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A new proof for the number of lozenge tilings of quartered hexagons

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ABSTRACT

It has been proven that the lozenge tilings of a quartered hexagon on the triangular lattice are enumerated by a simple product formula. In this paper we give a new proof for the tiling formula by using Kuo's graphical condensation. Our result generalizes a result of Proctor on enumeration of plane partitions contained in a "maximal staircase".

1. Introduction

A plane partition is a rectangular array of non-negative integers with weakly decreasing rows and columns. The number of plane partitions contained in a $b \times c$ rectangle with entries at most a is given by MacMahon's formula $\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}$ (see [5]). As a variation of this, Proctor proved a simple product formula for the number of plane partitions with entries at most a which are contained in a shape with row lengths $b, b-1, \ldots, b-c+1$ (see Corollary 4.1 in [6]).

A *lozenge tiling* of a region on the triangular lattice is a covering of the region by unit rhombi (or lozenges) so that there are no gaps or overlaps. We use notation $\mathrm{L}(R)$ for the number of lozenge tilings of a region R ($\mathrm{L}(\emptyset) := 1$). The plane partitions contained in a $b \times c$ rectangle with entries at most a are in bijection with lozenge tilings of the hexagon $H_{a,b,c}$ of sides a, b, c, a, b, c (in cyclic order, starting from the north side). In the view of this we have an equivalent form of Proctor's result as follows.

Theorem 1.1 (*Proctor [6]*). *Assume that a, b, c are non-negative integer so that $b \geq c$. Let $P_{a,b,c}$ be the region obtained from the hexagon $H_{a,b,c}$ by removing the "maximal staircase" from its east corner (see Fig. 1.1 for $P_{3,6,4}$). Then*

$$\mathrm{L}(P_{a,b,c}) = \prod_{i=1}^{c}\left[\prod_{j=1}^{b-c+1}\frac{a+i+j-1}{i+j-1}\prod_{j=1}^{b-c+i}\frac{2a+i+j-1}{i+j-1}\right], \tag{1.1}$$

where empty products are equal to 1 by convention.

One can find more variations and generalizations of Proctor's Theorem 1.1 in [1]. We consider next a different generalization of the theorem.

Let $R_{a,b,c}$ be the region described as in Fig. 1.2. To precise, $R_{a,b,c}$ consists of all unit triangles on the right of the vertical symmetry axis of the hexagon of sides $2a+1, b, c, 2a+b-c+1, c, b$ (in cyclic order, starting from the north side).

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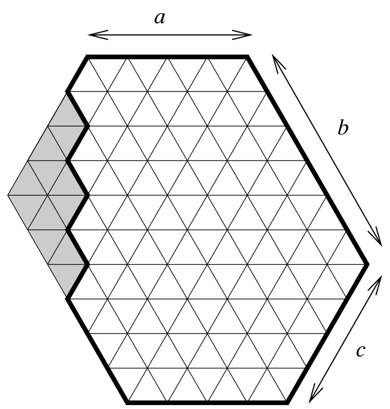

Fig. 1.1. Obtaining $P_{a,b,c}$ from the hexagon $H_{a,b,c}$ by removing a maximal staircase from the west corner.

