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Note

Around Erdős–Lovász problem on colorings of non-uniform hypergraphs

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ABSTRACT

The work deals with combinatorial problems concerning colorings of non-uniform hypergraphs. Let $H = (V, E)$ be a hypergraph with minimum edge-cardinality n. We show that if H is a simple hypergraph (i.e. every two distinct edges have at most one common vertex) and

$$\sum_{e\in E} r^{1-|e|} \leqslant c\sqrt{n},$$

for some absolute constant $c > 0$, then H is r-colorable. We also obtain a stronger result for triangle-free simple hypergraphs by proving that if H is a simple triangle-free hypergraph and

$$\sum_{e\in E} r^{1-|e|} \leqslant c \cdot n,$$

for some absolute constant $c > 0$, then H is r-colorable.

1. Introduction

The work deals with some aspects of the well-known problem of Erdős and Lovász concerning colorings of non-uniform hypergraphs. Let us recall the main definitions.

Let $H = (V, E)$ be a hypergraph. A vertex-coloring of a hypergraph H is said to be *proper* if none of the edges of H is monochromatic under this coloring. *The chromatic number* $\chi(H)$ is the minimum number of colors required for a proper coloring of H. A hypergraph is called *r-colorable* if $\chi(H) \leqslant r$. A hypergraph H is said to be *simple* if any two of its distinct edges have at most one common vertex, i.e.

$$\forall A, B \in E, A \neq B, \quad |A \cap B| \leqslant 1.$$

A triangle (*or* 3*-cycle*) in H is a triple of edges (A, B, C) such that $(A \cap B)\setminus C \neq \emptyset$, $(B \cap C)\setminus A \neq \emptyset$ and $(A \cap C)\setminus B \neq \emptyset$.

One of the most famous problems concerning colorings of hypergraphs is Property B problem stated by P. Erdős and A. Hajnal: find the value $m(n)$ equal to the minimum possible number of edges in an n-uniform hypergraph which is not 2-colorable. This classical problem was studied in a lot of papers by different authors (see the survey [5] for the details). We shall recall a fragment from its history.

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In 1973 Erdős and L. Lovász in their seminal paper [2] conjectured that

$$\frac{m(n)}{2^n} \to \infty \quad \text{as } n \to \infty.$$

Furthermore, they formulated a stronger conjecture concerning non-uniform hypergraphs. For a hypergraph $H = (V, E)$, let $f(H)$ denote the following function:

$$f(H) = \sum_{e \in E} 2^{-|e|}.$$

Erdős and Lovász proposed to consider the value $f(n)$ equal to the minimum possible value of $f(H)$ where H is 3-chromatic hypergraph with minimum edge-cardinality n. They conjectured that

$$f(n) \to \infty \quad \text{as } n \to \infty.$$

Both conjectures were proved by J. Beck in 1977–1978. He proved that $m(n) \geqslant 2^n n^{1/3-o(1)}$, but his lower bound for $f(n)$ was much weaker. Let us define the following sequence of functions: $g_0(x) = x$ and $g_k(x) = \log_2(g_{k-1}(x))$ for $k \geqslant 1$. For any natural x, let $\log^*(x) = \min\{k : g_k(x) \leqslant 1\}$. Using the function $\log^*(x)$ Beck's result (see [1]) can be formulated as follows:

$$f(n) \geqslant \frac{\log^*(n) - 100}{7}. \tag{1}$$

This inequality proves the conjecture of Erdős and Lovász, but the function $\log^*(n)$ grows very slowly since it is the inverse function for the tower of twos of height n:

$$n \to 2^{2^{\cdot^{\cdot^{2}}}} \ \text{— tower of height } n.$$

In 2008 L. Lu (see [3]) stated the following improvement of the lower bound (1):

$$f(n) \geqslant \left(\frac{1}{16} - o(1)\right)\frac{\ln n}{\ln \ln n}.$$

However his proof works only for simple hypergraphs, so he showed that if $f(H) = O(\ln n/\ln\ln n)$ and H is a simple hypergraph with edge cardinalities at least n then H is 2-colorable.

