

$$\delta_1\equiv_{\mathrm{t}}\delta_2\ :\Leftrightarrow\ (\delta_1\leqslant_{\mathrm{t}}\delta_2\text{ and }\delta_2\leqslant_{\mathrm{t}}\delta_1),$$

$$\delta_1\equiv_{\mathrm{c}}\delta_2\ :\Leftrightarrow\ (\delta_1\leqslant_{\mathrm{c}}\delta_2\text{ and }\delta_2\leqslant_{\mathrm{c}}\delta_1).$$

(3) An element $m\in M$ is called δ-computable iff $m=\delta(p)$ for some computable $p\in\mathbb{F}$.

Obviously, $\leqslant_{\mathrm{t}}$ and $\leqslant_{\mathrm{c}}$ are transitive and identitive, and $\equiv_{\mathrm{t}}$ and $\equiv_{\mathrm{c}}$ are equivalence relations. These reducibilities correspond to many-one reducibility from ordinary recursion theory. The significance of one-one reducibility for representations is not yet clear. The definition of a topological and a computable reducibility emphasize that we are developing a general topological theory and simultaneously a formally almost equivalent stronger computability theory. We can now formulate the fundamental characterization theorem for ψ. It corresponds to Rogers' equivalence theorem for effective Gödel numberings of the unary partial recursive functions.

2.13. Theorem. *Let* δ *be a representation of* $[\mathbb{F}\to\mathbb{B}]$. *Then* (1) *and* (2) *are equivalent:*

(1) $\delta\equiv_{\mathrm{c}}\psi$,

(2) δ *satisfies the* utm-*theorem and the (computable) translation lemma (see Theorem* 2.10).

The proof is easy and formally equivalent to the proof of the corresponding theorem for φ. Therefore, we leave it to the reader. Theorem 2.10 and Theorem 2.13 strongly indicate that ψ is the (up to equivalence) unique natural and (computably) 'effective' representation of the continuous functions from $\mathbb{F}$ to $\mathbb{B}$.

Many important theorems in ordinary recursion theory are proved by step counting arguments. Instead of direct step counting arguments often Kleene's T-predicate [3], abstract Blum complexity measures [1], or the projection theorem [11, p. 66] are used which serve the same purpose. For our Type 2 theory, we shall use the following lemma or the projection theorem (Theorem 4.3).

2.14. Lemma. *Let M be the machine for the universal function Γ_u from Definition* 2.8. *Then the following set T_u is decidable*:

$\{\langle i, j, k, m\rangle \mid$ *for any input $\langle p, q\rangle$ such that $\nu_N(i)\sqsubseteq p$ and $\nu_N(j)\sqsubseteq q$ within k steps the oracle machine M yields output $\nu_N(m)$ and reads at most the first* $\lg(\nu_N(i))$ *symbols of p and at most the first* $\lg(\nu_N(j))$ *symbols of q*$\}$.

Proof. Write $\nu_N(i)$ onto the first even places and $\nu_N(j)$ onto the first odd places of the input tape. Then try to execute k steps of computation of M and decide whether the given conditions are satisfied. $\square$

On the basis of Theorem 2.10 a rich theory of continuity and computability for ψ can be developed which corresponds to ordinary recursion theory for φ (see, e.g., [11]). In this paper we only want to present some focus points of the theory.

3. The standard representations of $[\mathbb{F}\to\mathbb{F}]$ and of $[\mathbb{F}\to\mathbb{N}]$

From the standard representation ψ of $[\mathbb{F}\to\mathbb{B}]$ two other representations will be derived: a representation $\tilde{\psi}$ of certain partial continuous functions from $\mathbb{F}$ to $\mathbb{F}$ and a representation χ of certain continuous functions from $\mathbb{F}$ to $\mathbb{N}$.

