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A generalisation of Johnson graphs with an application to triple factorisations

Seyed Hassan Alavi

Department of Mathematics, Faculty of Science, Bu-Ali Sina University, Hamedan, Iran

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ABSTRACT

In this paper, we introduce a new generalisation of Johnson graphs. The study of these graphs is linked to the study of intransitive triple factorisations $\mathrm{Sym}(\Omega) = ABA$ of the (finite) symmetric group, where the subgroups A and B are intransitive subgroups of $\mathrm{Sym}(\Omega)$. Indeed, we give combinatorial arguments to investigate the conditions under which such factorisations exist. We also use combinatorial arguments to study those conditions for which $\mathrm{Sym}(\Omega)$ is a *Geometric ABA*-group, that is to say, $\mathrm{Sym}(\Omega) = ABA$, $A \not\subseteq B$, $B \not\subseteq A$ and $AB \cap BA = A \cup B$.

1. Introduction

There are some generalisations of Johnson graphs in the literature with different applications, see for example [10,7]. In this paper, we introduce a new generalisation of the Johnson graphs arising from the study of triple factorisations $\mathrm{Sym}(\Omega) = ABA$ of symmetric groups in terms of their *intransitive* subgroups A and B. We call such factorisations *intransitive triple factorisations*. Note that triple factorisations are fundamental in group theory as well as in geometry.

Let Ω be a set of size of $n \geq 3$ (an n-set), and let m and k be positive integers less than n. Let also j be a positive integer such that $\max\{0, m + k - n\} \leq j \leq \min\{m, k\}$. We define a graph $\Gamma := \mathsf{J}(n, m, k, j)$ to be the graph whose vertices are distinct m-subsets of Ω and each edge between two vertices X and Z corresponds to a k-subset Y if $|X \cap Y| = |Z \cap Y| = j$. We observe that Γ has no loops but may have multiple edges (see Figs. 1–3). We show that the *Johnson graph* $\mathsf{J}(n, m)$ is a spanning subgraph of Γ (see Corollary 2.11), and so Γ may be viewed as a generalisation of the Johnson graph. The complement map (i.e., $T \mapsto \overline{T} = \Omega \setminus T$, for all t-subsets T of Ω) gives rise to an isomorphism between $\mathsf{J}(n, m, k, j)$ and $\mathsf{J}(n, n - m, n - k, n - m - k + j)$, see Lemma 2.3. Therefore in most cases we may focus on the case where $m \leq n/2$.

Although in Section 2, we study some combinatorial properties of $\mathsf{J}(n, m, k, j)$ as a useful tool to study the existence of intransitive triple factorisations, our interest is to find those conditions under which $\mathsf{J}(n, m, k, j)$ is both complete and simple. Indeed, each triple factorisation $G = ABA$ corresponds to a *collinearly complete* coset geometry $\mathsf{Cos}(G; A, B)$ (with A the stabiliser of a point p and B the stabiliser of a line ℓ incident with p) in which "each pair of points is incident with at least one line", see Section 3 for more details. Let now $G := \mathrm{Sym}(\Omega)$, $A := G_X$ and $B := G_Y$ with $|X \cap Y| = j$, where X and Y are an m-subset and a k-subset of Ω, respectively. Then G is a group of automorphisms of the graph $\mathsf{J}(n, m, k, j)$ which is the collinearity graph of the associated *coset geometry* $\mathsf{Cos}(G; A, B)$ of the triple factorisation $G = ABA$. So existence of an intransitive triple factorisation $\mathrm{Sym}(\Omega) = ABA$ is equivalent to the graph $\mathsf{J}(n, m, k, j)$ being complete. Therefore, studying the completeness of $\mathsf{J}(n, m, k, j)$ in Proposition 2.6 suggests a necessary and sufficient condition for the existence of intransitive triple factorisations in Theorem 1.1.

E-mail addresses: alavi.s.hassan@gmail.com, alavi.s.hassan@basu.ac.ir.

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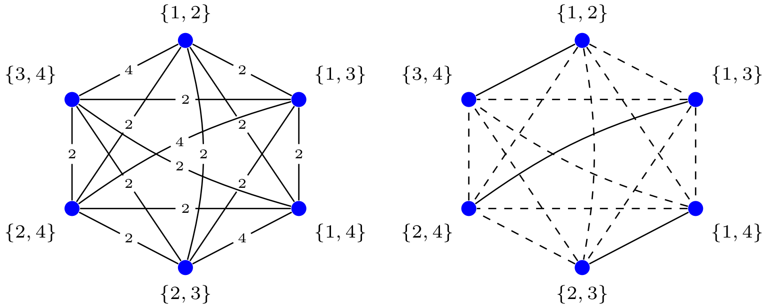
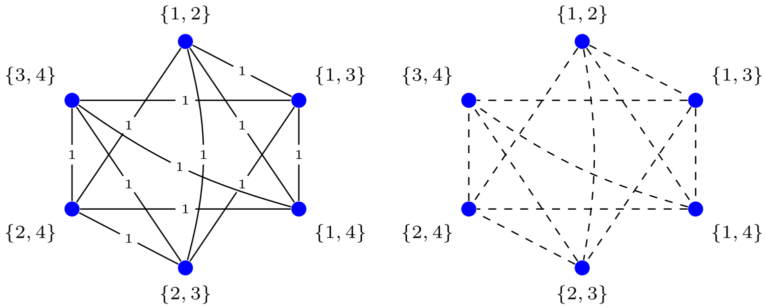


(a) The graph J(4, 2, 2, 1) is a multi-graph. The number on each edge represents the number of 2-subsets which meet vertices in 1 point. For example, there are four 2-subsets {1, 3}, {1, 4}, {2, 3}, {2, 4} which intersect both {1, 2} and {3, 4} in 1 point.

(b) The graph with dashed edges is the Johnson graph J(4, 2) which is a spanning subgraph of J(4, 2, 2, 1).

Fig. 1. The graph J(4, 2, 2, 1).



(a) The graph J(4, 2, 3, 1) is a simple graph but not complete.

(b) The graph with dashed edges is the Johnson graph J(4, 2) which is isomorphic to J(4, 2, 3, 1).

Fig. 2. The graph J(4, 2, 3, 1).

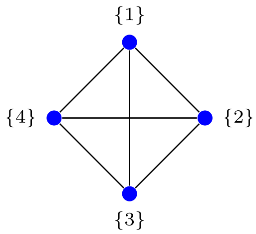


Fig. 3. The graph J(4, 1, 2, 0) is simple and complete. It is also isomorphic to J(4, 1, 2, 1) and the Johnson graph J(4, 1).

Theorem 1.1. *Let $n \geq 3$, m and k be positive integers such that $m, k < n$, and let $\Omega := \{1, \dots, n\}$. Let also $G = \mathrm{Sym}(\Omega)$, and let A and B be intransitive subgroups of G stabilising an m-subset X and a k-subset Y, respectively, with $j := |X \cap Y|$. Then $G = ABA$ if and only if $k + \min\{0, 2m - n\} \leq 2j \leq k + \max\{0, 2m - n\}$.*

Classifying triple factorisations $G = ABA$ seems to be out of reach in general (see for example, [4, Proposition 4.2]), however a reduction strategy has been introduced in [4] to the case where A is maximal (and core-free) in G. For geometric reasons, the subgroup B may also be assumed to be maximal and so both subgroups A and B have orders at least $|G|^{1/3}$. This motivated Alavi and Burness [3] to study large maximal subgroups H of finite simple groups G (i.e., $|H| \leq |G|^{1/3}$). In this direction, various triple factorisations of general linear groups $\mathrm{GL}(V)$ have been studied (see [1,2]). Triple factorisations $\mathrm{Sym}(\Omega) = ABA$ of symmetric groups with A and B conjugate subgroups have been studied in [8] and Theorem 1.1 focuses on intransitive factorisations of $\mathrm{Sym}(\Omega)$. We are also interested in a particular case of triple factorisations known as *Geometric ABA-group*, that is to say, $G = ABA$, $A \not\subseteq B$, $B \not\subseteq A$ and $AB \cap BA = A \cup B$. The notion of Geometric *ABA*-groups is introduced by Higman and McLaughlin and such factorisations are linked to studying flag-transitive linear spaces [9]. In fact, G is a Geometric *ABA*-group if and only if its associated collinearity graph is both complete and simple (see [9, Lemmas 1 and 3] and Section 3). Using this fact, obtaining the conditions under which $\mathrm{J}(n, m, k, j)$ is both complete and simple (Corollary 2.18) gives rise to those conditions for which the associated intransitive triple factorisation of $\mathrm{Sym}(\Omega)$ is a Geometric ABA-group, and vice versa:

Theorem 1.2. *Let $G = \mathrm{Sym}(\Omega)$, where $n := |\Omega| \geq 4$, and let m and k be positive integers less than n. Suppose that A and B are stabilisers of an m-subset X and a k-subset Y of Ω with $j := |X \cap Y|$, respectively. Then G is a Geometric ABA-group if and only if $A \cong S_{n-1}$ and $B \cong S_2 \times S_{n-2}$.*

In Section 2, we study some other combinatorial properties of $\mathsf{J}(n, m, k, j)$. For example, Corollary 2.23 suggests an upper bound for the distance between two vertices of $\mathsf{J}(n, m, k, j)$, and this leads us to an upper bound for the diameter of this graph. Our computations (see for example, Figs. 1–3) shows that these bounds are not sharp, and so we have the following unsolved Problem 1.3:

Problem 1.3. Let X and Z be two distinct vertices of the connected graph $\mathsf{J}(n, m, k, j)$ with $|X \cap Z| = i$. Then the distance $d(X, Z)$ between X and Z is equal to $\left\lceil \frac{m-i}{m-i_0} \right\rceil$, where $i_0 = \max\{0, 2m-n, 2j-k, 2m-n-(2j-k)\}$. Therefore the connected graph $\mathsf{J}(n, m, k, j)$ is of diameter $\left\lceil \frac{m-\max\{0,2m-n\}}{m-i_0} \right\rceil$.

In a connection with geometry, a collinearly complete rank 2 geometry has an associated Buekenhout diagram with point-diameter at most 3 (see [5]), and this gives only five possible values for the canonical parameters of the diagram: the point-diameter d_p, gonality g, and line-diameter d_ℓ. Thus $(d_p, g, d_\ell) \in \{(2, 2, 2), (3, 3, 3), (3, 3, 4), (3, 2, 3), (3, 2, 4)\}$. The geometries associated to $(2, 2, 2)$ are simply the *generalised di-gons*, and their automorphism groups G give degenerate factorisations $G = AB$. The geometries associated to $(3, 3, 3)$ and $(3, 3, 4)$ are flag-transitive linear spaces which have been classified up to the *one-dimensional affine* case [6]. Triple factorisations that we study in this paper and [1,2] are linked to the geometries with parameters $(3, 2, 3)$ and $(3, 2, 4)$.

2. Basic properties

In this section, we investigate various combinatorial properties of $\mathsf{J}(n, m, k, j)$. In what follows, we call a set (subset) of size t a t-set (t-subset).

Definition 2.1. Let Ω be an n-set with $n \geq 3$, and let m and k be positive integers less than n. For $1 \leq t \leq n$, denote by $\Omega(t)$ the set of all t-subsets of Ω. Let also j be a positive integer such that

$$\max\{0, m+k-n\} \leq j \leq \min\{m, k\}. \tag{1}$$

The graph $\Gamma := \mathsf{J}(n, m, k, j)$ is a multi-graph whose vertices are distinct m-subsets in $\Omega(m)$ and each edge between two vertices X and Z corresponds to $Y \in \Omega(k)$ with

$$|X \cap Y| = |Z \cap Y| = j. \tag{2}$$

We denote by $\mathsf{J}_s(n, m, k, j)$ the simple graph of Γ in which we draw an edge between two distinct vertices X and Z if there is a k-subset Y of Ω satisfying (2).

To simplify our arguments in the forthcoming sections, it is useful to introduce further notation:

Notation 2.2. *Let X and Z be two distinct vertices of $\mathsf{J}(n, m, k, j)$ with $I := X \cap Z$. Note that if $i := |I|$, then $\max\{0, 2m-n\} \leq i \leq m-1$. Define now*

$$\Omega_1 := \Omega \setminus (X \cup Z), \qquad X_1 := X \setminus I, \qquad Z_1 := Z \setminus I.$$

If Y is an edge between X and Z, then Y is a k-subset of Ω satisfying (2). Set $T := X \cap Y \cap Z$ with $t := |T|$, and define

$$U := X \cup Y \cup Z, \qquad Y_1 := Y \setminus (X \cup Z), \qquad J_1 := (X \cap Y) \setminus T, \qquad J_2 := (Z \cap Y) \setminus T.$$

Then $|\Omega_1| = n - 2m + i$, $|X_1| = |Z_1| = m - i$, $|U| = 2m + k - 2j - i + t$, $|Y_1| = k - 2j + t$ and $|J_r| = j - t$, for $r = 1, 2$ (see Fig. 4).

2.1. Algebraic properties

In this section, for studying $\mathsf{J}(n, m, k, j)$, we first give a reduction argument to the case where $m \leq n/2$. Then we show that the symmetric group $\mathrm{Sym}(\Omega)$ is a group of automorphisms of $\mathsf{J}(n, m, k, j)$. We also show that $\mathsf{J}(n, m, k, j)$ is vertex-transitive but not edge-transitive. In what follows, assume that Ω, n, m, k and j are as in Definition 2.1, and set $\Gamma := \mathsf{J}(n, m, k, j)$.

Lemma 2.3. *The graph $\mathsf{J}(n, m, k, j)$ is isomorphic to the graph $\mathsf{J}(n, \overline{m}, \bar{k}, \bar{j})$, where*

$$\overline{m} = n - m, \qquad \bar{k} = n - k \quad \text{and} \quad \bar{j} = n - m - k + j. \tag{3}$$

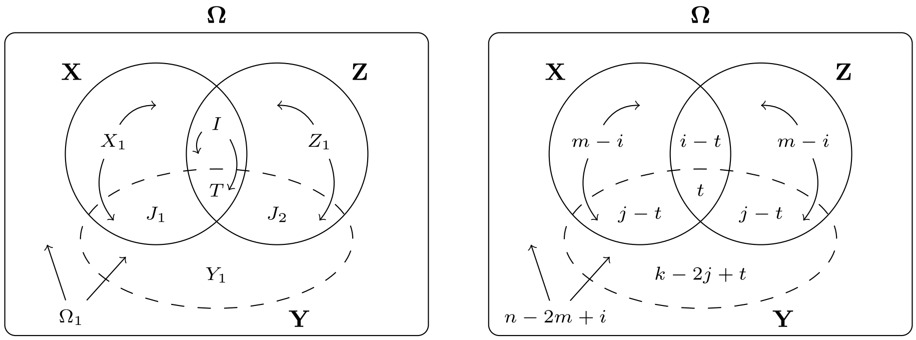


Fig. 4. Adjacent vertices X and Z (distinct m-subsets) in $\mathsf{J}(n, m, k, j)$ with $|X \cap Z| = i$ and an edge Y (a k-subset) between X and Z. The subsets $\Omega_1, I, T, X_1, Y_1, Z_1$ and J_r, for $r = 1, 2$, are as in Notation 2.2.

Proof. Set $\Gamma := \mathsf{J}(n, m, k, j)$ and $\overline{\Gamma} := \mathsf{J}(n, \overline{m}, \overline{k}, \overline{j})$, and consider the complement map $f : \Omega(m) \to \Omega(n-m)$ which sends each $X \in \Omega(m)$ to its complement $\overline{X} := \Omega \setminus X \in \Omega(n-m)$. Suppose that X and Z are adjacent in Γ. Then there exists a k-subset Y of Ω such that $|X \cap Y| = |Z \cap Y| = j$, and so

$$|\overline{X} \cap \overline{Y}| = |\overline{X \cup Y}| = n - |X \cup Y| = n - m - k + j = \overline{j},$$

and similarly $|\overline{Z} \cap \overline{Y}| = \overline{j}$. This shows that $\overline{X}$ and $\overline{Z}$ are adjacent in $\overline{\Gamma}$. Clearly, f is a bijection, and hence it is an isomorphism from Γ to $\overline{\Gamma}$. □

This, in particular, allows us to assume that $1 \le m \le n/2$ in the most of our arguments below.

Lemma 2.4. $\mathrm{Sym}(\Omega) \le \mathrm{Aut}(\Gamma)$.