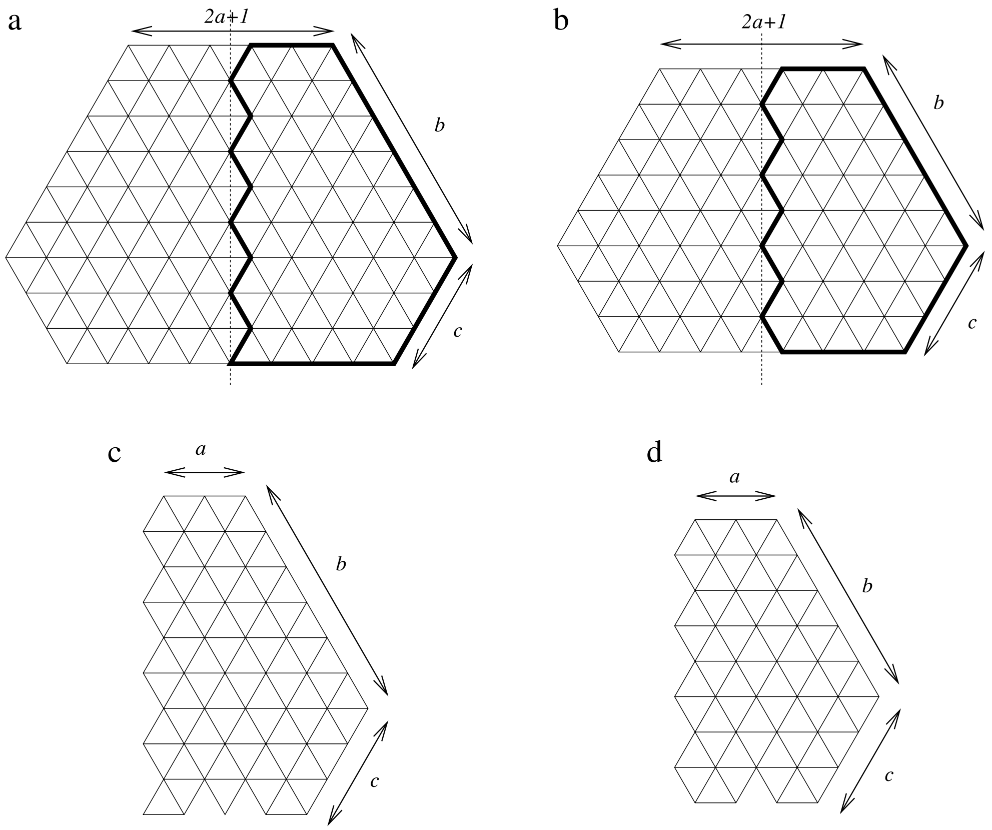

Fig. 1.2. Obtaining the region $R_{a,b,c}$ from a hexagon ((a) and (b)). Two examples of the region $R_{a,b,c}(s_1, s_2, \ldots, s_k)$ ((c) and (d)).

Fig. 1.2(a) illustrates the region $R_{2,6,3}$ and Fig. 1.2(b) shows the region $R_{2,5,3}$ (see the ones restricted by the bold contours). We are interested in the region $R_{a,b,c}$ with k up-pointing unit triangles removed from the base ($k = \lfloor \frac{b-c+1}{2} \rfloor$). If the positions of the triangles removed are $s_1, s_2, \ldots, s_k$, then we denote by $R_{a,b,c}(s_1, s_2, \ldots, s_k)$ the resulting region (see Fig. 1.2(b) and (c) for $R_{2,3,6}(2, 3)$ and $R_{2,5,3}(2)$, respectively). The number of lozenge tilings of the region $R_{a,b,c}(s_1, \ldots, s_k)$ is given by the theorem stated below.

Theorem 1.2. *Assume a, b, c are non-negative integers. If $b - c = 2k - 1$ for some non-negative integer k, then*

$$\mathrm{L}\big(R_{a,b,c}(s_1, s_2, \ldots, s_k)\big) = \prod_{1 \le i < j \le k+c} \frac{s_j - s_i}{j - i} \frac{s_i + s_j - 1}{i + j - 1}, \tag{1.2}$$

where $s_{k+i} := a + \frac{b-c+1}{2} + i$, for $i = 1, 2, \ldots, c$.

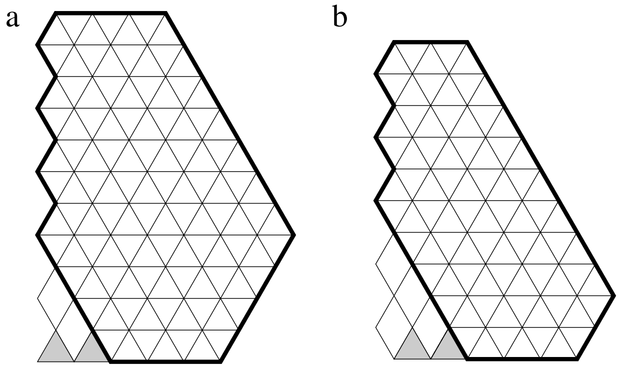

Fig. 1.3. Obtaining the region $P_{a,b,c}$ from the region $R_{a,b,c}(s_1,\dots,s_k)$.