3.1. Definition. (1) Define a set $[\mathbb{F}\to\mathbb{N}]$ of partial functions from $\mathbb{F}$ to $\mathbb{N}$ and a surjective function $\chi:\mathbb{F}\to[\mathbb{F}\to\mathbb{N}]$ as follows:

$$\chi(p)(q):=\begin{cases}\text{div} & \text{if } \psi(p)(q)=\varepsilon\in B,\\ \text{the first number of the sequence } \psi(p)(q) & \\ \qquad\text{otherwise,}\end{cases}$$

for all $p, q\in\mathbb{F}$.

(2) Define a set $[\mathbb{F}\to\mathbb{F}]$ of partial functions from $\mathbb{F}$ to $\mathbb{F}$ and a surjective function $\tilde{\psi}:\mathbb{F}\to[\mathbb{F}\to\mathbb{F}]$ as follows:

$$\tilde{\psi}(p)(q):=\begin{cases}\psi(p)(q) & \text{if } \psi(p)(q)\in\mathbb{F},\\ \text{div} & \text{otherwise,}\end{cases}$$

for all $p, q\in\mathbb{F}$.

The definition extends well-known concepts of computable operators and functionals to a uniform topological description. The χ-computable elements from $[\mathbb{F}\to\mathbb{N}]$ are exactly the recursive functionals on $\mathbb{F}$ defined by Rogers [11, Section 15.3]. The total computable functions from $[\mathbb{F}\to\mathbb{F}]$ are exactly the restrictions of general recursive operators to $\mathbb{F}$ [11, Section 9.8]. In the Type 2 theory developed here, computability of operators on $\mathbb{P}(=\{f:\mathbb{N}\dashrightarrow\mathbb{N}\})$ or $2^{\mathbb{N}}$ can be derived via admissible representations of appropriate cpo's [14]. This is beyond the scope of this paper.

As in the case of partial recursive functions which have the recursively enumerable sets as their natural domains, also the functions from $[\mathbb{F}\to\mathbb{N}]$ and $[\mathbb{F}\to\mathbb{F}]$ have natural domains.

3.2. Definition. (1) Let a representation ω of the set of open subsets of $\mathbb{F}$ be defined by $\omega(p):=\bigcup\{[\nu_N(j)]\mid j+1\in\text{range}(p)\}$ for any $p\in\mathbb{F}$.

(2) Let a representation ξ of the set of G_δ-subsets of $\mathbb{F}$ be defined by $\xi(p)=\bigcap_i\bigcup_j\{[\nu_N(j)]\mid\langle i,j\rangle+1\in\text{range}(p)\}$ for any $p\in\mathbb{F}$.

By the following theorem, the domains of the functions from $[\mathbb{F}\to\mathbb{N}]$ are the open subsets and from $[\mathbb{F}\to\mathbb{F}]$ are the G_δ-subsets of $\mathbb{F}$. We prove a computably effective version.

3.3. Theorem. (1) *Define a representation ω' by $\omega'(p):=\text{dom}(\chi(p))$ for all $p\in\mathbb{F}$. Then $\omega\equiv_c\omega'$.*

(2) *Define a representation ξ' by $\xi'(p):=\text{dom}(\tilde{\psi}(p))$ for all $p\in\mathbb{F}$. Then $\xi\equiv_c\xi'$.*

Property (1) corresponds to the characterization of the recursively enumerable sets as domains of the partial recursive functions on the one hand and as the ranges of the total recursive functions on the other hand.

Proof. (1) "$\omega\leq_c\omega'$": Let M be an oracle Turing machine which, on input $\langle p,q\rangle$, $p\in\mathbb{F}$, $q\in\mathbb{F}$, works in stages as follows.