Proof. Let $g \in \mathrm{Sym}(\Omega)$. Then X and Z are adjacent if and only if there is $Y \in \Omega(k)$ such that $|X \cap Y| = |Z \cap Y| = j$. This holds if and only if $|X^g \cap Y^g| = |Z^g \cap Y^g| = j$, or equivalently, X^g and Z^g are adjacent. □

Proposition 2.5. (a) *Γ is vertex-transitive;*
(b) *If $X \in \Omega(m)$ and $G = \mathrm{Sym}(\Omega)$, then the G_X-orbits on $\Omega(m)$ are of the form*

$$\Delta_i = \{Z \in \Omega(m) \mid |Z \cap X| = i\}, \tag{4}$$

for $\max\{0, 2m - n\} \le i \le m$. Hence Γ is not G-arc-transitive;
(c) *Γ is not G-edge-transitive.*

Proof. (a) This part follows from Lemma 2.4 and the fact that $\mathrm{Sym}(\Omega)$ acts transitively on the set $\Omega(m)$ via $\{x_1, \ldots, x_m\}^g := \{x_1^g, \ldots, x_m^g\}$, for all $g \in \mathrm{Sym}(\Omega)$.
(b) It is well-known that $\mathrm{Sym}(\Omega)$ has rank $m + 1$ in its action on the set of m-subspaces of Ω, and hence this part follows immediately by looking at the permutation character and applying the Young's rule.
(c) Let $g \in \mathrm{Aut}(\Gamma)$. If X and Z are distinct vertices in Γ, then $|X \cap Z| = |X^g \cap Z^g|$, and so if X and Z are adjacent with $|X \cap Z| = i$, then the G-orbit of the edge XZ contains exactly those edges $X'Z'$ of Γ with $|X' \cap Z'| = i$. □

2.2. Completeness

Proposition 2.6. *The graph $\mathsf{J}(n, m, k, j)$ is a complete graph if and only if $k + \min\{0, 2m - n\} \le 2j \le k + \max\{0, 2m - n\}$.*

Proof. Let $\Gamma := \mathsf{J}(n, m, k, j)$, and let X and Z be vertices of Γ with $|X \cap Z| = i$. In order for Γ to be complete, there must be enough room for there to be a k-subset Y meeting both X and Z in j points (see Fig. 4). Therefore a k-set Y exists if and only if for every possible i, the complement Ω_1 of $X \cup Z$ contains at least $k - 2j + i$ elements, that is, $0 \le k - 2j + i \le n - 2m + i$. Note that $\max\{0, 2m - n\} \le i < m$. Therefore Γ is complete if and only if $\min\{0, n - 2m\} \le k - 2j \le \max\{0, n - 2m\}$. This proves the result. □

2.3. Connectivity

Lemma 2.7. *Let X and Z be distinct vertices of $\mathsf{J}(n, m, k, j)$. Then*

(a) *X and Z are adjacent if and only if $(k, j) \ne (m, m), (n - m, 0)$ and*

$$\max\{0, 2m - n, 2j - k, 2m - n - (2j - k)\} \le |X \cap Z| \le m - 1; \tag{5}$$

(b) *if Y is an edge between X and Z, then*

$$\max\{0, i+j-m, 2j-k\} \leq |X \cap Y \cap Z| \leq \min\{i, j, n+2j+i-2m-k\}. \tag{6}$$

Proof. Let $\Gamma := \mathsf{J}(n, m, k, j)$. By Lemma 2.3, we may assume that $m \leq n/2$.

(a) Suppose that X and Z are adjacent vertices of Γ. Let I, T, Ω_1, U, Y_1 and J_r, for $r = 1, 2$, are as in Notation 2.2. Since X and Z are adjacent, there exists a k-subset Y of Ω such that $|X \cap Y| = |Z \cap Y| = j$. If $(k, j) = (m, m)$, then we must have $Y = X$ and $Y = Z$, and so $X = Z$, which is a contradiction. If $(k, j) = (n-m, 0)$, then Y meets neither X, nor Z. So Y is a subset of Ω_1 of size $n-2m+i$. This implies that $k = n-m \leq n-2m+i$, or equivalently, $m \leq i$, which is also a contradiction. Thus $(k, j) \neq (m, m), (n-m, 0)$. Since U is a subset of Ω, we have that $2m+k-2j-i \leq 2|X|+|Y|-2|J_1|-|J_2|+|T| = |U| \leq |\Omega| = n$, and so $2m-n-2j+k \leq i$. Moreover, $J_1 \cup J_2$ is a subset of Y and $T \subseteq I$. Then $2j-i = 2j-|I| \leq |J_1|+|J_2|-|T| = |J_1 \cup J_2| \leq |Y| = k$, and so $2j-k \leq i$. Note that $0 \leq i \leq m-1$. Therefore (5) holds.

Conversely, suppose that X and Z are vertices in Γ with $(k, j) \neq (m, m), (n-m, 0)$ and $i = |X \cap Z|$ satisfying (5). Suppose also Ω_1, I, X_1 and Z_1 are as in Notation 2.2. In each of the following cases, we find a k-subset Y of Ω satisfying (2), and hence X and Z will be adjacent.

Let $j = 0$. Then by (5), we have that $k+2m-n = k+2m-2j-n \leq i$, and so $k \leq n-2m+i = |\Omega_1|$, and hence we can choose a k-subset Y of Ω_1. Note that $|X \cap Y| = |Z \cap Y| = 0 = j$. Then, in this case, X and Z are adjacent in Γ.

Let now $1 \leq j \leq m-i$. By (5), we have that $2m-n-(2j-k) \leq i$, and so $k-2j \leq n-2m+i = |\Omega_1|$. If $k-2j \geq 0$, we can choose a $(k-2j)$-subset Y_1 of Ω_1 (if $k-2j = 0$, we simply choose $Y_1 = \varnothing$). As $j \leq m-i$, we also can choose j-subsets J_1 and J_2 of X_1 and Z_1, respectively. Then the subset $Y := J_1 \cup J_2 \cup Y_1$ is of size k, and $|X \cap Y| = |J_1| = j$ and $|Z \cap Y| = |J_2| = j$, and hence X and Z are adjacent in Γ. If $k-2j < 0$, then $t := 2j-k \geq 1$. Note by (5) that $t = 2j-k \leq i$, and so we can take a t-subset T of I. Since $0 \leq k-j = j-t \leq j \leq m-i$, we choose $(j-t)$-subsets J_1 and J_2 of X_1 and Z_1, respectively. Set $Y = T \cup J_1 \cup J_2$. Then $|Y| = t+2(j-t) = 2j-t = 2j-(2j-k) = k$. Moreover, $|X \cap Y| = |T \cup J_1| = j$ and $|Z \cap Y| = |T \cup J_2| = j$. Therefore X and Z are adjacent in Γ.

Let finally $j > m-i$. Set $a := k-j-m+i$. Since $j \geq m+k-n$, we have that $a = k-j-m+i \leq k-(m+k-n)-m+i = n-2m+i = |\Omega_1|$. If $a \geq 0$, then we can choose an a-subset Y_1 of Ω_1. Since $j \leq m$, it follows that $j-(m-i) \leq i = |I|$, and so we choose a $(j-m+i)$-subset T of I. Set $Y := T \cup X_1 \cup Z_1 \cup Y_1$. Then $|Y| = k$ and $|Y \cap X| = |T \cup X_1| = |T| + |X_1| = (j-m+i)+(m-i) = j$. Similarly, $|Y \cap Z| = j$. Thus X and Z are adjacent in Γ. If $a < 0$, then $k-j < m-i$, and since $j > m-i$, it follows that $2j-k > 0$, and so by (5), we have that $0 < 2j-k \leq i$. Note that $0 < 2j-k \leq \min\{i, j\}$. Then we can choose a $(2j-k)$-subset T of I, and since $k-j < m-i$, we can also take $(k-j)$-subsets J_1 and J_2 of X_1 and Z_1, respectively. Define $Y = T \cup J_1 \cup J_2$. Then $|Y| = (2j-k)+2(k-j) = k$, $|X \cap Y| = |T \cup J_1| = (2j-k)+k-j = j$ and $|Z \cap Y| = |T \cup J_2| = (2j-k)+k-j = j$. This shows that X and Z are adjacent.