If $b-c=2k$ for some non-negative integer k, then

$$\mathrm{L}\big(R_{a,b,c}(s_1,s_2,\dots,s_k)\big)=\prod_{i=1}^{k+c}\frac{s_i}{2i-1}\prod_{1\le i<j\le k+c}\frac{s_j-s_i}{j-i}\frac{s_i+s_j}{i+j},\tag{1.3}$$

where $s_{k+i}:=a+\frac{b-c}{2}+i$, for $i=1,2,\dots,c$.

We note that by specializing $k=b-c$ and $s_i=i$, for $i=1,2,\dots,k$, the region $P_{a,b,c}$ is obtained from $R_{a,b,c}(1,2,\dots,k)$ by removing forced lozenges on the lower-left corner (see Fig. 1.3). Thus

$$\mathrm{L}\big(R_{a,b,c}(1,2,\dots,k)\big)=\mathrm{L}\big(P_{a,b,c}\big),$$

and Proctor's Theorem 1.1 follows from Theorem 1.2.

Remark 1.3. We enumerated the lozenge tilings of the region $R_{a,b,c}(s_1,s_2,\dots,s_k)$ in [3] under the name *quartered hexagon* (see Theorem 3.1, Equations (3.1) and (3.2)). In particular, the region $R_{a,b,c}(s_1,s_2,\dots,s_k)$ is obtained from the quartered hexagon $QH_{b+c,n}(s_1,s_2,\dots,s_k,n-c+1,n-c+2,\dots,n)$ $(n:=\lfloor\frac{2a+1+b+c}{2}\rfloor)$ by removing several forced lozenges (see Figures 2.8 and 2.9(a) in [3]). Thus, we still call our $R_{a,b,c}$-type regions *quartered hexagons*. In [3], we identified the lozenge tilings of $R_{a,b,c}(s_1,s_2,\dots,s_k)$ with certain families of non-intersecting paths on $\mathbb{Z}^2$, then used the Lindström–Gessel–Viennot Theorem (see e.g. [4], Lemma 1; or [7], Theorem 1.2) to turn the number of path families to the determinant of a matrix whose entries are binomial coefficients, and evaluated the determinant.

A *perfect matching* of a graph G is a collection of edges so that each vertex of G is incident to exactly one selected edge. The *dual graph* G of a region R on the triangular lattice is the graph whose vertices are unit triangles in R and whose edges connect precisely two unit triangles sharing an edge. We have a bijection between the tilings of a region R and the perfect matchings of its dual graph G. We use notation $\mathrm{M}(G)$ for the number of perfect matchings of a graph G.

In this paper, we give a new inductive proof of Theorem 1.2 by using Proctor's Theorem 1.1 as a base case. The method that we use in the proof is the *graphical condensation method* first introduced by Eric Kuo [2]. In particular, we will employ the following theorem in our proof.

Theorem 1.4 (*Kuo [2]*). *Let $G=(V_1\cup V_2,E)$ be a planar bipartite graph, and V_1 and V_2 its vertex classes. Assume that x,y,z,t are four vertices appearing on a face of G in a cyclic order. Assume in addition that $a,b,c\in V_1$, $d\in V_2$, and $|V_1|=|V_2|+1$. Then*

$$\mathrm{M}(G-\{y\})\,\mathrm{M}(G-\{x,z,t\})=\mathrm{M}(G-\{x\})\,\mathrm{M}(G-\{y,z,t\})+\mathrm{M}(G-\{t\})\,\mathrm{M}(G-\{x,y,z\}).\tag{1.4}$$

2. Proof of Theorem 1.2

We only prove (1.2), as (1.3) can be obtained by a perfectly analogous manner.

It is easy to see that if $a=0$ then the region $R_{a,b,c}(s_1,s_2,\dots,s_k)$ has only one tiling (see Fig. 2.1(a)). On the other hand, since now $\{s_1,\dots,s_k\}=[k]$,[1] the right-hand side of the equality (1.2) is also equal to 1, then (1.2) holds for $a=0$. Moreover, if $b=0$, then $c=1$, $k=0$, and the region has the form as in Fig. 2.1(b). In this case, the region has also a unique tiling; and it is easy to verify that the right hand side of (1.2) equals 1. Thus, we can assume in the remaining of the proof that $a,b\ge 1$.