Stage $\langle i,k\rangle$: If $p(i)\neq 0$ and $\nu_N(p(i)-1)=q^{[k]}$, then write 0. Since f_M is computable, by Theorem 2.10(2) there is a computable function $\Gamma:\mathbb{F}\to\mathbb{B}$ with $f_M\langle p,q\rangle=\psi_{\Gamma(p)}(q)$ for all $p, q\in\mathbb{F}$. Then

$$\begin{aligned} q\in\omega(p) &\Leftrightarrow (\exists i,k)(p(i)\neq 0\wedge\nu_N(p(i)-1)=q^{[k]})\\ &\Leftrightarrow \psi_{\Gamma(p)}(q)\neq\varepsilon \Leftrightarrow q\in\text{dom}(\chi_{\Gamma(p)}),\end{aligned}$$

therefore $\omega\leq_c\omega'$.

"$\omega' \leq_c \omega$": Let T_u be the set from Lemma 2.14 Define $\Sigma:\mathbb{F}\to\mathbb{F}$ by

$$\Sigma(p)\langle i,j,k,m\rangle := \begin{cases} j+1 & \text{if } \nu_N(i)\sqsubseteq p \wedge \langle i,j,k,m\rangle\in T_u \wedge \nu_N(m)\neq\varepsilon,\\ 0 & \text{otherwise,}\end{cases}$$

for all $p\in\mathbb{F}$, $i, j, k, m\in\mathbb{N}$. Then, Σ is computable and

$$\begin{aligned} q\in\omega(\Sigma(p)) &\Leftrightarrow (\exists i,j,k,m)\\ &\qquad(\langle i,j,k,m\rangle\in T_u \wedge \nu_N(i)\sqsubseteq p\wedge \nu_N(j)\sqsubseteq q\wedge \nu_N(m)\neq\varepsilon)\\ &\Leftrightarrow \Gamma_u\langle p,q\rangle\neq\varepsilon\\ &\Leftrightarrow q\in\mathrm{dom}(\chi_p)\end{aligned}$$

for any $p, q\in\mathbb{F}$.

(2) "$\xi\leq_c\xi'$": Let M be an oracle Turing machine which on input $\langle p,q\rangle$ tries to compute $\Sigma\langle p,q\rangle(i)$ in Stage i ($i=0,1,2,\ldots$) as follows:

$$\Sigma\langle p,q\rangle(i) := \min\{\langle j,k\rangle \mid p(k)=\langle i,j\rangle+1\wedge \nu_N(j)\sqsubseteq q\},$$

and if $\Sigma\langle p,q\rangle(i)$ does not exist, then $\Sigma\langle p,q\rangle(i')$ does not exist for all $i'>i$. By Theorem 2.10(2), $\Sigma\langle p,q\rangle=\psi_{\Gamma(p)}(q)$ for some computable Γ with $\Gamma(\mathbb{F})\subseteq\mathbb{F}$. Then

$$\begin{aligned} q\in\xi(p) &\Leftrightarrow (\forall i)(\exists j)(\exists k)(p(k)=\langle i,j\rangle+1\wedge\nu_N(j)\sqsubseteq q)\\ &\Leftrightarrow (\forall i)\Sigma\langle p,q\rangle(i)\text{ exists}\\ &\Leftrightarrow q\in\mathrm{dom}(\tilde\psi_{\Gamma(p)})=\xi'\Gamma(p)\end{aligned}$$

for all $p, q\in\mathbb{F}$.

"$\xi'\leq_c\xi$": Let T_u be the set from Lemma 2.14. Define $\Sigma:\mathbb{F}\to\mathbb{F}$ by

$$\Sigma(p)\langle i,j,k,m,n\rangle := \begin{cases}\langle i,j\rangle+1 & \text{if } \langle k,j,m,n\rangle\in T_u\wedge\nu_N(k)\sqsubseteq p\wedge \lg(\nu_N(n))>i,\\ 0 & \text{otherwise.}\end{cases}$$

