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Characterization of the finite C-MH-homogeneous graphs

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ABSTRACT

A graph G is C-MH-homogeneous, or C-MH, if every monomorphism between finite connected induced subgraphs of G extends to a homomorphism from G into itself. Similarly, G is C-IH if every isomorphism between finite connected induced subgraphs of G extends to a homomorphism from G into itself. In this paper, the finite C-MH graphs are characterized and a new family of finite C-IH graphs is discussed.

1. Introduction

In this paper, we work with recent generalizations of the definition of a homogeneous graph, characterizing a class of finite graphs satisfying one of the generalized definitions. A graph G is *homogeneous* if every isomorphism between any two finite induced subgraphs of G extends to an automorphism of G. The concept of a general homogeneous structure was defined in Fraïssé's 1953 paper [3], but classification results for graphs were not obtained until the late 1970s, when the finite homogeneous graphs were classified by Gardiner [4] and the countable homogeneous graphs were characterized by Lachlan and Woodrow [10].

Incentive for further work was provided in the 2005 paper by Kechris, Pestov, and Todorcevic [9], in which the theory of homogeneous structures is applied to topological dynamics. Generalizations of the definition were first suggested in the 2006 paper [1] by Cameron and Nešetřil. For instance, they defined an MH graph as one where every monomorphism between any two of its finite induced subgraphs extends to a homomorphism from the graph into itself. Similarly, we can define IH, IM, II, MM, MI, HH, HM, and HI graphs, with I, M, and H representing "isomorphism", "monomorphism", and "homomorphism", respectively. These families of graphs are collectively called the *homomorphism–homogeneous* graphs. In particular, note that II graphs are another name for homogeneous graphs. With these new definitions in place, the task of characterizing these families began. For instance, an investigation into countable MM, MH, and HH graphs was carried out by Rusinov and Schweitzer in [12].

A more recent generalization builds off the work of Gardiner in [5] and Gray and Macpherson in [6], characterizing what are now known as the C-II graphs. A graph is C-II if every isomorphism between any two of its finite *connected* induced subgraphs extends to an automorphism of the graph. In [11], Lockett combined this generalization with the definitions developed by Cameron and Nešetřil, introducing C-HH graphs, C-MH graphs, C-IH graphs, and so on, where the initial mappings considered are defined on finite, *connected* induced subgraphs. Such graphs are referred to as the *connected-homomorphism–homogeneous* graphs. This generalization allowed for additional classification research, and Lockett characterized the finite C-HI, C-MI, and C-HH graphs in [11].

It is interesting to note that the classes of homomorphism–homogeneous graphs form a partial order under containment, given in Fig. 1 for both the general and finite cases. The poset in the finite case is simpler because any monomorphism from

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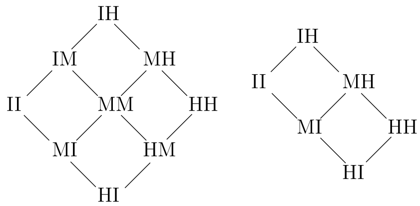

Fig. 1. The partial orders of general (left) and finite (right) homomorphism–homogeneous graphs.

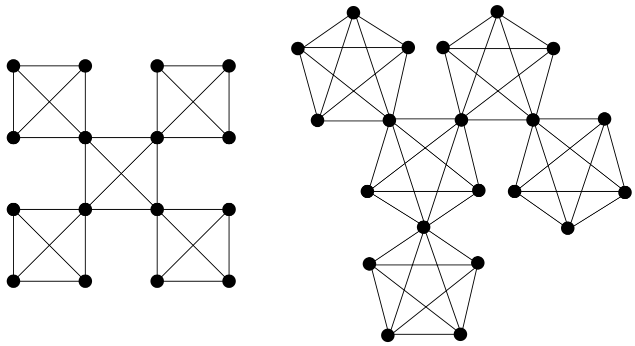

Fig. 2. A K_4-treelike graph and a K_5-treelike graph.

a finite graph into itself must also be an isomorphism, so that IM = II, for instance. Likewise, similar posets exist for the connected-homomorphism–homogeneous graphs. Lastly, observe that every MH graph is a C-MH graph, every HH graph is a C-HH graph, and so on, as follows directly from the definitions.

This paper completes the characterization of the finite C-MH graphs, identified as an open problem in [11], leaving only the class C-IH to be investigated in the case of finite graphs. Since the C-HH graphs are a subclass of the C-MH graphs, Lockett's list of finite C-HH graphs is tacitly included in the results of this paper. As such, we will state her characterizations below.

Before stating Lockett's results, a few of the graphs mentioned in the results require some explanation. For fixed n, a *K_n-treelike graph* is a connected graph constructed from copies of K_n (called *components*), where pairs of components are joined by taking a unique pair of vertices, one vertex from each component, and identifying them, in such a way that no new cycles are constructed. Examples of these graphs, each with five components, are given in Fig. 2. As observed in [11], for $n \geq 2$, they may also be characterized as the connected graphs such that all induced cycles are triangles and the neighbour set of each vertex is a disjoint union of copies of K_{n-1}. Note also that complete graphs are K_n-treelike graphs with only one component and that K_1 is the only K_1-treelike graph.