[1] We use the notation $[k]$ for the set $\{1,2,\dots,k\}$ of all positive integers not exceed k.

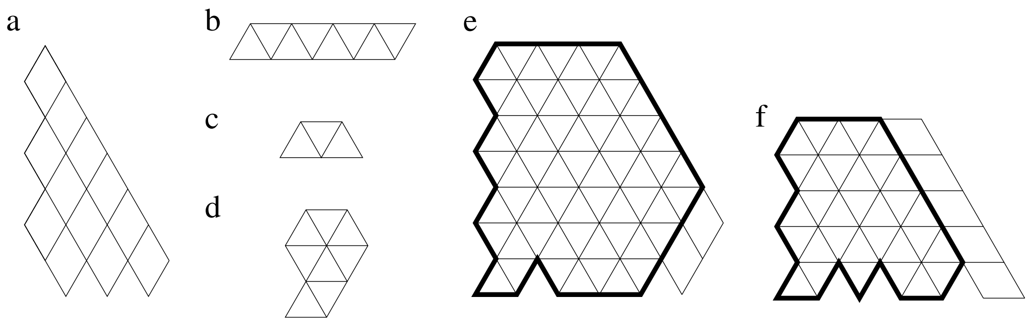

Fig. 2.1. Several special cases of $R_{a,b,c}(s_1, \ldots, s_k)$.

We will prove (1.2) by induction on $a + b$.

If $a + b \leq 2$, then we have $a = b = 1$. It is easy to see that there are only 2 possible shapes for the region $R_{a,b,c}$ (i.e. the region before removed triangles from the base) as in Fig. 2.1(c) and (d). Then it is routine to verify (1.2) for $a = b = 1$.

For the induction step, we assume that (1.2) is true for any region with $a + b < l$, for some $l \geq 3$, then we need to show that it is also true for any region $R_{a,b,c}(s_1, \ldots, s_k)$ with $a + b = l$.

Let $A := \{s_1, s_2, \ldots, s_k\}$ be a set of positive integers, we define the operators Δ and $\bigstar$ by setting

$$\Delta(A) := \prod_{1 \leq i < j \leq k} (s_j - s_i) \tag{2.1}$$

and

$$\bigstar(A) := \prod_{1 \leq i < j \leq k} (s_i + s_j - 1). \tag{2.2}$$

Then one can re-write (1.2) in terms of the above operators as:

$$\mathrm{L}\big(R_{a,b,c}(s_1, \ldots, s_k)\big) = \frac{\Delta(S)}{\Delta([k+c])} \frac{\bigstar(S)}{\bigstar([k+c])}, \tag{2.3}$$

where $S := \{s_1, s_2, \ldots, s_{k+c}\}$ and $[k+c] := \{1, 2, \ldots, k+c\}$. From this stage we will work on this new form of the equality (1.2).

We first consider three special cases as follows:

(i) If $c = 0$, then by considering forced lozenges as in Fig. 2.1(f), we get

$$\mathrm{L}\big(R_{a,b,0}(s_1, s_2, \ldots, s_k)\big) = \mathrm{L}\big(R_{a-q,b-1,1}(s_1, s_2, \ldots, s_k)\big), \tag{2.4}$$

where $q = a + \frac{b-c+1}{2} - a_k$. Then (2.3) follows from the induction hypothesis for the region on the right-hand side of (2.4).

(ii) If $k = 0$, then $b = c - 1$; and we get the region $P_{a,b,b}$ is obtained from the region $R_{a,b,b+1}(\emptyset)$ by removing forced lozenges along its base. Thus, (2.3) follows from Proctor's Theorem 1.1.

(iii) Let $d := a + \frac{b-c+1}{2}$ (so $s_{k+i} = d + i$). We consider one more special case when $a_k = d$. By removing forced lozenges again, one can transform our region into the region $R_{a,b-1,c+1}(s_1, \ldots, s_{k-1})$, then we get again (2.3) by induction hypothesis for the latter region (see Fig. 2.1(e)).

From now on, we assume that our region $R_{a,b,c}(s_1, \ldots, s_k)$ has the two parameters k and c positive (so $b = c+2k-1 \geq 2$), and that $a_k < d$.