Then Σ is computable and

$$\begin{aligned} q\in\xi'(p) &\Leftrightarrow (\forall i)\psi_p(q)(i)\text{ exists}\\ &\Leftrightarrow (\forall i)(\exists j,k,m,n)\\ &\qquad(\nu_N(j)\sqsubseteq q\wedge\nu_N(k)\sqsubseteq p\wedge\langle k,j,m,n\rangle\in T_u\wedge\lg(\nu_N(n))>i)\\ &\Leftrightarrow (\forall i)(\exists j)(\nu_N(j)\sqsubseteq q\wedge\langle i,j\rangle+1\in\mathrm{range}(\Sigma(p)))\\ &\Leftrightarrow q\in\xi\Sigma(p)\end{aligned}$$

for all $p, q\in\mathbb{F}$. □

The following theorem characterizes $\mathbb{F}\to\mathbb{N}$ ($[\mathbb{F}\to\mathbb{F}]$) and shows that 'essentially' every continuous function $\mathbb{F}\dashrightarrow\mathbb{N}$ ($\mathbb{F}\dashrightarrow\mathbb{F}$) is represented by χ ($\tilde\psi$).

3.4. Theorem. (1) $[\mathbb{F}\to\mathbb{N}]$ *is the set of all continuous* $\Sigma:\mathbb{F}\dashrightarrow\mathbb{N}$ *such that* $\mathrm{dom}(\Sigma)$ *is open.*

(2) *For any continuous* $\Sigma: \mathbb{F} \dashrightarrow \mathbb{N}$ *there is some* $\Sigma' \in [\mathbb{F} \to \mathbb{N}]$ *which extends* Σ.

(3) $[\mathbb{F} \to \mathbb{F}]$ *is the set of all continuous* $\Sigma: \mathbb{F} \dashrightarrow \mathbb{F}$ *such that* $\mathrm{dom}(\Sigma)$ *is a* G_δ*-subset of* $\mathbb{F}$.

(4) *For any continuous* $\Sigma: \mathbb{F} \dashrightarrow \mathbb{F}$ *there is some* $\Sigma' \in [\mathbb{F} \to \mathbb{F}]$ *which extends* Σ.

Proof. (1) Let $\Sigma \in [\mathbb{F} \to \mathbb{N}]$. Then $\mathrm{dom}(\Sigma)$ is open by Theorem 3.3. Define $H: \mathbb{B} \to \mathbb{N}$ by $H(x) :=$ (the first symbol of x if $x \neq \varepsilon$, div otherwise). Then H is continuous. Since $\Sigma = H\Gamma$ for some continuous $\Gamma: \mathbb{F} \to \mathbb{B}$, Σ is continuous. On the other hand, let $\Sigma: \mathbb{F} \dashrightarrow \mathbb{N}$ be continuous and let $\mathrm{dom}(\Sigma)$ be open.

Define γ: $W(\mathbb{N}) \to W(\mathbb{N})$ by

$$\gamma(w) := \begin{cases} n & \text{if } [w] \subseteq \mathrm{dom}(\Sigma) \text{ and } \Sigma([w]) = \{n\}, \\ \mathrm{div} & \text{otherwise.} \end{cases}$$

Obviously, γ is isotone, hence $H\bar{\gamma} \in [\mathbb{F} \to \mathbb{N}]$. For any $p \in \mathbb{F}$, $n \in \mathbb{N}$ we have

$$\begin{aligned} \Sigma(p) = n \;&\Leftrightarrow\; (\exists w)([w] \in \mathrm{dom}(\Sigma) \wedge \Sigma([w]) = \{n\} \wedge p \in [w]) \\ &\qquad \text{(since } \Sigma \text{ is continuous and } \mathrm{dom}(\Sigma) \text{ is open)} \\ &\Leftrightarrow\; (\exists w)(\gamma(w) = n \wedge w \sqsubseteq p) \\ &\Leftrightarrow\; H\bar{\gamma}(p) = n. \end{aligned}$$

We obtain $\Sigma = H\bar{\gamma} \in [\mathbb{F} \to \mathbb{N}]$.

