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Comment

Comments on the paper "Relativistic generalized uncertainty principle"

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ABSTRACT

We point out some misleading results reported in the recent study made by Todorinov et al. (2019), concerning a relativistic extension of the generalized uncertainty principle (GUP). We derive, in this frame, the correct deformed Klein–Gordon (KG) and Dirac equations, valid up to the first order in the deformation parameter, and discuss their minimal coupling to external fields.

In a recent paper published in this Journal [1], V. Todorinov et al. considered a Lorentz covariant deformed algebra, where the commutation relations between the four-position and the four-momentum operators are given by

$$[X^{\mu}, P^{\nu}] = i\hbar \left\{ \left(1 + (\varepsilon - \alpha)\gamma^2 P^{\rho} P_{\rho}\right) \eta^{\mu\nu} + (\beta + 2\varepsilon)\gamma^2 P^{\mu} P^{\nu} \right\} \tag{1}$$

where α, β and ε are dimensionless parameters and $\gamma = 1/cM_{pl}$, with M_{pl} the Planck mass. $\eta^{\mu\nu} = diag(-, +, +, +)$ is the metric tensor in the $(3+1)$-dimensional Minkowski space–time. It should be noted that the commutation relations in Eq. (1) are exactly the same as those studied by Quesne and Tkachuk in Refs. [2,3], who used the notations β and β' to designate the two independent deformation parameters, such as $\beta = (\varepsilon - \alpha)\gamma^2$ and $\beta' = (\beta + \varepsilon)\gamma^2$. Then the authors focused on the particular case where $\varepsilon = \alpha$, and adopted the following representations of X^{μ} and P^{μ}:

$$X^{\mu} = x^{\mu} \left(1 - \alpha\gamma^2 p^{\rho} p_{\rho}\right), \quad P^{\mu} = p^{\mu} \left(1 + \alpha\gamma^2 p^{\rho} p_{\rho}\right) \tag{2}$$

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where the small letters $x=\left(x^0,\mathbf{x}\right)$ and $p=\left(p^0,\mathbf{p}\right)$ are used, throughout this work, to designate the canonically conjugate operators of four-position and four-momentum, respectively. These operators satisfy the commutation relations

$$[x^\mu,p^\nu]=i\hbar\eta^{\mu\nu} \tag{3}$$

Moreover, henceforth, we shall simply use the symbol α in order to refer to $\alpha\gamma^2$. Now, the deformed free KG equation, valid up to the first order in α, follows merely from the invariance of the squared physical momentum, that is from the relation $P^\mu P_\mu=-m^2c^2$, with m the mass of the particle. Thus, using Eq. (2) for the realization of the operator P^μ, and retaining only terms of first order in α, this relation results in the equation

$$\left\{p^\mu p_\mu+2\alpha\left(p^\mu p_\mu\right)^2\right\}\psi=-m^2c^2\psi \tag{4}$$

where ψ is the KG wave function. Then, Eq. (4) is equivalent to

$$\left\{\left(1-4\alpha p_0^2\right)\mathbf{p}^2+2\alpha\mathbf{p}^4-\left(1-2\alpha p_0^2\right)p_0^2\right\}=-m^2c^2\psi \tag{5}$$

Furthermore, the KG equation, in the position-space representation, stems from the substitutions $p^0=\frac{i\hbar}{c}\frac{\partial}{\partial t}$ and $\mathbf{p}=-i\hbar\boldsymbol{\nabla}$ into Eq. (5). This yields

$$\begin{aligned}&\left\{\left(1+2\alpha\frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2}\right)\frac{\partial^2}{\partial t^2}-c^2\left(1+4\alpha\frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2}\right)\boldsymbol{\nabla}^2\right.\\&\qquad\left.+2\alpha\hbar^2c^2\boldsymbol{\nabla}^4+\frac{m^2c^4}{\hbar^2}\right\}\psi=0\end{aligned} \tag{6}$$

This equation differs drastically from that obtained in Ref. [1]. Therein Eq. (4) was regarded as a simple algebraic equation and then solved for $p^\mu p_\mu$, to the first order in α. This had led to an inadequate GUP-modified KG equation. The latter is very similar to the non-deformed one, but just with a modified mass (Eq. (20) in Ref. [1]).

Next, we discuss the derivation of the GUP-deformed KG equation in the presence of a minimally-coupled electromagnetic field, with a four-potential A^μ. This will involve the replacement $P^\mu\to P^\mu-qA^\mu$ into the free GUP-deformed KG equation, where q is the electric charge of the considered particle. Hence, one has to start from the equation

$$\left(P^\mu-qA^\mu\right)\left(P_\mu-qA_\mu\right)\psi=-m^2c^2\psi \tag{7}$$

Then, we use Eq. (2) and expand to the first order in α. This gives

$$\begin{aligned}&\left\{p^\mu p_\mu+2\alpha\left(p^\mu p_\mu\right)^2+q^2A^\mu A_\mu-q\left(2A^\mu p_\mu+\left[p^\mu,A_\mu\right]\right)\left(1+\alpha p^\rho p_\rho\right)\right.\\&\qquad\left.-\alpha qp^\mu\left(p_\rho\left[p^\rho,A_\mu\right]+\left[p^\rho,A_\mu\right]p_\rho\right)\right\}\psi=-m^2c^2\psi\end{aligned} \tag{8}$$