Now we consider the region R obtained from $R_{a,b,c}(s_1, \ldots, s_k)$ by recovering the unit triangle at the position s_1 on the base. We now apply Kuo's Theorem 1.4 to the dual graph G of R, where the unit triangles corresponding to the four vertices x, y, z, t are chosen as in Fig. 2.2 (see the shaded triangles). In particular, the triangles corresponding to x and y are at the positions s_1 and d on the base; and the ones corresponding to z, t are on the upper-right corner of the region.

One readily sees that the six regions that have dual graphs appearing in the Eq. (1.4) of Kuo's Theorem have some lozenges, which are forced into any tilings. Luckily, by removing such forced lozenges, we still get new regions of $R_{a,b,c}$-type. In particular, after removing forced lozenges from the region corresponding to $G - \{x\}$, we get the region

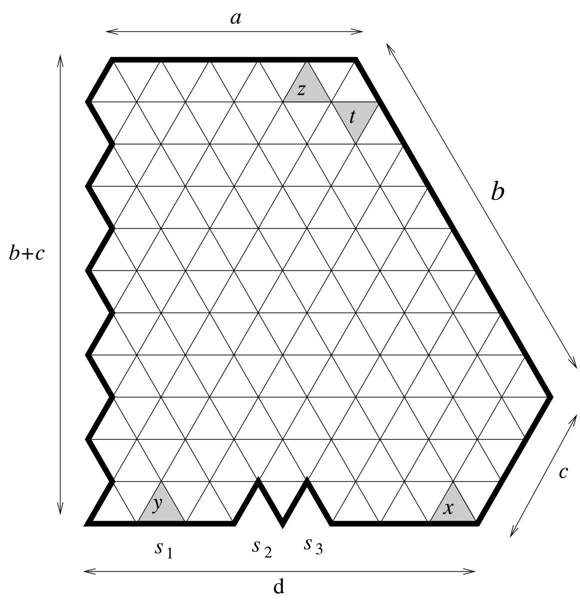

Fig. 2.2. Region to which we apply Kuo's graphical condensation.

$R_{a,b-1,c+1}(s_2, s_3, \dots, s_k)$ (see the region restricted by bold contour in Fig. 2.3(a)). This implies that

$$\mathrm{M}(G - \{x\}) = \mathrm{L}\big(R_{a,b-1,c+1}(s_2, s_3, \dots, s_k)\big). \tag{2.5}$$

Similarly, we have five more equalities corresponding to other graphs in (1.4):

$$\mathrm{M}(G - \{y\}) = \mathrm{L}\big(R_{a,b,c}(s_1, s_2, \dots, s_k)\big) \quad \text{(see Fig. 2.3(b))}, \tag{2.6}$$
$$\mathrm{M}(G - \{z\}) = \mathrm{L}\big(R_{a+1,b-2,c}(s_2, s_3, \dots, s_k)\big) \quad \text{(see Fig. 2.3(c))}, \tag{2.7}$$
$$\mathrm{M}(G - \{y, z, t\}) = \mathrm{L}\big(R_{a,b-2,c-1}(s_1, s_2, \dots, s_k)\big) \quad \text{(see Fig. 2.3(d))}, \tag{2.8}$$
$$\mathrm{M}(G - \{x, z, t\}) = \mathrm{L}\big(R_{a,b-2,c}(s_2, s_3, \dots, s_k)\big) \quad \text{(see Fig. 2.3(e))}, \tag{2.9}$$
$$\mathrm{M}(G - \{x, y, t\}) = \mathrm{L}\big(R_{a-1,b,c}(s_1, s_2, \dots, s_k)\big) \quad \text{(see Fig. 2.3(f))}. \tag{2.10}$$

Plugging the above six equalities (2.5)–(2.10) in (1.4), we have the following recurrence

$$\mathrm{L}\big(R_{a,b,c}(s_1, s_2, \dots, s_k)\big)\,\mathrm{L}\big(R_{a,b-2,c}(s_2, s_3, \dots, s_k)\big) = \mathrm{L}\big(R_{a,b-1,c+1}(s_2, s_3, \dots, s_k)\big)\,\mathrm{L}\big(R_{a,b-2,c-1}(s_1, s_2, \dots, s_k)\big) + \mathrm{L}\big(R_{a+1,b-2,c}(s_2, s_3, \dots, s_k)\big)\,\mathrm{L}\big(R_{a-1,b,c}(s_1, s_2, \dots, s_k)\big). \tag{2.11}$$