(b) Let I, T, Ω_1, Y_1 and J_r, for $r = 1, 2$ be as in Notation 2.2. Recall that $|I| = i$, $|T| = t$, $|\Omega_1| = n-2m+i$, $|Y_1| = k-2j+t$ and $|J_r| = j-t$, for $r = 1, 2$. Since $T \subseteq I$ and $T \subseteq X \cap Y$, it follows that $t \leq \min\{i, j\}$. Since also Y_1 is a subset of Ω_1, we have that $k-2j+t \leq n-2m+i$, or equivalently, $t \leq n+2j+i-2m-k$. Thus $t \leq \min\{i, j, n+2j+i-2m-k\}$. Note that $T = I \cap J_1$ and $I \cup J_1 \subseteq X$. Then $t = |T| = |I| + |J_1| - |I \cup J_1| \geq i+j-m$. Moreover $T = J_1 \cap J_2$ and $J_1 \cup J_2 \subseteq Y$. So $t \geq 2j-k$. Since $t \geq 0$, we conclude that $t \geq \max\{0, i+j-m, 2j-k\}$. Hence (6) holds. □

Definition 2.8. Let $\Omega = \{1, \ldots, n\}$ with $n \geq 3$ positive integer, and let m be a positive integer less than n. Suppose that i is a positive integer such that $\max\{0, 2m-n\} \leq i \leq m-1$. For $0 \leq r \leq m-i$, define $U_r = I \cup V_r$, where

$$I = \begin{cases} \{1, \ldots, i\}, & \text{if } 1 \leq i \leq m-1; \\ \varnothing, & \text{if } i = 0. \end{cases}$$

$$V_r = \begin{cases} \{i+r+1, \ldots, m\} \cup \{m+1, \ldots, m+r\}, & \text{if } 1 \leq r \leq m-i-1; \\ \{m+1, \ldots, 2m-i\}, & \text{if } r = m-i; \\ \{i+1, \ldots, m\}, & \text{if } r = 0. \end{cases}$$

Note, for each possible r, that U_r is an m-subset of Ω.

Lemma 2.9. *Let Ω, n, m, i and U_r be as in Definition 2.8, and let k and j be positive integers such that $(k, j) \neq (m, m), (n-m, 0)$. Then, for each r with $0 \leq r \leq m-i-1$, we have that $|U_r \cap U_{r+1}| = m-1$ and $(U_0, \ldots, U_{2m-i})$ is a path in $\mathsf{J}(n, m, k, j)$.*

Proof. By Definition 2.8, for $0 \leq r \leq m-i-1$, we observe that

$$U_r \cap U_{r+1} = I \cup \begin{cases} \{i+r+2, \ldots, m\} \cup \{m+1, \ldots, m+r\}, & \text{if } 1 \leq r \leq m-i-2; \\ \{m+1, \ldots, 2m-i-1\}, & \text{if } r = m-i-1; \\ \{i+2, \ldots, m\}, & \text{if } r = 0. \end{cases} \tag{7}$$

This shows that $|U_r \cap U_{r+1}| = m-1$. Then, by Lemma 2.7(a), for each r, two vertices U_r and U_{r+1} are adjacent, and consequently, $(U_0, \ldots, U_{2m-i})$ is a path. □

Proposition 2.10. *The graph $\mathsf{J}(n, m, k, j)$ is connected if and only if $(k, j) \neq (m, m), (n-m, 0)$.*

Proof. Let $\Gamma := \mathsf{J}(n, m, k, j)$. If $(k, j) = (m, m), (n-m, 0)$, then obviously Γ is a null graph, and hence it is not connected. Conversely, suppose that $(k, j) \neq (m, m), (n-m, 0)$. By Lemma 2.3, we only need to focus one the case where $m \le n/2$. Let X and Z be two distinct vertices of $\mathsf{J}(n, m, k, j)$, and let $I := X \cap Z$. Set $i := |I|$. Then $0 = \max\{0, 2m-n\} \le i \le m$. By Proposition 2.5(b), we may assume that $X = U_0$ and $Z = U_{m-i}$, where U_0 and U_{m-i} are as in Definition 2.8. Hence Lemma 2.9 introduces a path between X and Z. □

Corollary 2.11. *The Johnson graph* $\mathsf{J}(n, m)$ *is a spanning subgraph of the connected graph* $\mathsf{J}(n, m, k, j)$*, for every* j *satisfying* (1).

Proof. Note that the Johnson graph $\mathsf{J}(n, m)$ has the same vertex set as $\mathsf{J}(n, m, k, j)$. Since $\mathsf{J}(n, m, k, j)$ is connected, it follows from Proposition 2.10 that $(k, j) \neq (m, m), (n-m, 0)$. Let X and Z be two distinct vertices of $\mathsf{J}(n, m, k, j)$ with $m-1$ elements in common. By Proposition 2.5(b), we may assume that $X = U_0$ and $Z = U_1$, where U_0 and U_1 are as in Definition 2.8, and so by Lemma 2.9, X and Z are adjacent, for every possible j as in (1). This proves the result. □

Corollary 2.12. *Let* X *be an* m*-subset of* Ω*, and let* k *be positive integer such that* $1 \le k \le n := |\Omega|$*. Let also* j *be as in* (1)*. Suppose that* $(k, j) \neq (m, m), (n-m, 0)$*. Then there exist an* m*-subset* Z *and a* k*-subset* Y *of* Ω *such that* $|Y \cap X| = |Y \cap Z| = j$*.*

Proof. Without loss of generality we may assume that $X = \{1, \dots, m\}$. Set $Z = \{1 \dots, m-1, m+1\}$. Then $X = U_0$ and $Z = U_1$, where U_0 and U_1 are as in Definition 2.8, and so the assertion follows from Lemma 2.9. □

2.4. Simplicity

Lemma 2.13. *Let* X *and* Z *be two distinct vertices of* $\mathsf{J}(n, m, k, j)$ *with* $i := |X \cap Z|$*. If* $\mathbf{w}(X, Z)$ *is the number of edges between* X *and* Z*, then*

$$\mathbf{w}(X, Z) = \sum_{t=t_0}^{t_1} \binom{i}{t} \binom{m-i}{j-t}^2 \binom{n-2m+i}{k-2j+t}, \tag{8}$$

where $t_0 := \max\{0, i+j-m, 2j-k\}$ *and* $t_1 := \min\{i, j, n+2j+i-2m-k\}$*.*

Proof. Let Y be an edge which joins X and Z. Then $Y = T \cup Y_1 \cup J_1 \cup J_2$, where $I, T, U, \Omega_1, X_1, Z_1$ and J_r, for $r = 1, 2$, are as in Notation 2.2 (see also Fig. 4). Note that the number of edges between X and Z is the number of distinct such k-subsets Y. Therefore to construct such k-subsets we need to choose $\binom{i}{t}$ number of t-subsets T of I with t as in (6). Next, for each possible t as in (6), we must choose $\binom{m-i}{j-t}^2$ number of $(j-t)$-subsets J_1 and J_2 of X_1 and Z_1, respectively, and finally, we have to choose $\binom{n-2m+i}{k-2j+t}$ number of $(k-2j+t)$-subsets Y_1 of Y. The assertion follows from counting principals and (6). □

Lemma 2.14. *Let* X *and* Z *be adjacent vertices in* $\mathsf{J}(n, m, k, j)$ *with* $n \ge 4$ *and* $m \ge 2$*. If* $|X \cap Z| < m-1$*, then* $\mathsf{J}(n, m, k, j)$ *is not simple.*

Proof. Assume contrary and let $\Gamma := \mathsf{J}(n, m, k, j)$ be simple. By Lemma 2.3, we only need to focus on the case where $m \le n/2$. As $n \ge 4$, we have that $m \le n/2 = n-2+(4-n)/2 \le n-2$. Set $i := |X \cap Z|$. Then by Proposition 2.5(b), we may assume that $X = U_0$ and $Z = U_{m-i}$ defined as in Definition 2.8. As Γ is simple, $\mathbf{w}(X, Z) = 1$, where $\mathbf{w}(X, Z)$ is the number of edges between X and Z. Suppose that Y is the unique k-subset of Ω with $|X \cap Y| = |Z \cap Y| = j$. Let $T := X \cap Y \cap Z$ and $t := |T|$ (see Fig. 4). Note by Lemma 2.13 that $t \in \{0, i\}$.

(i) Suppose $i = 0$. By Lemma 2.7(b), we have that $t \le \min\{i, j\} = 0$, and so $t = 0$. Thus Lemma 2.13 implies that

$$\begin{aligned}\mathbf{w}(X, Z) &= \binom{i}{t}\binom{m-i}{j-t}^2\binom{n-2m+i}{k-2j+t}\\ &= \binom{m}{j}^2\binom{n-2m}{k-2j}.\end{aligned}$$

Since $\mathbf{w}(X, Z) = 1$ and $1 \le k < n$, we conclude that $(k, j) = (n-2m, 0)$ or $(2m, m)$. In each case, we find adjacent vertices X' and Z' with $\mathbf{w}(X', Z') \ge 2$ which leads us to a contradiction. Since $0 = i \le m-2$, we can take $X' := U_0 = X$ and $Z' := U_1 = \{1, \dots, m-1, m+1\}$ as in Definition 2.8. Then $i' := |X' \cap Z'| = m-1$, and so Lemma 2.9 implies that X' and Z' are adjacent in Γ.