The first new result of the current paper improves the bound of Lu for simple hypergraphs and provides a sufficient condition for r-colorability in terms of the function $f_r(H) = \sum_{e\in E} r^{1-|e|}$.

Theorem 1. *Suppose $n \geqslant 3$, $r \geqslant 2$ and $H = (V, E)$ is a simple hypergraph with minimum edge-cardinality n. If*

$$f_r(H) = \sum_{e \in E} r^{1-|e|} \leqslant c\sqrt{n}, \tag{2}$$

for some absolute constant $c > 0$, then H is r-colorable.

In the case of triangle-free simple hypergraphs the author of the paper showed in [6] that the inequality

$$f_r(H) \leqslant \frac{1}{2}\left(\frac{n}{\ln n}\right)^{2/3}$$

implies r-colorability of a triangle-free simple hypergraph H with minimum edge-cardinality n. The second theorem of the paper refines this result as follows.

Theorem 2. *Suppose $n \geqslant 3$, $r \geqslant 2$ and $H = (V, E)$ is a simple triangle-free hypergraph with minimum edge-cardinality n. If*

$$f_r(H) = \sum_{e \in E} r^{1-|e|} \leqslant c \cdot n, \tag{3}$$

for some absolute constant $c > 0$, then H is r-colorable.

In the next section we shall prove Theorem 1. Section 3 is devoted to the proof of Theorem 2. In both proofs we follow the ideas of random recoloring from the remarkable paper [4] of J. Radhakrishnan and A. Srinivasan.

2. Proof of Theorem 1

We have to show the existence of a proper r-coloring for the hypergraph $H = (V, E)$. Let us consider the following construction of a random r-coloring of the vertex set V:

- color every vertex randomly, independently and uniformly with r colors $\{0, 1, \dots, r-1\}$ (initial coloring),
- take a random ordering σ of the vertex set V with uniform distribution (independent of the random coloring),
- in every monochromatic edge recolor the first vertex (according to σ) as follows: if the initial color is α then recolor with color $\alpha + 1$ (modulo r).

Suppose now that the described random r-coloring contains a monochromatic edge A of some color α. Due to the construction this edge was not completely colored with α in the initial coloring. Thus, there are several vertices in A which were recolored with α during the recoloring stage. Let $v_1, \dots, v_L$ denote these vertices. Our recoloring rule implies that all of them were colored with color $\alpha - 1$ in the initial coloring and, for every $i = 1, \dots, L$, there was an initially monochromatic edge B_i containing v_i such that v_i is the first vertex of B_i in the random ordering σ.

Further we shall use the notation $\sigma(w, B)$ for the number of the vertex w in the vertex subset B according to σ. Now, consider different values of the parameter L.

Case 1. $L = 1$. We have two edges A and B_1 with one common vertex v_1, $A \setminus \{v_1\}$ is colored with α in the initial coloring, B_1 is monochromatic of color $\alpha - 1$. Thus, the probability of such bad event $\mathcal{A}$ can be estimated as follows:

$$\begin{aligned}\Pr(\mathcal{A}) &\leqslant \sum_{\alpha=0}^{r-1}\sum_{A\in E}\sum_{B_1\in E} r^{1-|A|}r^{-|B_1|}\Pr\left(\sigma(v_1, B_1) = 1\right)\\ &= \sum_{A\in E}\sum_{B_1\in E} r^{1-|A|}r^{1-|B_1|}\frac{1}{|B_1|} \leqslant |\text{since}|B_1| \geqslant n| \leqslant \frac{1}{n}\sum_{A\in E}\sum_{B_1\in E} r^{1-|A|}r^{1-|B_1|}\\ &\leqslant \frac{1}{n}(f_r(H))^2 \leqslant |(2)| \leqslant c^2. \end{aligned} \tag{4}$$