(2) Assume that $\Sigma: \mathbb{F} \dashrightarrow \mathbb{N}$ is continuous. Define γ: $W(\mathbb{N}) \to W(\mathbb{N})$ by

$$\gamma(w) := \begin{cases} n & \text{if } \Sigma([w]) = \{n\}, \\ \mathrm{div} & \text{otherwise.} \end{cases}$$

As in (1) one easily shows that $H\bar{\gamma}$ extends Σ.

(3), (4) Let $\Sigma \in [\mathbb{F} \to \mathbb{F}]$. Then $\mathrm{dom}(\Sigma)$ is a G_δ-set by Theorem 3.3 and Σ is continuous as a restriction of a continuous function. On the other hand, let $\Sigma: \mathbb{F} \dashrightarrow \mathbb{F}$ be continuous. As in the proof of Lemma 2.1(2) one easily shows that some continuous $\bar{\gamma}: \mathbb{F} \to \mathbb{B}$ extends Σ.

Let Σ' be the restriction of $\bar{\gamma}$ to $\bar{\gamma}^{-1}(\mathbb{F})$. Then $\Sigma' \in [\mathbb{F} \to \mathbb{F}]$ extends Σ. This proves (4). Assume in addition that $\mathrm{dom}(\Sigma)$ is a G_δ-set. By Theorem 3.3 there is some $\Delta \in [\mathbb{F} \to \mathbb{F}]$ with $\mathrm{dom}(\Delta) = \mathrm{dom}(\Sigma)$. Obviously, $\mathrm{dom}(\Delta) \subseteq \mathrm{dom}(\Sigma')$. There is some γ_1: $W(\mathbb{N}) \to W(\mathbb{N})$ such that $\bar{\gamma}_1$ extends Δ.

Define γ_2 by

$$\gamma_2(w) := \begin{cases} \gamma(w) & \text{if } \lg(\gamma(w)) \leq \lg(\gamma_1(w)), \\ \text{the prefix of length } \lg(\gamma_1(w)) \text{ of } \gamma_2(w) & \\ \qquad \text{otherwise.} & \end{cases}$$

Then Σ is the restriction of $\bar{\gamma}_2$ to $\mathbb{F}$, hence $\Sigma \in [\mathbb{F} \to \mathbb{F}]$. $\square$

The representations $\chi: \mathbb{F} \to [\mathbb{F} \to \mathbb{N}]$ and $\tilde{\psi}: \mathbb{F} \to [\mathbb{F} \to \mathbb{F}]$ satisfy the smn- and the utm-theorem (Theorem 2.10) and the immediate consequences.

3.5. Theorem. *The representations $\tilde{\psi}$ and χ satisfy the* utm-*theorem and the translation lemma* (*Theorem* 2.10), *the uniform* smn-*theorem and the continuous translation lemma* (*Corollary* 2.11), *and the equivalence theorem* (*Theorem* 2.13).

Proof. There are functions H_1 and H_2 such that $\chi_p(q) = H_1(\psi_p(q))$ and $\tilde{\psi}_p(q) = H_2(\psi_p(q))$. Using H_1 and H_2, Theorem 2.10 can be transformed to the corresponding theorems for χ and $\tilde{\psi}$. The other statements are consequences. $\square$

As for any representation, a function $\Gamma \in [\mathbb{F} \to \mathbb{F}]$ ($\Gamma \in [\mathbb{F} \to \mathbb{N}]$) is called computable, iff $\Gamma = \tilde{\psi}_p$ ($\Gamma = \chi_p$) for some computable $p \in \mathbb{F}$. It is easy to see that $\Gamma(p)$ is a computable function if $\Gamma \in [\mathbb{F} \to \mathbb{F}]$ is computable and $p \in \mathrm{dom}(\Gamma)$ is computable. Using Theorem 2.10 one easily shows that $\psi_p \tilde{\psi}_p = \psi_{\Sigma\langle p,q\rangle}$ for some (total) computable $\Sigma : \mathbb{F} \to \mathbb{F}$. The corresponding equation holds, if ψ is substituted by $\tilde{\psi}$. There are several other obvious properties which we do not mention here.