If $(k, j) = (n-2m, 0)$, then $t_0' := \max\{0, i'+j-m, 2j-k\} = \max\{0, -1, -k\} = 0$ and $t_1' := \min\{i', j, n+2j+i'-2m-k\} = \min\{0, m-1\} = 0$, and so by Lemma 2.13, we have that

$$\begin{aligned}\mathbf{w}(X', Z') &= \binom{i'}{0}\binom{m-i'}{j}^2\binom{n-2m+i'}{k-2j}\\ &= \binom{m-1}{0}\binom{1}{0}^2\binom{n-m-1}{n-2m} = \binom{n-m-1}{n-2m}.\end{aligned}$$

As $m \geq 2$, it follows that $(n-m-1)-(n-2m) = m-1 \geq 1$, and since $n-2m = k \geq 1$, we conclude that $\mathbf{w}(X', Z') \geq 2$, which is a contradiction.

If $(k,j) = (2m,m)$, then $t_0' := \max\{0, i'+j-m, 2j-k\} = \max\{0, m-1\} = m-1$, and since $4 \leq 2m = k < n$, we have that $(n-m-1)-(m-1) \geq 1$, and so $t_1' := \min\{i', j, n+2j+i'-2m-k\} = \min\{m-1, n-m-1\} = m-1$. Then Lemma 2.13 implies that

$$\begin{aligned}\mathbf{w}(X', Z') &= \binom{m-1}{m-1}\binom{1}{1}^2\binom{n-m-1}{m-1}\\ &= \binom{n-m-1}{m-1}.\end{aligned}$$

Since $n-m-1 > m-1 \geq 1$, it follows that $\mathbf{w}(X', Z') \geq 2$, which is a contradiction.

(ii) Suppose now $i \neq 0$. Since $\mathbf{w}(X, Z) = 1$, by Lemma 2.13, we must have $i = t$. Let $\Gamma_1 := \mathsf{J}(n-i, m-i, k-t, j-t)$. Let also X_1 and Z_1 be as in Notation 2.2 (see Fig. 4). Then $m_1 := |X_1| = |Z_1| = m-i$, and so we may view X_1 and Z_1 as adjacent vertices of Γ_1. Since $i \leq m-2$ and $n \geq 2m$, it follows that $n_1 := n-i \geq n-(m-2) \geq 2m-m+2 = m+2 \geq 4$ and $m_1 = m-i \geq m-(m-2) = 2$. Moreover, $i_1 := |X_1 \cap Z_1| = 0 \leq m-2$. Hence we can apply part (i) to the graph Γ_1, for X_1 and Z_1. Therefore we obtain m_1-subsets X' and Z' of $\Omega' := \Omega \setminus I$ with $\mathbf{w}(X', Z') \geq 2$. Therefore there exist at least two $(k-t)$-subsets Y_1' and Y_2' of Ω' such that $|X' \cap Y_r'| = |Z' \cap Y_r'| = j-t$, for $r = 1, 2$. Set $X'' := X' \cup I$, $Z'' := Z' \cup I$ and $Y_r'' := I \cup Y_r'$, for $r = 1, 2$. Since $t = i$, we have that $|X'' \cap Y_r''| = |X' \cap Y_r''| + |I \cap Y_r''| = (j-t)+i = j$, and similarly $|Z'' \cap Y_r''| = j$, for $r = 1, 2$. This shows that $\mathbf{w}(X'', Z'') \geq 2$ in Γ, which is a contradiction. □

Corollary 2.15. *If the graph $\mathsf{J}(n, m, k, j)$ is simple with $n \geq 4$ and $m \geq 2$, then it is the Johnson graph $\mathsf{J}(n, m)$.*

Proof. By Corollary 2.11, the Johnson graph $\mathsf{J}(n, m)$ is a spanning subgraph of $\mathsf{J}(n, m, k, j)$, for every j satisfying (1). If X and Z are adjacent vertices of the simple graph $\mathsf{J}(n, m, k, j)$, then by Lemma 2.14, we must have $|X \cap Z| = m-1$. Thus X and Y are adjacent in $\mathsf{J}(n, m)$. □

Theorem 2.16. *Let $n \geq 3$ be a positive integer and $1 \leq m \leq n/2$. Then $\mathsf{J}(n, m, k, j)$ is a simple graph if and only if $(k, j) \in \{(n-m-1, 0), (n-m+1, 1), (m+1, m), (m-1, m-1)\}$.*

Proof. If $n = 3$, then $(m, k, j) = (1, 1, 0)$, and so $\mathsf{J}(3, 1, 1, 0)$ is the cycle graph C_3 which is a simple graph. In what follows, we assume that $n \geq 4$. Suppose also $\Gamma := \mathsf{J}(n, m, k, j)$ with

$$(k, j) \in \mathcal{A} := \{(n-m-1, 0), (n-m+1, 1), (m+1, m), (m-1, m-1)\}. \tag{9}$$

Let X and Z be adjacent vertices of Γ with $i := |X \cap Z|$. Using Lemma 2.13, we show that the number $\mathbf{w}(X, Z)$ of edges between X and Z is 1. Set

$$t_0 := \max\{0, i+j-m, 2j-k\} \quad \text{and} \quad t_1 := \min\{i, j, n+2j+i-2m-k\}. \tag{10}$$

If $m = 1$, then $i = 0$, and since $1 \leq k < n$, by (9), we have that $(k, j) = (n-2, 0)$ or $(2, 1)$. In both cases, we observe that $t_0 = t_1 = 0$, where t_0 and t_1 are as in (10). Hence Lemma 2.13 implies that $\mathbf{w}(X, Z) = 1$. If $m \geq 2$, then since $m \leq n/2$ and $n \geq 4$, it follows that $m \leq n-2$. Let $(k, j) = (n-m-1, 0)$. Then $2j-k = -(n-m-1) < 0$ and $2m-n-(2j-k) = 2m+(n-m-1)-n = m-1$, and so $m-1 = \max\{0, 2j-k, 2m-n-(2j-k)\} \leq i \leq m-1$ by Lemma 2.7(a). This implies that $i = m-1$. Thus $i+j-m = (m-1)-m = -1$ and $n+2j+i-2m-k = n+(m-1)-2m-(n-m-1) = 0$, and so $t_0 = t_1 = 0$, where t_0 and t_1 are as in (10). It follows from Lemma 2.13 that

$$\mathbf{w}(X, Z) = \binom{m-1}{0}\binom{1}{0}^2\binom{n-m-1}{n-m-1} = 1.$$

Hence, Γ is simple. By a similar argument for other possibilities of $(k, j) \in \mathcal{A}$, we conclude that $\mathbf{w}(X, Z) = 1$, and hence Γ is simple.

Conversely, suppose that Γ is simple. Then for adjacent vertices X and Z, we have that $\mathbf{w}(X, Z) = 1$, that is to say, there exists exactly one k-subset Y of Ω such that $|X \cap Y| = |Z \cap Y| = j$. Set $i := |X \cap Z|$, $T := X \cap Y \cap Z$ and $t := |T|$.

Let $m = 1$. Then $i = 0$ and $j \in \{0, 1\}$. As $t \leq \min\{i, j\} = 0$, it follows that $t = 0$. Suppose $j = 0$. Then Lemma 2.13 implies that Γ is simple if and only if $\binom{n-2}{k} = \binom{n-2m+i}{k-2j+t} = \mathbf{w}(X, Z) = 1$. Since $k \geq 1$, this is equivalent to $k = n-2$. Hence $(k, j) = (n-2, 0)$. Suppose now $j = 1$. Similarly, by Lemma 2.13, simplicity of Γ implies that $\binom{n-2}{k-2} = \mathbf{w}(X, Z) = 1$, and so $k-2 = 0$ or $n-2$. The latter case does not hold as $k < n$. Thus $k = 2$, and hence $(k, j) = (1, 2) = (m, m+1)$.

Let now $m \geq 2$. Note that $m \leq n-2$ (as $m \leq n/2$ and $n \geq 4$). If $i \leq m-2$, then by Lemma 2.14, the graph Γ is not simple which is a contradiction. Thus $i = m-1$. By Proposition 2.5(b), we may assume that $X = \{1, \ldots, m-1, m\}$ and $Z := \{1, \ldots, m-1, m+1\}$. Note that Γ is simple if and only if $\mathbf{w}(X, Z) = 1$, or equivalently, by Lemma 2.13,

$$t \in \{0, i\}, \quad j-t \in \{0, m-i\}, \quad k-2j+t \in \{0, n-2m+i\} \quad \text{and} \quad t_0 = t_1, \tag{11}$$

where t_0 and t_1 satisfy (10).