Case 2. $2 \leqslant L \leqslant 2c\sqrt{n}$. Since hypergraph is simple then there can be at most one edge of H containing all the vertices $v_1, \dots, v_L$. Let $A(v_1, \dots, v_L)$ denote such an edge if it exists. So, we have L monochromatic (of a color $\alpha - 1$) edges $B_1, \dots, B_L$, their first vertices $v_1, \dots, v_L$ and monochromatic (of a color α) edge $A(v_1, \dots, v_L)$. Thus, the probability of such bad event $\mathcal{B}$ satisfies:

$$\Pr(\mathcal{B}) \leqslant \sum_{L=2}^{2c\sqrt{n}}\sum_{\alpha=0}^{r-1}\sum_{B_1,\dots,B_L\in E} r^{-|\cup_{i=1}^{L}B_i|}\sum_{v_1,\dots,v_L: v_i\in B_i,\, \exists A(v_1,\dots,v_L)}\Pr\left(\forall i: \sigma(v_i, B_i) = 1\right) r^{L-|A(v_1,\dots,v_L)|}. \tag{5}$$

It is clear that $|A(v_1, \dots, v_L)| \geqslant n$. Due to the simplicity of the hypergraph we have

$$\left|\bigcup_{i=1}^{L} B_i\right| \geqslant \sum_{i=1}^{L}|B_i| - \binom{L}{2}.$$

Finally, by the independence of $\sigma(v, B)$ and $\sigma(w, C)$ for not intersecting sets B and C one easily gets

$$\begin{aligned}&\Pr\left(\forall i: \sigma(v_i, B_i) = 1\right) \leqslant \Pr\left(\forall i: \sigma(v_i, B_i \setminus (B_1 \cup \dots \cup B_{i-1})) = 1\right)\\ &= \prod_{i=1}^{L}\frac{1}{|B_i \setminus (B_1 \cup \dots \cup B_{i-1})|} \leqslant \prod_{i=1}^{L}\frac{1}{|B_i| - i + 1}.\end{aligned}$$

So, the inequality (5) implies that

$$\begin{aligned}\Pr(\mathcal{B}) &\leqslant \sum_{L=2}^{2c\sqrt{n}}\sum_{\alpha=0}^{r-1}\sum_{B_1,\dots,B_L\in E} r^{\binom{L}{2}-\sum_{i=1}^{L}|B_i|}\sum_{v_1,\dots,v_L: v_i\in B_i,\, \exists A(v_1,\dots,v_L)}\prod_{i=1}^{L}\frac{1}{|B_i| - i + 1}r^{L-n}\\ &\leqslant \sum_{L=2}^{2c\sqrt{n}}\sum_{B_1,\dots,B_L\in E} r^{1+L+\binom{L}{2}-\sum_{i=1}^{L}|B_i|-n}\prod_{i=1}^{L}\frac{|B_i|}{|B_i| - i + 1}\end{aligned}$$

(since $\binom{L}{2} + 1 \leqslant L^2/2 \leqslant 2c^2 n$ and $|B_i| \geqslant n$)

$$\begin{aligned}&\leqslant \sum_{L=2}^{2c\sqrt{n}}\sum_{B_1,\dots,B_L\in E} r^{L-\sum_{i=1}^{L}|B_i|} r^{(2c^2-1)n}\left(\frac{n}{n-L}\right)^L\\ &\leqslant \sum_{L=2}^{2c\sqrt{n}}\frac{1}{L!}\sum_{B_1\in E}\dots\sum_{B_L\in E}\left(\prod_{i=1}^{L} r^{1-|B_i|}\right) r^{(2c^2-1)n}\left(\frac{n}{n-2c\sqrt{n}}\right)^L\end{aligned}$$

(assuming $c < 1/2$ and, consequently, $2c\sqrt{n} < n/2$, $n/(n - 2c\sqrt{n}) < 2$)

$$\leqslant \sum_{L=2}^{2c\sqrt{n}} \frac{1}{L!}(f_r(H))^L r^{(2c^2-1)n} 2^L \leqslant e^{2f_r(H)} r^{(2c^2-1)n} \leqslant |(2) \quad \text{and} \quad r \geqslant 2| \leqslant e^{2c\sqrt{n}} 2^{(2c^2-1)n}. \tag{6}$$