The five regions other than $R_{a,b,c}(s_1, s_2, \dots, s_k)$ in the above recurrence (2.11) have their $(a + b)$-parameter less than l. Therefore, by induction hypothesis, we get

$$\mathrm{L}\big(R_{a,b-1,c+1}(s_2, \dots, s_k)\big) = \frac{\Delta(S \cup \{d\} - \{s_1\})}{\Delta([k+c])}\,\frac{\bigstar(S \cup \{d\} - \{s_1\})}{\bigstar([k+c])}, \tag{2.12}$$
$$\mathrm{L}\big(R_{a,b-2,c-1}(s_1, \dots, s_k)\big) = \frac{\Delta(S - \{s_{k+c}\})}{\Delta([k+c-1])}\,\frac{\bigstar(S - \{s_{k+c}\})}{\bigstar([k+c-1])}, \tag{2.13}$$
$$\mathrm{L}\big(R_{a+1,b-2,c}(s_2, \dots, s_k)\big) = \frac{\Delta(S - \{s_1\})}{\Delta([k+c-1])}\,\frac{\bigstar(S - \{s_1\})}{\bigstar([k+c-1])}, \tag{2.14}$$
$$\mathrm{L}\big(R_{a-1,b,c}(s_1, \dots, s_k)\big) = \frac{\Delta(S \cup \{d\} - \{s_{k+c}\})}{\Delta([k+c])}\,\frac{\bigstar(S \cup \{d\} - \{s_{k+c}\})}{\bigstar([k+c])}, \tag{2.15}$$
$$\mathrm{L}\big(R_{a,b-2,c}(s_2, \dots, s_k)\big) = \frac{\Delta(S \cup \{d\} - \{s_1, s_{k+c}\})}{\Delta([k+c-1])}\,\frac{\bigstar(S \cup \{d\} - \{s_1, s_{k+c}\})}{\bigstar([k+c-1])}. \tag{2.16}$$