4. Open and computably open sets

In this section we show by examples how results from ordinary (Type 1) recursion theory can be transferred to Type 2 theory. Usually, there are two versions for any theorem, a topological ("t") version where only continuous operators are considered and a computable ("c") version where computable operators are considered. For example, we already have introduced topological reducibility, $\leq_t$, and computable reducibility, $\leq_c$, and we have proved a computable translation lemma (Theorem 2.10(2)) and a topological translation lemma (Corollary 2.11(2)) for the representation ψ.

By Theorem 3.3, the open subsets of $\mathbb{F}$ can be represented by ω', where $\omega'(p) = \mathrm{dom}(\chi(p))$. There is a formal correspondence to the numbering W of the recursively enumerable subsets of $\mathbb{N}$ defined by $W_i := \mathrm{dom}(\varphi^i)$. This implies that many concepts and theorems for r.e. sets can be transferred to open or computably open subsets of $\mathbb{F}$. For example, Theorem 3.3(1) corresponds to the fact that the r.e. sets can be defined as ranges of total recursive functions or as domains of partial recursive functions.

4.1. Definition. Let $A \subseteq \mathbb{F}$. Then:
- A is called t-open iff A is open.
- A is called c-open iff $A = \omega'(p)$ for some computable $p \in \mathbb{F}$.
- A is called t-clopen iff A and $\mathbb{F} \backslash A$ are open.
- A is called c-clopen iff A and $\mathbb{F} \backslash A$ are c-open.

The c-clopen sets are also called recursive.

The clopen sets formally correspond to the recursive sets in Type 1 recursion theory. Using oracle Turing machines the following lemma can easily be proved.

4.2. Lemma. *A set* $A \subseteq \mathbb{F}$ *is* t-*clopen* (c-*clopen*) *iff* $A = \Gamma^{-1}\{0\}$ *for some continuous (computable) function* $\Gamma: \mathbb{F} \to \mathbb{N}$.

Notice that Γ must be a total function. Another basic theorem, which connects open and clopen sets, is the projection theorem. For $i \in \mathbb{N}$ and $p \in \mathbb{F}$ we shall denote the function $q = (i, p(0), p(1), \ldots)$ by $\langle i, p\rangle$.

4.3. Theorem. (1) *There are computable total functions* Σ, $\Sigma' \in [\mathbb{F} \to \mathbb{F}]$ *such that, for any* $p \in \mathbb{F}$,

$$\omega(p) = \{q \mid (\exists i \in \mathbb{N})\, \langle i, q\rangle \in \omega\Sigma(p)\} \quad \textit{and} \quad \omega\Sigma'(p) = \mathbb{F}\backslash\omega\Sigma(p).$$

(2) *There is some computable total function* $\Gamma \in [\mathbb{F} \to \mathbb{F}]$ *such that, for any* $p \in \mathbb{F}$,

$$\{q \mid (\exists i \in \mathbb{N})\, \langle i, q\rangle \in \omega(p)\} = \omega\Gamma(p).$$

From (1) we conclude that any t-open (c-open) set is the projection of a t-clopen (c-clopen) set. From (2) we conclude that the projection of any t-open (c-open) set is t-open (c-open).

Proof. (1) By the translation lemma for χ there are computable total functions Δ, $\Delta': \mathbb{F} \to \mathbb{F}$ with $\chi_{\Delta(p)}\langle i, q\rangle = (0$ if $p(i) \neq 0$ and $\nu_N(p(i) - 1) \sqsubseteq q$, div otherwise) and $\chi_{\Delta'(p)}\langle i, q\rangle = ($div if $p(i) \neq 0$ and $\nu_N(p(i) - 1) \sqsubseteq q$, 0 otherwise). By Theorem 3.3(1), $\omega' = \omega\Pi$ for some $\Pi: \mathbb{F} \to \mathbb{F}$. Then $\Sigma := \Pi\Delta$ and $\Sigma' := \Pi\Delta'$ have the desired properties.