Suppose $t = 0$. Then by (11), we have that $j \in \{0, 1\}$. Let $j = 0$. Since $k \geq 1$, we must have $k - 2j + t = n - 2m + i$, or equivalently, $k = n - m - 1$. Note that $t_0 = 0 = t_1$. Thus $(k, j) = (n - m - 1, 0)$. Let now $j = 1$. If $k - 2j + t = 0$, then $k = 2$, and so $t_0 = 0 \neq 1 = t_1$ (as $m \leq n - 2$) which is a contradiction. If $k - 2j + t = n - 2m + i$, then $k = n - m + 1$, and so $t_0 = 0 = t_1$. Hence $(k, j) = (n - m + 1, 1)$.

Suppose now $t = i = m - 1$. Then by (11), we have that $j \in \{m - 1, m\}$. Let $j = m - 1$. If $k - 2j + i = n - 2m + t$, then $k = n - 2$, and so $t_0 = m - 2 \neq m - 1 = t_1$ which is a contradiction. Thus $k - 2j + t = 0$. Then $k = m - 1$, and so $t_0 = m - 1 = t_1$. Therefore $(k, j) = (m - 1, m - 1)$. Let now $j = m$. If $k - 2j + i = n - 2m + t$, then $k = n$ which is a contradiction. If $k - 2j + t = 0$, then $k = m + 1$, and so $t_0 = m - 1 = t_1$. Therefore $(k, j) = (m + 1, m)$. □

Remark 2.17. By Theorem 2.16 and Lemma 2.3, we also obtain isomorphic simple graphs $\mathsf{J}(n, n - m, \bar{k}, \bar{j})$, where $(\bar{k}, \bar{j}) = (m + 1, 1), (m - 1, 0), (n - m - 1, n - m - 1), (n - m + 1, n - m)$.

Corollary 2.18. *Let $n \geq 3$ be a positive integer. Then $\mathsf{J}(n, m, k, j)$ is complete and simple if and only if $(m, k, j) \in \{(1, n - 2, 0), (1, 2, 1), (n - 1, 2, 1), (n - 1, n - 2, n - 2)\}$.*

Proof. It is obvious when $n = 3$. Let $\Gamma := \mathsf{J}(n, m, k, j)$ with $n \geq 4$. Suppose first $(m, k, j) \in \{(1, n - 2, 0), (1, 2, 1)\}$. Then (m, k, j) satisfies $k + \min\{0, 2m - n\} \leq 2j \leq k + \max\{0, 2m - n\}$. Thus Proposition 2.6 implies that Γ is complete. Moreover, Γ is $\mathsf{J}(n, m, n - m - 1, 0)$ or $\mathsf{J}(n, m, m + 1, m)$ with $m = 1$, and so by Theorem 2.16, Γ is also simple. Suppose now $(m, k, j) \in \{(n - 1, 2, 1), (n - 1, n - 2, n - 2)\}$. Then $(\overline{m}, \bar{k}, \bar{j})$ is $(1, n - 2, 0)$ or $(1, 2, 1)$, where $\overline{m}, \bar{k}, \bar{j}$ are as in (3), and so by Lemma 2.3 and above argument, we have that Γ is complete and simple.

Conversely, suppose that Γ is complete and simple. Let first $m \leq n/2$. Since Γ is simple, by Theorem 2.16, we have that $(k, j) \in \{(n - m - 1, 0), (n - m + 1, 1), (m + 1, m), (m - 1, m - 1)\}$. Since also Γ is complete and $m \leq n/2$, Proposition 2.6 implies that

$$k + 2m - n \leq 2j \leq k. \tag{12}$$

Let $(k, j) = (n - m - 1, 0)$. Then (12) follows that $(n - m - 1) + 2m - n \leq 0 \leq n - m - 1$, and so $m \leq \min\{1, n - 1\}$. Since $n \geq 3$, we observe that $\min\{1, n - 1\} = 1$, and so $m = 1$. This implies that $k = m - m - 1 = n - 2$, and hence $(m, k, j) = (1, n - 2, 0)$.

Let $(k, j) = (m + 1, m)$. By (12), we have that $(m + 1) + 2m - n \leq 2m \leq m + 1$, and so $m \leq \min\{1, n - 1\}$. Since $n \geq 3$ and $m \geq 1$, we observe that $m = 1$ which implies that $k = 2$ and $j = 1$, and hence $(m, k, j) = (1, 2, 1)$.

Let $(k, j) = (n - m + 1, 1)$. Then (12) implies that $(n - m + 1) + 2m - n \leq 2 \leq n - m + 1$, and so $m \leq \min\{1, n - 1\} = 1$. It follows that $k = n$, which is a contradiction. Similarly, if $(k, j) = (m - 1, m - 1)$, then $(m - 1) + 2m - n \leq 2m - 2 \leq m - 1$, and so $m \leq \min\{1, n - 1\} = 1$. Hence $k = m - 1 = 0$ which is a contradiction.

Let now $n/2 < m < n$, and let $\overline{m}, \bar{k}, \bar{j}$ be as in (3). Since Γ is complete and simple, it follows from Lemma 2.3 that $\mathsf{J}(n, \overline{m}, \bar{k}, \bar{j})$ is complete and simple with $\overline{m} \leq n/2$, and so by above argument $(\overline{m}, \bar{k}, \bar{j}) = (n - 1, 2, 1), (n - 1, n - 2, n - 2)$, or equivalently, $(m, k, j) = (n - 1, 2, 1), (n - 1, n - 2, n - 2)$. □

2.5. Girth and valency

Proposition 2.19. *The connected graph $\mathsf{J}(n, m, k, j)$ with $n \geq 3$ is of girth 3.*

Proof. By Lemma 2.3, we may assume that $m \leq n/2$. If $n = 3$, then $(m, k, j) = (1, 1, 0)$, and so $\mathsf{J}(3, 1, 1, 0)$ is the cycle graph C_3 which is of girth 3. If $n \geq 4$, then $m \leq n/2 = n - 2 + (4 - n)/2 \leq n - 2$, and so by Proposition 2.5(b), we can choose three distinct m-subsets X, Z and T, where

$$X = \{1, \ldots, m\}, \qquad Z = \{1, \ldots, m - 1, m + 1\} \quad \text{and} \quad T = \{1, \ldots, m - 1, m + 2\}.$$

Note that each pair of these three m-subsets has $m - 1$ points in common. Then by Corollary 2.11, we observe that X, Z and T are pairwise adjacent, and hence $\mathsf{J}(n, m, k, j)$ has a triangle and is of girth 3. □

Recall that the vertices of the graph $\mathsf{J}_s(n, m, k, j)$ are m-subsets of Ω, and two distinct vertices X and Z are adjacent if there exists a k-subset Y such that $|X \cap Y| = |Z \cap Y| = j$.

Proposition 2.20. *The graph $\mathsf{J}_s(n, m, k, j)$ is regular of valency*

$$\sum_{i=i_0}^{m-1} \binom{m}{i} \binom{n - m}{m - i}, \tag{13}$$

where $i_0 = \max\{0, 2m - n, 2j - k, 2m - n - (2j - k)\}$.