Case 3. $L > 2c\sqrt{n}$. Our recoloring procedure implies that the number of recolored vertices does not exceed the number of monochromatic edges in the initial coloring. Let X denote this random variable. So, if at least $2c\sqrt{n}$ vertices in the edge A change their colors then $X \geqslant 2c\sqrt{n}$. The probability of the third bad event $\mathcal{C}$ can be estimated as follows:

$$\Pr(\mathcal{C}) \leqslant \Pr(X \geqslant 2c\sqrt{n}) \leqslant |\text{Markov inequality}| \leqslant \frac{\mathsf{E}X}{2c\sqrt{n}} = \frac{f_r(H)}{2c\sqrt{n}} \leqslant |(2)| \leqslant \frac{1}{2}. \tag{7}$$

Let us finish the proof. Using (5)–(7), we see that the probability that our recoloring procedure does not provide any new monochromatic edge is at most

$$\Pr(\mathcal{A}) + \Pr(\mathcal{B}) + \Pr(\mathcal{C}) \leqslant c^2 + e^{2c\sqrt{n}} 2^{(2c^2-1)n} + \frac{1}{2}. \tag{8}$$

Setting $c = 1/8$ is sufficient to make the expression on the right-hand side of (8) less than 1 for arbitrary $n \geqslant 3$. Thus, with positive probability there will be no monochromatic edges after the recoloring procedure and, consequently, hypergraph H is r-colorable.

3. Proof of Theorem 2

As in the previous theorem we will construct a random r-coloring for the vertex set of the hypergraph $H = (V, E)$. The coloring procedure of V goes as follows:

- color every vertex randomly, independently and uniformly with r colors $\{0, 1, \ldots, r-1\}$ (initial coloring),
- take a random ordering σ of the vertex set V with uniform distribution (independent of the initial coloring),
- an edge A is called *almost monochromatic* with *dominating color* α if only one vertex $v \in A$ is not colored with α in the initial coloring. In this case v is said to be *dangerous* of a color α,
- consider every vertex due to the order provided by σ and check two properties:
  – there is a monochromatic edge of some color α containing this vertex and no recoloring has been made yet in the edge,
  – the vertex is not dangerous of a color $\alpha + 1$ (modulo r).
  If both conditions hold then recolor the vertex with color $\alpha + 1$. Otherwise, skip it and consider the next one.

Suppose that the coloring procedure does not succeed and there is a monochromatic edge A of some color α at the end of it. Now we have two possibilities: either A was monochromatic of α in the initial coloring and no recoloring happened or A was not monochromatic of α but became monochromatic during the recoloring process. Let us consider these cases separately. As previously, $\sigma(w, B)$ denote the number of the vertex w in the subset B.

Case 1. Suppose A is monochromatic of a color α in the initial coloring. Then we should try to recolor its vertices according to σ. Let $v_1, \ldots, v_h$ be the first h vertices of A. Since no recoloring happens, all these vertices should be dangerous of a color $\alpha + 1$, so there are h edges $B_1, \ldots, B_h$ such that v_i is the only vertex of B_i not colored with $\alpha + 1$. Since H is simple and triangle-free all the edges $B_1, \ldots, B_h$ are pairwise disjoint and the vertices $v_1, \ldots, v_h$ are uniquely defined by them and A. Thus, the probability of the first bad event $\mathcal{A}$ can be estimated as follows:

$$\Pr(\mathcal{A}) \leqslant \sum_{\alpha=0}^{r} \sum_{A \in E} r^{-|A|} \sum_{B_1 \in E} \ldots \sum_{B_h \in E} r^{h - \sum_{i=1}^{h} |B_i|} \Pr(\sigma(v_i, A) = i, \forall i = 1, \ldots, h)$$