(2) Let M be an oracle Turing machine which with the input $\langle p, q\rangle$ operates in stages $n = 0, 1, 2, \ldots$, and in Stage $n = \langle i, j\rangle$ writes "0" onto the output tape if $(p(j) \neq 0$ and $\nu_N(p(j) - 1) \sqsubseteq \langle i, q\rangle)$ and writes nothing otherwise. By the translation lemma for χ and by Theorem 3.3(1) there is some computable total $\Gamma: \mathbb{F} \to \mathbb{F}$ such that

$$\begin{aligned} q \in \omega\Gamma(p) &\Leftrightarrow f_M\langle p, q\rangle \neq \varepsilon \\ &\Leftrightarrow (\exists i)(\exists j)(p(j) \neq 0 \text{ and } \nu_N(p(j) - 1) \sqsubseteq \langle i, q\rangle) \\ &\Leftrightarrow (\exists i)\langle i, q\rangle \in \omega(p). \qquad \square \end{aligned}$$

Theorem 4.3 corresponds to the uniform version of the projection theorem for recursively enumerable sets.

In Definition 2.12 we have introduced t- and c-reducibility for representations. Any total representation δ of $\{0, 1\}$ can be considered as a characteristic function of $\delta^{-1}\{1\}$. Then, reducibility of characteristic functions is equivalent to reducibility between sets A, $B \subseteq \mathbb{F}$ defined by $A \leqslant_t B$ $(A \leqslant_c B)$ iff $A = \Gamma^{-1}B$ for some total (computable) $\Gamma \in [\mathbb{F} \to \mathbb{F}]$. Also, 1-reducibility by injective functions and isomorphism can be defined. In contrast to Myhill's theorem for subsets of $\mathbb{N}$, 1-equivalence and isomorphism do not seem to be equivalent. However, the cylinder theorem which shows the connection between reducibility and 1-reducibility is formally the same as for subsets of $\mathbb{N}$, where A is t-cylinder, iff $A \equiv_{t1} \langle B, \mathbb{F}\rangle$ for some $B \subseteq \mathbb{F}$ (similar for c-cylinders). The proof can be almost copied from the corresponding proof in ordinary recursion theory (see, e.g., [11]).

In Type 2 recursion theory the halting problem can be defined and turns out to be equivalent to the self applicability problem. It is c-complete in the class of t-open subsets of $\mathbb{F}$, its complement is c-productive w.r.t. the representation ω.

4.4. Definition. (1) $K_\chi := \{p \in \mathbb{F} \mid p \in \operatorname{dom} \chi_p\}$ (self applicability problem).

(2) $K_\chi^\circ := \{\langle p, q\rangle \in \mathbb{F} \mid p \in \operatorname{dom} \chi_q\}$ (halting problem).

(3) $A \subseteq \mathbb{F}$ is t-complete (c-complete) in $X \subseteq 2^{\mathbb{F}}$, iff $A \in X$ and $B \leqslant_t A$ $(B \leqslant_c A)$ for any $B \in X$.

(4) $A \subseteq \mathbb{F}$ is t-productive (c-productive) w.r.t. ω iff there is some total (computable) function $\Gamma \in [\mathbb{F} \to \mathbb{F}]$ such that $\omega(q) \subseteq A \Rightarrow \Gamma(q) \in A \backslash \omega(q)$ for any $q \in \mathbb{F}$.

Some properties of K_χ are summarized in the following theorem.

4.5. Theorem

(1) $K_\chi \equiv_c K_\chi^\circ$.

(2) K_χ *is* c-*open.*

(3) $\bar{K}_\chi$ *is not* t-*open.*

(4) K_χ *is* c-*complete in the class of* t-*open sets.*

(5) $\bar{K}_\chi$ *is productive w.r.t.* ω.