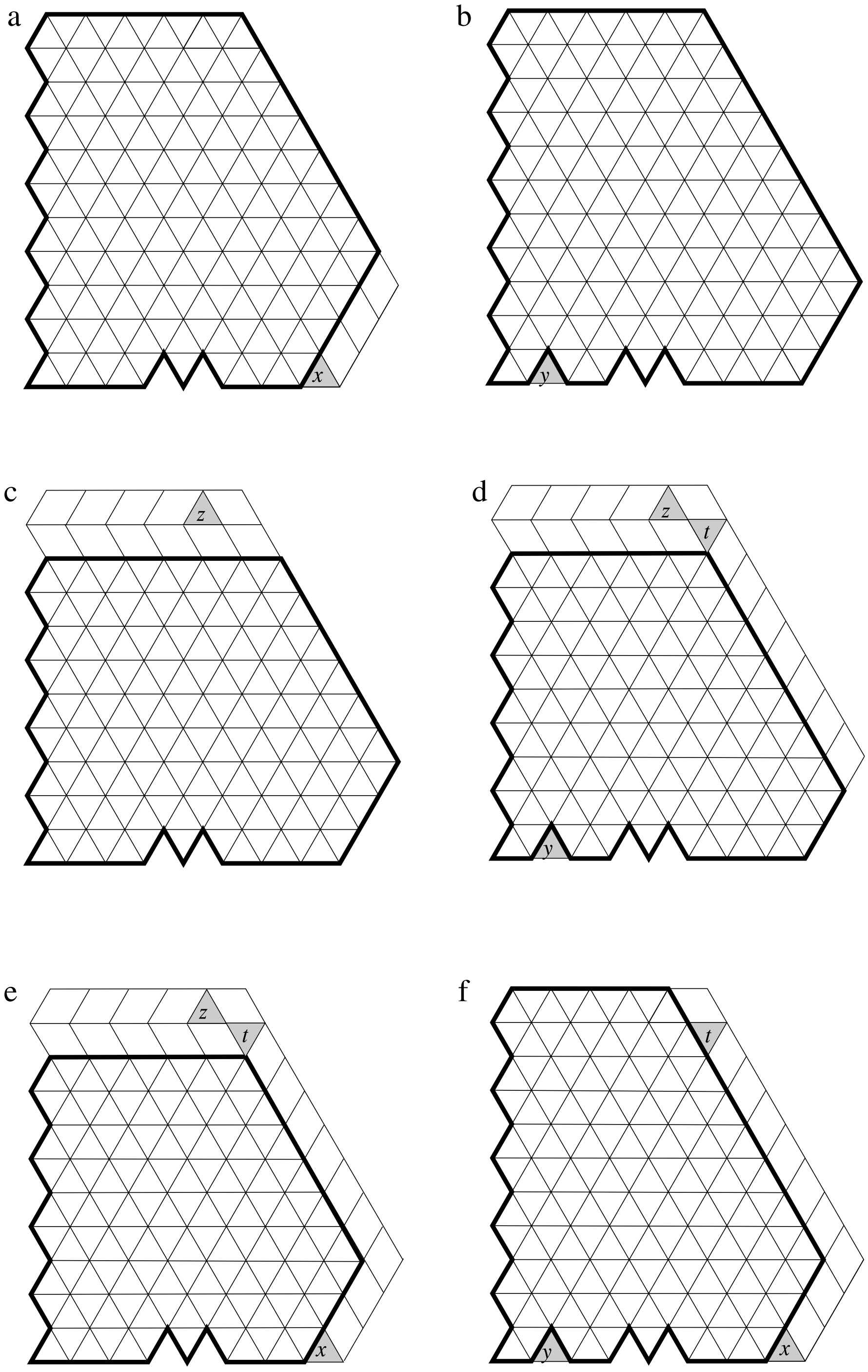

Fig. 2.3. Obtaining the recurrence for $\mathrm{L}(R_{a,b,c}(s_1,\ldots,s_k))$.

By the above five equalities (2.12)–(2.16) and the recurrence (2.11), we only need to show that

$$\begin{aligned}1 &= \frac{\Delta(S\cup\{d\}-\{s_1\})\Delta(S-\{s_{k+c}\})}{\Delta(S)\Delta(S\cup\{d\}-\{s_1,s_{k+c}\})}\frac{\bigstar(S\cup\{d\}-\{s_1\})\bigstar(S-\{s_{k+c}\})}{\bigstar(S)\bigstar(S\cup\{d\}-\{s_1,s_{k+c}\})}\\ &\quad+\frac{\Delta(S\cup\{d\}-\{s_{k+c}\})\Delta(S-\{s_1\})}{\Delta(S)\Delta(S\cup\{d\}-\{s_1,s_{k+c}\})}\frac{\bigstar(S\cup\{d\}-\{s_{k+c}\})\bigstar(S-\{s_1\})}{\bigstar(S)\bigstar(S\cup\{d\}-\{s_1,s_{k+c}\})},\end{aligned}\tag{2.17}$$

and (2.3) follows.

First, we want to simplify the first ratio in the first term on the right-hand side of (2.17). We re-write it as

$$\frac{\Delta(S\cup\{d\}-\{s_1\})\Delta(S-\{s_{k+c}\})}{\Delta(S)\Delta(S\cup\{d\}-\{s_1,s_{k+c}\})}=\frac{\frac{\Delta(S\cup\{d\}-\{s_1\})}{\Delta(S)}\frac{\Delta(S-\{s_{k+c}\})}{\Delta(S)}}{\frac{\Delta(S\cup\{d\}-\{s_1,s_{k+c}\})}{\Delta(S)}}. \tag{2.18}$$

The ratio $\frac{\Delta(S\cup\{d\}-\{s_1\})}{\Delta(S)}$ has its numerator and denominator almost the same, except for some terms involving s_1 or d. Canceling out all common terms of the numerator and the denominator, we have

$$\frac{\Delta(S\cup\{d\}-\{s_1\})}{\Delta(S)}=\frac{\prod_{i=2}^{k}(d-s_i)\prod_{i=1}^{c}(s_{k+i}-d)}{\prod_{i=2}^{k+c}(s_i-s_1)}.$$