$$\leqslant \sum_{A \in E} r^{1-|A|} \sum_{B_1 \in E} \ldots \sum_{B_h \in E} r^{h - \sum_{i=1}^{h} |B_i|} \frac{(|A| - h)!}{|A|!} \leqslant |\text{since} |A| \geqslant n|$$

$$\leqslant \frac{(n-h)!}{n!} \sum_{A \in E} r^{1-|A|} \sum_{B_1 \in E} \ldots \sum_{B_h \in E} r^{h - \sum_{i=1}^{h} |B_i|} = \frac{(n-h)!}{n!} (f_r(H))^{h+1}$$

(using (3))

$$\leqslant \frac{(n-h)!}{n!} (cn)^{h+1} \leqslant c^{h+1} n \left(\frac{n}{n-h}\right)^h. \tag{9}$$

Case 2. Suppose now that the edge A is not completely monochromatic of α in the initial coloring, but at the end of the recoloring procedure it is monochromatic of this color. Let $v_1, \ldots, v_L$ denote the vertices of A not colored with α in the initial coloring. Due to our recoloring procedure A should not be almost monochromatic with dominating color α, thus, $L \geqslant 2$. All these vertices are recolored with α during the recoloring process, consequently, there are L edges $B_1, \ldots, B_L$ such that, for any $i = 1, \ldots, L$,

- $v_i \in B_i$,
- B_i is monochromatic of a color $\alpha - 1$ in the initial coloring and
- when v_i is considered B_i is still monochromatic of $\alpha - 1$.

Since H is simple and triangle-free, all the edges $B_1, \ldots, B_L$ are pairwise disjoint and the vertices $v_1, \ldots, v_L$ are uniquely defined by them and A. Moreover, we can assume that $\sigma(v_i, B_i) \leqslant h$ for any $i = 1, \ldots, L$. Otherwise, when $\sigma(v_i, B_i) > h$ the edge B_i remains monochromatic after the consideration of its first h vertices and this situation was analyzed in the Case 1.

The estimation of the probability of the second bad event depends on the value of L. If $L \leqslant (n \ln 2)/\ln n$ then as in the previous theorem we shall sum up over the vertices $v_1, \ldots, v_L$ instead of A, since the simplicity of the hypergraph implies the uniqueness of the edge containing all of them. Thus, the probability of the such bad event can be bounded from above as follows:

$$\mathrm{Pr}(\mathcal{B}) \leqslant \sum_{L=2}^{(n\ln 2)/\ln n} \sum_{\alpha=0}^{r-1} \sum_{B_1,\ldots,B_L \in E} r^{-\sum_{i=1}^{L}|B_i|} \sum_{v_1,\ldots,v_L: v_i \in B_i,\, \exists A(v_1,\ldots,v_L)} \mathrm{Pr}\left(\forall i : \sigma(v_i, B_i) \leqslant h\right) r^{L-|A(v_1,\ldots,v_L)|}$$

(since $|A(v_1, \ldots, v_L)| \geqslant n$)

$$\leqslant \sum_{L=2}^{(n\ln 2)/\ln n} \sum_{B_1,\ldots,B_L \in E} r^{-\sum_{i=1}^{L}|B_i|} r^{L+1-n} \sum_{v_1,\ldots,v_L: v_i \in B_i,\, \exists A(v_1,\ldots,v_L)} \mathrm{Pr}\left(\forall i : \sigma(v_i, B_i) \leqslant h\right)$$

$$\leqslant \sum_{L=2}^{(n\ln 2)/\ln n} \sum_{B_1,\ldots,B_L \in E} r^{-\sum_{i=1}^{L}|B_i|} r^{L+1-n} \sum_{v_1,\ldots,v_L:\, v_i \in B_i} \prod_{i=1}^{L} \frac{h}{|B_i|}$$