The proofs are very easy and formally equivalent to the corresponding proofs from ordinary recursion theory (see, e.g., [11]). Several questions are still open. Is m-completeness equivalent to 1-completeness? Is creativity equivalent to completeness? Is productivity of $A \subseteq \mathbb{F}$ equivalent to $\bar{K}_\chi \leqslant A$? Is productivity via partial functions equivalent to Definition 4.4(4)?

The concept of effective inseparability can easily be transferred to $\mathbb{F}$.

4.6. Definition. $A, B \subseteq \mathbb{F}$ are called t- (c-)effectively inseparable iff there is some total (computable) function $\Gamma \in [\mathbb{F} \to \mathbb{F}]$ such that

$$(A \subseteq \omega'(p) \wedge B \subseteq \omega'(q) \wedge \omega'(p) \cap \omega'(q) = \emptyset) \Rightarrow \Gamma\langle p, q\rangle \in \mathbb{F} \backslash (\omega'(p) \cup \omega'(q))$$

for all $p, q \in \mathbb{F}$.

The following theorem corresponds to a similar theorem in ordinary recursion theory.

4.7. Theorem. (1) $\{p \mid \chi_p(p) = 0\}$ *and* $\{p \mid \chi_p(p) = 1\}$ *are* c-*effectively inseparable.*

(2) *If* A_0 *and* A_1 *are* t- (c-)*effectively inseparable and* $A_0 \subseteq B_0$ *and* $A_1 \subseteq B_1$, *then* B_0 *ands* B_1 *are* t- (c-)*effectively inseparable.*

(3) *If* $\Gamma : \mathbb{F} \to \mathbb{F}$ *is continuous (computable) and* A_0 *and* A_1 *are* t- (c-)*effectively inseparable, then* $\Gamma(A_0)$ *and* $\Gamma(A_1)$ *are* t- (c-)*effectively inseparable.*

We only indicate the proof of (1).

Proof. By the computable translation lemma there is a computable function $\Gamma:\mathbb{F}\to\mathbb{F}$ such that

$$\chi_{\Gamma\langle p,q\rangle}(r)=\begin{cases}1 & \text{if } r\in\operatorname{dom}\chi_p \wedge r\notin\operatorname{dom}\chi_q,\\ 0 & \text{if } r\notin\operatorname{dom}\chi_p \wedge r\in\operatorname{dom}\chi_q,\\ \in\{0,1\} & \text{if } r\in\operatorname{dom}\chi_p \wedge r\in\operatorname{dom}\chi_q,\\ \text{div} & \text{otherwise.}\end{cases}$$

Then Γ has the desired properties. □

Theorem 4.7 is useful for the study of precomplete representations, i.e., representations which satisfy the recursion theorem (cf. [5]. Especially the representations ψ, χ, $\tilde{\psi}$, ω and ξ are precomplete. Representations will be investigated from a general point of view on pp. 35-53 of this issue by Kreitz and Weihrauch [15].

5. Conclusion

We have introduced three function classes together with standard representations $\psi:\mathbb{F}\to[\mathbb{F}\to\mathbb{B}]$, $\tilde{\psi}:\mathbb{F}\to[\mathbb{F}\to\mathbb{F}]$, and $\chi:\mathbb{F}\to[\mathbb{F}\to\mathbb{N}]$. These representations admit a theory which is formally very similar to ordinary recursion theory based on the standard numbering φ of the partial recursive functions. An essential feature of this theory is that it splits into two versions, a purely topological version and a more special recursion theoretical version. Thus, it demonstrates very clearly that topology is fundamental for computability theory. The definitions coincide as far as possible with standard definitions of computable operators and functionals given earlier. The purpose of this paper is to lay a basis for a concise, general, and simple theory of continuity and computability on $\mathbb{F}$, for a general theory of representations and for constructive and computable analysis and mathematics. Representations will be investigated in [15].

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