Similarly, we get

$$\frac{\Delta(S-\{s_{k+c}\})}{\Delta(S)}=\frac{1}{\prod_{i=1}^{k+c-1}(s_{k+c}-s_i)}$$

and

$$\frac{\Delta(S\cup\{d\}-\{s_1,s_{k+c}\})}{\Delta(S)}=\frac{\prod_{i=2}^{k}(d-s_i)\prod_{i=1}^{c-1}(s_{k+i}-d)}{\prod_{i=2}^{k+c}(s_i-a_1)\prod_{i=2}^{k+c-1}(s_{k+c}-s_i)}.$$

Thus, we obtain

$$\frac{\Delta(S\cup\{d\}-\{s_1\})\Delta(S-\{s_{k+c}\})}{\Delta(S)\Delta(S\cup\{d\}-\{s_1,s_{k+c}\})}=\frac{\frac{\Delta(S\cup\{d\}-\{s_1\})}{\Delta(S)}\frac{\Delta(S-\{s_{k+c}\})}{\Delta(S)}}{\frac{\Delta(S\cup\{d\}-\{s_1,s_{k+c}\})}{\Delta(S)}}=\frac{s_{k+c}-d}{s_{k+c}-s_1}. \tag{2.19}$$

By the same trick, we can simply the first ratio in the second term on the right-hand side of (2.17) as

$$\frac{\Delta(S\cup\{d\}-\{s_{k+c}\})\Delta(S-\{s_1\})}{\Delta(S)\Delta(S\cup\{d\}-\{s_1,s_{k+c}\})}=\frac{\frac{\Delta(S\cup\{d\}-\{s_{k+c}\})}{\Delta(S)}\frac{\Delta(S-\{s_1\})}{\Delta(S)}}{\frac{\Delta(S\cup\{d\}-\{s_1,s_{k+c}\})}{\Delta(S)}}=\frac{d-s_1}{s_{k+c}-s_1}. \tag{2.20}$$

Next, we simply the second ratio in each term on the right-hand side of (2.3). By replacing the operator Δ by the operator $\bigstar$, the whole simplifying-process works in the same way with each term (s_j-s_i) replaced by (s_i+s_j-1). Thus, we get

$$\frac{\bigstar(S\cup\{d\}-\{s_1\})\bigstar(S-\{s_{k+c}\})}{\bigstar(S)\bigstar(S\cup\{d\}-\{s_1,s_{k+c}\})}=\frac{s_{k+c}+d-1}{s_{k+c}+s_1-1} \tag{2.21}$$

and

$$\frac{\bigstar(S\cup\{d\}-\{s_{k+c}\})\bigstar(S-\{s_1\})}{\bigstar(S)\bigstar(S\cup\{d\}-\{s_1,s_{k+c}\})}=\frac{d+s_1-1}{s_{k+c}+s_1-1}. \tag{2.22}$$

Finally, by (2.19)–(2.22), we can simplify Eq. (2.17) to

$$1=\frac{s_{k+c}-d}{s_{k+c}-s_1}\frac{s_{k+c}+d-1}{s_{k+c}+s_1-1}+\frac{d-s_1}{s_{k+c}-s_1}\frac{d+s_1-1}{s_{k+c}+s_1-1}, \tag{2.23}$$

which is obviously true with $s_{k+c}=d+c$. This means that (2.17) holds, and so does (1.2).

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References

[1] M. Ciucu, C. Krattenthaler, Enumeration of lozenge tilings of hexagons with cut off corners, J. Combin. Threory Ser. A 100 (2002) 201–231.
[2] E.H. Kuo, Applications of graphical condensation for enumerating matchings and tilings, Theoret. Comput. Sci. 319 (2004) 29–57.
[3] T. Lai, Enumeration of tilings of quartered Aztec rectangles, Electron. J. Combin. 21 (2014) P4.46.
[4] B. Lindström, On the vector representations of induced matroids, Bull. Lond. Math. Soc. 5 (1973) 85–90.
[5] P.A. MacMahon, Combinatory Analysis, Vol. 2, Cambridge Univ. Press, 1916, Reprinted by Chelsea, New York, 1960.
[6] R. Proctor, Odd symplectic group, Invent. Math. 92 (1998) 307–332.
[7] J.R. Stembridge, Nonintersecting paths, Pfaffians and plane partitions, Adv. Math. 83 (1990) 96–131.