$$\leqslant \sum_{L=2}^{(n\ln 2)/\ln n} \sum_{B_1,\ldots,B_L \in E} r^{-\sum_{i=1}^{L}|B_i|} r^{L+1-n} h^L \leqslant \sum_{L=2}^{(n\ln 2)/\ln n} \frac{r^{1-n}h^L}{L!} \sum_{B_1 \in E} \cdots \sum_{B_L \in E} r^{L-\sum_{i=1}^{L}|B_i|}$$

$$\leqslant \sum_{L=2}^{(n\ln 2)/\ln n} \frac{(f_r(H)h)^L}{L!} r^{1-n} \leqslant (f_r(H))^{(n\ln 2)/\ln n} r^{1-n} \sum_{L=2}^{(n\ln 2)/\ln n} \frac{h^L}{L!}$$

(using (3))

$$\leqslant (cn)^{(n\ln 2)/\ln n} r^{1-n} e^h = c^{(n\ln 2)/\ln n} e^{(\ln 2)n} r^{1-n} e^h$$

(since $r \geqslant 2$)

$$\leqslant c^{(n\ln 2)/\ln n} 2^n 2^{1-n} e^h = 2c^{(n\ln 2)/\ln n} e^h. \qquad (10)$$

If $L > (n \ln 2)/\ln n$ then it would be better for us to sum up over the edges A. In this case we have the following estimate for the probability of the bad event:

$$\mathrm{Pr}(\mathcal{C}) \leqslant \sum_{L>(n\ln 2)/\ln n} \sum_{\alpha=0}^{r-1} \sum_{A \in E} \sum_{B_1,\ldots,B_L \in E} r^{L-|A|} r^{-\sum_{i=1}^{L}|B_i|} \mathrm{Pr}\left(\forall i : \sigma(v_i, B_i) \leqslant h\right)$$

$$= \sum_{L>(n\ln 2)/\ln n} \sum_{A \in E} \sum_{B_1,\ldots,B_L \in E} r^{1-|A|} r^{L-\sum_{i=1}^{L}|B_i|} \prod_{i=1}^{L} \frac{h}{|B_i|}$$

(since $|B_i| \geqslant n$ for any i)

$$\sum_{L>(n\ln 2)/\ln n} \left(\frac{h}{n}\right)^L \sum_{A \in E} \sum_{B_1,\ldots,B_L \in E} r^{1-|A|} r^{L-\sum_{i=1}^{L}|B_i|}$$

$$\leqslant \sum_{L>(n\ln 2)/\ln n} \left(\frac{h}{n}\right)^L \frac{1}{L!} \sum_{A \in E} \sum_{B_1 \in E} \cdots \sum_{B_L \in E} r^{1-|A|} r^{L-\sum_{i=1}^{L}|B_i|}$$

$$\leqslant \sum_{L>(n\ln 2)/\ln n} \left(\frac{h}{n}\right)^L \frac{(f_r(H))^{L+1}}{L!} \leqslant |(3)| \leqslant \sum_{L>(n\ln 2)/\ln n} \left(\frac{h}{n}\right)^L \frac{(cn)^{L+1}}{L!} = n \sum_{L>(n\ln 2)/\ln n} \frac{h^L}{L!} c^{L+1}$$

(assuming $c < 1$)

$$\leqslant nc^{(n\ln 2)/\ln n}e^h. \tag{11}$$

Let us finish the proof. Using (9)–(11), we obtain that the probability that our recoloring procedure does not succeed is at most

$$\Pr(\mathcal{A}) + \Pr(\mathcal{B}) + \Pr(\mathcal{C}) \leqslant c^{h+1}n\left(\frac{n}{n-h}\right)^h + (n+2)c^{(n\ln 2)/\ln n}e^h. \tag{12}$$

By setting $h = \lceil \ln n \rceil$ and taking sufficiently small $c > 0$ we can make the expression on the right-hand side of (12) strictly less than 1 for arbitrary $n \geqslant 3$. Thus, with positive probability there will be no monochromatic edges after the recoloring procedure and, consequently, hypergraph H is r-colorable.

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