Proof. Let X be a vertex of $\Gamma := \mathsf{J}_s(n, m, k, j)$, and let Z be an m-subset adjacent to X with $|X \cap Z| = i$. For each possible i as in (5), there exist $\binom{m}{i}\binom{n-m}{m-i}$ subsets Z adjacent to X which intersect X at i points. Hence (13) is the total number of m-subsets adjacent to X. □

2.6. Distance and diameter

Definition 2.21. Let $\Omega = \{1, \ldots, n\}$ with $n \geq 3$ positive integer, and let m be a positive integer less than n. Suppose that i and i_0 are positive integers such that $i < i_0$ and

$$\max\{0, 2m-n\} \leq i \leq m-1;$$
$$\max\{0, 2m-n, 2j-k, 2m-n-(2j-k)\} \leq i_0 \leq m-1.$$

Set $\ell := \lceil \frac{m-i}{m-i_0} \rceil$. For $0 \leq r \leq \ell$, define $U_r = I \cup V_r$, where

$$I = \begin{cases} \{1, \ldots, i\}, & \text{if } 1 \leq i \leq m-1; \\ \varnothing, & \text{if } i = 0. \end{cases}$$

$$V_r = \begin{cases} \{i + r(m-i_0) + 1, \ldots, m\} \cup \{m+1, \ldots, m + r(m-i_0)\}, & \text{if } 1 \leq r \leq \ell - 1; \\ \{m+1, \ldots, 2m-i\}, & \text{if } r = \ell; \\ \{i+1, \ldots, m\}, & \text{if } r = 0. \end{cases}$$

Lemma 2.22. *Let i, i_0, ℓ and U_r be as in Definition 2.21. Then U_r is an m-subset of Ω, for each possible r. If $0 \leq r < s \leq \ell$, then*

$$|U_r \cap U_s| = \begin{cases} m + (r-s)(m-i_0) & \text{if } 1 \leq r < s \leq \ell - 1; \\ i + r(m-i_0), & \text{if } 1 \leq r \leq \ell - 1 \text{ and } s = \ell; \\ m - s(m-i_0), & \text{if } r = 0 \text{ and } 1 \leq s \leq \ell - 1. \\ i, & \text{if } r = 0 \text{ and } s = \ell. \end{cases} \tag{14}$$

Furthermore, $p := (U_0, \ldots, U_\ell)$ is a walk in $\mathsf{J}(n, m, k, j)$. In particular, if $s - r \geq 2$ and $i_0 = \max\{0, 2m-n, 2j-k, 2m-n-(2j-k)\}$, then p is a path.

Proof. It is clear that U_r is an m-subset of Ω when $r = 0$ or ℓ. If $1 \leq r \leq \ell - 1$, set $R_r := \{i + r(m-i_0) + 1, \ldots, m\}$ and $S_r := \{m+1, \ldots, m + r(m-i_0)\}$. Since $r \leq \ell - 1 \leq \frac{m-i}{m-i_0} - 1$, we have that $m - (i + r(m-i_0) + 1) + 1 = m - i - r(m-i_0) \geq m - i - (\frac{m-i}{m-i_0} - 1)(m-i_0) \geq m - i_0 \geq 1$ which implies that R_r is well-defined. Similarly, S_r is well-defined as $1 \leq r < \frac{m-i}{m-i_0}$ and $m + r(m-i_0) - (m+1) + 1 = r(m-i_0) \geq 1$ and $m + r(m-i_0) < m + (\frac{m-i}{m-i_0})(m-i_0) = 2m - i < n$. Note that R_r and S_r are disjoint subsets of Ω of size $m - i - r(m-i_0)$ and $r(m-i_0)$, respectively. Since $V_r = R_r \cup S_r$, we have that $|U_r| = |I| + |V_r| = |I| + |R_r| + |S_r| = i + [m - i - r(m-i_0)] + r(m-i_0) = m$. Therefore U_r is an m-subset of Ω, for each $0 \leq r \leq \ell$. Now we observe that

$$U_r \cap U_s = I \cup \begin{cases} \{i + s(m-i_0) + 1, \ldots, m\} \cup \\ \quad \{m+1, \ldots, m + r(m-i_0)\}, & \text{if } 1 \leq r < s \leq \ell - 1; \\ \{m+1, \ldots, m + r(m-i_0)\}, & \text{if } 1 \leq r \leq \ell - 1 \text{ and } s = \ell; \\ \{i + s(m-i_0) + 1, \ldots, m\}, & \text{if } r = 0 \text{ and } 1 \leq s \leq \ell - 1. \\ \varnothing, & \text{if } r = 0 \text{ and } s = \ell. \end{cases}$$

This shows that (14) holds. Since $i < i_0$, it follows that $m - i > m - i_0$, and so $\ell = \lceil \frac{m-i}{m-i_0} \rceil > 1$. Therefore $|U_r \cap U_s| = i_0$ when $s - r = 1$, and hence, by Lemma 2.7, we conclude that $p = (U_0, \ldots, U_\ell)$ is a path.

Suppose that $i_0 = \max\{0, 2m-n, 2j-k, 2m-n-(2j-k)\}$ and $s - r \geq 2$. It suffices to show that U_r and U_s are not adjacent.

If $1 \leq r < s \leq \ell - 1$, then as $s - r \geq 2$, we have that $|U_r \cap U_s| = m + (r-s)(m-i_0) \leq 2i_0 - m < i_0$, and so Lemma 2.7 implies that U_r and U_s are not adjacent.

If $s = \ell$ and $r \geq 1$, then since $s - r \geq 2$, we have that $1 \leq r \leq \ell - 2$. Note that $\ell - 1 < (m-i)/(m-i_0) < \ell$. So $r \leq \ell - 2 < \frac{m-i}{m-i_0} - 1 \leq \ell - 1$, and so (14) implies that $|U_r \cap U_s| = i + r(m-i_0) < i + (\frac{m-i}{m-i_0} - 1)(m-i_0) = i_0$, and again by Lemma 2.7, we conclude that U_r and U_s are not adjacent.

If $r = 0$, then either $2 \leq s \leq \ell - 1$, or $s = \ell \geq 2$. By (14), either $|U_r \cap U_s| = m - s(m-i_0) \leq 2i_0 - m < i_0$, or $|U_r \cap U_s| = i < i_0$, respectively. Now Lemma 2.7 implies that U_r and U_s are not adjacent. □

Lemma 2.22 suggests an upper bound for the distance $d(X, Z)$ between two vertices X and Z in $\mathsf{J}(n, m, k, j)$ and hence an upper bound for its diameter. Figs. 1–3 are small examples showing that these bounds are achieved. Indeed, the author believes that the diameter of $\mathsf{J}(n, m, k, j)$ is equal to the bound in Corollary 2.23.

Corollary 2.23. *Let X and Z be two vertices of the connected graph $\mathsf{J}(n, m, k, j)$ with $|X \cap Z| = i$. Then*

$$d(X, Z) \leq \left\lceil \frac{m-i}{m-i_0} \right\rceil,$$

where $i_0 = \max\{0, 2m - n, 2j - k, 2m - n - (2j - k)\}$. *Therefore the diameter of the connected graph* $\mathsf{J}(n, m, k, j)$ *is bounded by*

$$\left\lceil \frac{m - \max\{0, 2m - n\}}{m - i_0} \right\rceil.$$

Proof. By Proposition 2.5(b), we may assume that $X = U_0$ and $Z = U_\ell$ with $\ell = \lceil \frac{m-i}{m-i_0} \rceil$, where i_0 is as in Definition 2.21. So by Lemma 2.22, $(U_0, \dots, U_\ell)$ is a path between X and Z, and it is of minimum length when $i_0 = \max\{0, 2m - n, 2j - k, 2m - n - (2j - k)\}$. □

3. Geometric triple factorisations

This section is devoted to proving Theorem 1.2. We first establish a natural connection between triple factorisations $\mathrm{Sym}(\Omega) = ABA$ and the graphs $\mathsf{J}(n, m, k, j)$ defined as in Definition 2.1. We need first to mention some basic definitions in geometry.

A *rank* 2 *geometry* $\mathcal{G} = (\mathbb{P}, \mathbb{L}, *)$ consists of a set $\mathbb{P}$ of points, a set $\mathbb{L}$ of lines and an incidence relation $*$ between them. A *flag* of $\mathcal{G}$ is an incident point and line pair. A *geometry isomorphism* f from $\mathcal{G}_1 = (\mathbb{P}_1, \mathbb{L}_1, *_1)$ to $\mathcal{G}_2 = (\mathbb{P}_2, \mathbb{L}_2, *_2)$ is a bijection from the elements $\mathbb{P}_1 \cup \mathbb{L}_1$ of $\mathcal{G}_1$ onto the elements $\mathbb{P}_2 \cup \mathbb{L}_2$ of $\mathcal{G}_2$ such that

(i) incidence is preserved: $x *_1 y \iff f(x) *_2 f(y)$, and
(ii) points are sent to points, lines are sent to lines: $f(\mathbb{P}_1) = \mathbb{P}_2$ and $f(\mathbb{L}_1) = \mathbb{L}_2$.

An automorphism of $\mathcal{G} = (\mathbb{P}, \mathbb{L}, *)$ is a geometry isomorphism of $\mathcal{G}$ onto itself. The group of all automorphisms of a rank 2 geometry $\mathcal{G}$, denoted by $\mathrm{Aut}(\mathcal{G})$, is the *full automorphism group* of $\mathcal{G}$. Let now $G \le \mathrm{Aut}(\mathcal{G})$. Then G acts on the set of flags of $\mathcal{G}$ via $(p, \ell)^g = (p^g, \ell^g)$, for all flags (p, ℓ) of $\mathcal{G}$ and $g \in G$. The group G is *flag-transitive* (respectively, *point-transitive*, *line-transitive*) if G acts transitively on the set of flags (respectively, the set of points, the set of lines) of $\mathcal{G}$. A rank 2 geometry $\mathcal{G} = (\mathbb{P}, \mathbb{L}, *)$ is said to be *collinearly complete* (respectively, a *linear space*) if each pair of distinct points is incident with at least (respectively, exactly) one line.