Particularly, in the case where $qA^\mu=\left(V\left(\mathbf{X}\right)/c,0\right)$, with $V\left(\mathbf{X}\right)$ a static potential energy (i.e., independent of X^0), Eq. (9) reduces to

$$\begin{aligned}&\left\{c^2p^\mu p_\mu+2\alpha c^2\left(p^\mu p_\mu\right)^2-V\left(\mathbf{X}\right)^2+2cV\left(\mathbf{X}\right)\left(1+\alpha p^\rho p_\rho\right)p^0\right.\\&\qquad\left.+\alpha c\left(p_\rho\left[p^\rho,V\left(\mathbf{X}\right)\right]+\left[p^\rho,V\left(\mathbf{X}\right)\right]p_\rho\right)p^0\right\}\psi=-m^2c^4\psi\end{aligned} \tag{9}$$

In addition, according to Eq. (2), the physical three-position operator $\mathbf{X}$, is now a function of the non-deformed three-position operator $\mathbf{x}$ and the four-momentum operator p. For instance, the distance $R=|\mathbf{X}|$, expanded to the first order in α, reads

$$R=\sqrt{\mathbf{X}^2}=r\left(1-\alpha p^\rho p_\rho+i\hbar\frac{\alpha}{r^2}\mathbf{x}.\mathbf{p}\right) \tag{10}$$

Thus, the development of the Coulomb potential $V\left(\mathbf{X}\right)=-\kappa/|\mathbf{X}|$, valid to the leading order in α, turns to be

$$V\left(\mathbf{X}\right)=-\left(1+\alpha p^\rho p_\rho-i\hbar\frac{\alpha}{r^2}\mathbf{x}.\mathbf{p}\right)\frac{\kappa}{r} \tag{11}$$

All the above considerations were not part of the procedure followed by V. Todorinov et al. [1], when studying the effect of the GUP on the relativistic hydrogen atom using the KG equation.

Furthermore, throughout their work, the authors of [1] have employed the relation

$$E = E_0 \left(1 + \alpha p^\rho p_\rho\right) = E_0 \left(1 - \alpha m^2 c^2\right) \tag{12}$$

in order to calculate the GUP corrections to the physical energy E from those to the non-deformed energy, denoted by E_0. However, we believe this is an incorrect reasoning. As a matter of fact, the physical energy E, corresponding to a stationary state ψ, should be worked out as the expectation value of the operator energy P^0, in the state ψ. Then, using Eq. (2), this yields

$$E = \left\langle P^0 \right\rangle_\psi = E_0 \left(1 - \alpha E_0^2\right) + \alpha E_0 c^2 \left\langle \mathbf{p}^2 \right\rangle_\psi \tag{13}$$

where E_0 is the non-deformed energy associated to the state ψ. That is ψ is of the form $\psi = e^{-iE_0 t/\hbar}\phi$ where ϕ is a time-independent state.

We turn now to discuss briefly the derivation of the GUP-deformed Dirac equation. For a free particle of mass m, this equation could be directly written as

$$\gamma^\mu P_\mu \psi = -mc\psi \tag{14}$$

where γ^μ are the usual 4×4 Dirac matrices generating a Clifford algebra. Then, in the presence of a minimally coupled electromagnetic field A^μ, this equation becomes

$$\gamma^\mu \left(P_\mu - qA_\mu\right) \psi = -mc\psi \tag{15}$$

In particular for $qA^\mu = (V(\mathbf{X})/c, 0)$, where $V(\mathbf{X})$ is X^0-independent, Eq. (15) leads to

$$\left\{\left(1 + \alpha p^\rho p_\rho\right)\left(c\tilde{\boldsymbol{\alpha}}.\mathbf{p} - p^0\right) + V(\mathbf{X})\right\}\psi = -\tilde{\beta} mc^2 \psi \tag{16}$$

where we have introduced the notations $\tilde{\beta} = \gamma^0$ and $\tilde{\boldsymbol{\alpha}}^k = \tilde{\beta}\gamma^k$. In this way, the GUP-deformed Dirac equation, written in the position representation, reads

$$\left\{i\hbar\left(1 + \alpha\hbar^2\left(\frac{\partial^2}{\partial t^2} - c^2\boldsymbol{\nabla}^2\right)\right)\left(c\tilde{\boldsymbol{\alpha}}.\boldsymbol{\nabla} + \frac{\partial}{\partial t}\right) - V\left(\mathbf{x}, \boldsymbol{\nabla}, \frac{\partial}{\partial t}\right)\right\}\psi = \tilde{\beta} mc^2 \psi \tag{17}$$

Remember that the operator $\mathbf{X}$, when written in the position representation, entails a function of the non-deformed position vector $\mathbf{x}$, $\boldsymbol{\nabla}$ and $\frac{\partial}{\partial t}$. Eq. (17) is completely different from the version of the deformed Dirac equation used in Ref. [1] to study the Dirac hydrogen atom in the context of the GUP.

Finally, we should mention that when all the above considerations are taken into account, solutions of the KG equation and the Dirac equation with the Coulomb potential turn out to be not trivial at all.

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References

[1] V. Todorinov, P. Bosso, S. Das, Ann. Phys. 405 (2019) 92.
[2] C. Quesne, V.M. Tkachuk, Czech. J. Phys. 56 (2006) 1269.
[3] C. Quesne, V.M. Tkachuk, J. Phys. A, Mat. Gen. 39 (2006) 10909.