Example 3.1 (*Coset Geometries*)**.** Let G be a group, and let A and B be proper subgroups of G. Let also $\mathbb{P}$ and $\mathbb{L}$ be the set of right cosets of A and B in G, respectively. These sets together with the incidence $*$ defined by $Ax \cap By \neq \varnothing$ possess a rank 2 geometry called *coset geometry* $\mathsf{Cos}(G; A, B)$ associated to the group G with subgroups A and B. In particular, G is a flag-transitive group of automorphisms of this geometry.

Although, by Proposition 3.2, each triple factorisation naturally introduces a coset geometry, not every coset geometry gives rise to a triple factorisation. For example, let $G = \mathrm{Sym}(\{1, 2, \dots, 5\})$, $A = \langle (4, 5) \rangle$ and $B = \langle (1, 2, 3) \rangle$. Then $G \neq ABA$ while $\mathsf{Cos}(G; A, B)$ is a G-flag-transitive rank 2 geometry.

Proposition 3.2 ([9, *Lemmas 1 and 3*])**.** *Let $\mathcal{G}$ be a rank* 2 *geometry and $G \le \mathrm{Aut}(\mathcal{G})$. Then G acts transitively on the flags of $\mathcal{G}$ if and only if $\mathcal{G} \cong \mathsf{Cos}(G; A, B)$, where A is the stabiliser of a point p and B is the stabiliser of a line ℓ incident with p. Moreover,*

(a) $\mathsf{Cos}(G; A, B)$ *is collinearly complete if and only if $G = ABA$;*
(b) $\mathsf{Cos}(G; A, B)$ *is linear space if and only if G is a Geometric ABA-group, that is, $G = ABA$, $A \not\subseteq B$, $B \not\subseteq A$ and $AB \cap BA = A \cup B$.*

For a rank 2 geometry $\mathcal{G}$, we may draw its *collinearity graph* $\mathsf{J}(\mathcal{G})$ whose vertices are points of $\mathcal{G}$ and each edge between two vertices p and q corresponds to a line passes through them. Note that such graphs may have multiple edges but no loops.

Example 3.3. Let n, m, k and j be a positive integers such that $1 \le m, k < n$ and $\max\{0, m + k - n\} \le j \le \min\{m, n\}$. Let also Ω be an n-set. Suppose that $\mathbb{P} := \Omega(m)$ and $\mathbb{L} := \Omega(k)$ are the set of all m-subsets of Ω and the set of all k-subsets of Ω, respectively (if $m = k$, we simply take $\mathbb{L}$ as a copy of $\mathbb{P}$.) Define the incidence relation $*_j$ on $\mathbb{P} \cup \mathbb{L}$ by $X *_j Y$ if and only if $|X \cap Y| = j$, for $X \in \mathbb{P}$ and $Y \in \mathbb{L}$. This incidence gives rise to the rank 2 geometry $\mathcal{J} := (\mathbb{P}, \mathbb{L}, *_j)$ whose collinearity graph is the graph $\mathsf{J}(n, m, k, j)$ defined as in Definition 2.1. Moreover, by Lemma 2.4 and Corollary 2.12, excluding the cases where $(k, j) \neq (m, m), (n - m, 0)$, the group $G := \mathrm{Sym}(\Omega)$ acts transitively as an automorphism group on the set of flags of $\mathcal{J}$. Therefore, $\mathcal{J}$ is geometrically isomorphic to the coset geometry $\mathsf{Cos}(G, A, B)$, where $A := G_X$ and $B := G_Y$ with $X \in \mathbb{P}$ and $Y \in \mathbb{L}$ and $|X \cap Y| = j$. In other words, A and B are intransitive subgroups of $\mathrm{Sym}(\Omega)$.

Note that the collinearity graph of a collinearly complete rank 2 geometry is a complete graph. Since the geometry $\mathcal{J}$ introduced in Example 3.3 is flag-transitive and $\mathsf{J}(n, m, k, j)$ is the collinearity graph of $\mathcal{J}$, Proposition 3.2 may be restated for $\mathsf{J}(n, m, k, j)$ as follows:

Corollary 3.4. *Let n, m, k and j be as in Example 3.3. Let also $G = \mathrm{Sym}(\Omega)$, $A := G_X$ and $B := G_Y$, where $|X \cap Y| = j$ with X and Y an m-subset and k-subset of Ω, respectively. Then*

(a) *$G = ABA$ if and only if $\mathsf{J}(n, m, k, j)$ is a complete graph. Moreover, G is a Geometric ABA-group if and only if $\mathsf{J}(n, m, k, j)$ is complete and simple;*
(b) *$G = ABA$ if and only if $G = \overline{A}\,\overline{B}\,\overline{A}$, where $\overline{A} := G_{\overline{X}}$ and $\overline{B} := G_{\overline{Y}}$, where $\overline{X}$ and $\overline{Y}$ are complements of X and Y, respectively. Moreover, G is a Geometric ABA-group if and only if G is Geometric $\overline{A}\,\overline{B}\,\overline{A}$-group.*

Proof. Part (a) follows from Proposition 3.2 and part (b) immediately follows from part (a) and the fact that $G_X = G_{\overline{X}}$ and $G_Y = G_{\overline{Y}}$. □

3.1. Proof of Theorem 1.2

We are now ready to prove Theorem 1.2.

Proof. Suppose that $G := \mathrm{Sym}(\Omega)$ and that A and B are stabilisers of an m-subset X and a k-subset Y of Ω with $|X \cap Y| = j$, respectively. By Corollary 3.4(b), we may assume that $m \leq n/2$. Assume now that G is a Geometric *ABA*-group. Then by Corollary 3.4(a), the associated graph $\mathsf{J}(n, m, k, j)$ is complete and simple, and so Corollary 2.18 implies that $(m, k, j) = (1, n-2, 0), (1, 2, 1)$. Therefore $A \cong S_{n-1}$ and $B \cong S_2 \times S_{n-2}$. The converse also follows from Corollary 3.4(a) and Corollary 2.18. □

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References

[1] S.H. Alavi, J. Bamberg, C.E. Praeger, Triple factorisations of classical groups by maximal parabolic subgroups and their associated geometries (2015) in preparation.
[2] S.H. Alavi, J. Bamberg, C.E. Praeger, Triple factorisations of the general linear group and their associated geometries, Linear Algebra Appl. 469 (2015) 169–203. http://dx.doi.org/10.1016/j.laa.2014.11.018.
[3] S.H. Alavi, T.C. Burness, Large subgroups of simple groups, J. Algebra 421 (2015) 187–233. http://dx.doi.org/10.1016/j.jalgebra.2014.08.026.
[4] S.H. Alavi, C.E. Praeger, On triple factorizations of finite groups, J. Group Theory 14 (3) (2011) 341–360. http://dx.doi.org/10.1515/JGT.2010.052.
[5] F. Buekenhout, Diagrams for geometries and groups, J. Combin. Theory Ser. A 27 (2) (1979) 121–151. http://dx.doi.org/10.1016/0097-3165(79)90041-4.
[6] F. Buekenhout, A. Delandtsheer, J. Doyen, P.B. Kleidman, M.W. Liebeck, J. Saxl, Linear spaces with flag-transitive automorphism groups, Geom. Dedicata 36 (1) (1990) 89–94. http://dx.doi.org/10.1007/BF00181466.
[7] A.D. Cannon, J. Bamberg, C.E. Praeger, A classification of the strongly regular generalised Johnson graphs, Ann. Comb. 16 (3) (2012) 489–506. http://dx.doi.org/10.1007/s00026-012-0142-9.
[8] M. Giudici, J. James, Factorisations of groups into three conjugate subgroups (2015) in preparation.
[9] D.G. Higman, J.E. McLaughlin, Geometric *ABA*-groups, Illinois J. Math. 5 (1961) 382–397.
[10] F.-T. Hu, J.-W. Wang, J.-M. Xu, A new class of transitive graphs, Discrete Math. 310 (4) (2010) 877–886. http://dx.doi.org/10.1016/j.disc.2009.10.004.