Following [11], a bipartite graph with parts X and Y where $|X| \leq |Y|$ is said to have a *perfect complement matching* if, for each $x \in X$, we can choose a vertex $y_x \in Y$ such that $x \not\sim y_x$, with the additional condition that the mapping $x \mapsto y_x$ is injective, i.e. for $x \neq x'$, we have $y_x \neq y_{x'}$. If G is a finite connected bipartite graph with parts X and Y such that $2 \leq |X| \leq |Y| = n$ and G has a perfect complement matching, then G is a *PCM(n) graph*. A graph is *PCM(n)-free* if it does not embed a PCM(n) graph. Here, as elsewhere in this paper, a graph G *embeds* a graph H if H is isomorphic to an induced subgraph of G.

A *bipartite complement of a perfect matching* is a bipartite graph with a perfect complement matching where $|X| = |Y|$. To be explicit, it is a bipartite graph with parts X and Y, where $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$, with $x_i \sim y_j$ if and only if $i \neq j$. Following [11], we note that such a graph may be constructed as $\overline{L(K_{2,n})}$, and we will use this notation to refer to these graphs. Finally, a *domino graph* is a pair of 4-cycles sharing a common edge.

Now, we can state Lockett's results:

Theorem 1 (*Connected C-HH Characterization, Lockett [11]*)**.** *A finite connected graph is C-HH if and only if it is one of the following:*

(i) *a K_n-treelike graph;*
(ii) *a domino-free graph such that all induced cycles are squares;*
(iii) *a bipartite graph such that each part has a common neighbour;*
(iv) *the bipartite complement of a perfect matching $\overline{L(K_{2,n})}$ $(n \geq 3)$.*

Theorem 2 (*General C-HH Characterization, Lockett [11]*)**.** *A finite graph is C-HH if and only if it is a finite disjoint union of finite connected C-HH graphs $\bigcup_{i=1}^{k} G_i$ such that one of the following holds:*

(a) *for fixed n, each G_i is a K_n-treelike graph;*
(b) *each G_i is a nontrivial domino-free graph such that all induced cycles are squares;*

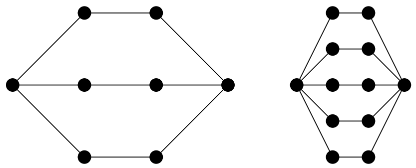

Fig. 3. The bridge graphs B_3 and B_5.

(c) *each G_i is a bipartite graph such that each part has a common neighbour;*
(d) *for fixed $n \geq 3$, some of the components are copies of $\overline{L(K_{2,n})}$, and all other components G_i are bipartite PCM(n)-free graphs such that each part has a common neighbour.*

The main goal of this paper is to characterize the finite C-MH graphs. First, we classify the connected graphs, and then we use this to classify all finite C-MH graphs. Naturally, this leads to two different theorems: one for connected graphs and one for the general case. Before stating them, we must introduce a family of graphs appearing in the theorems.

Definition 1. For any integer $n \geq 1$, the *bridge graph* B_n consists of n paths of length 3, all sharing the same endpoints, with no other shared vertices and no edges between the paths.

Examples of bridge graphs are given in Fig. 3. Note that the bridge graphs are a special type of *theta graph*. A theta graph $\Theta(l_1, \ldots, l_t)$ is a graph consisting of two vertices joined by t internally disjoint paths of lengths $l_1, \ldots, l_t$ respectively. Using this notation, the bridge graph B_n is the theta graph $\Theta(l_1, \ldots, l_n)$, where $l_1 = l_2 = \cdots = l_n = 3$.

Now, we can state the characterization of finite connected C-MH graphs:

Theorem 3 (*Connected C-MH Characterization*)**.** *A finite connected graph is C-MH if and only if it is one of the following:*

(i) *a finite connected C-HH graph;*
(ii) *a cycle C_n $(n \geq 3)$;*
(iii) *a bridge graph B_n.*

It is interesting to note that only the bridge graphs B_n have not previously been identified as C-MH. After classifying the connected C-MH graphs, we prove the following theorem:

Theorem 4 (*General C-MH Characterization*)**.** *A finite graph is C-MH if and only if it satisfies one of the following:*

(i) *for fixed n, each component is a K_n-treelike graph;*
(ii) *for fixed n, each component is an odd cycle C_{2n+1};*
(iii) *each component is a bipartite graph where each part has a common neighbour;*
(iv) *for a fixed $n \geq 3$, at least one component is a copy of $\overline{L(K_{2,n})}$, and all components that are not copies of $\overline{L(K_{2,n})}$ are bipartite PCM(n)-free graphs where each part has a common neighbour;*
(v) *at least one component is a bridge graph B_n, where n may differ for each graph, and all components that are not bridge graphs B_n are domino-free bipartite graphs where all induced cycles are squares and each part has a common neighbour;*
(vi) *each component is a nontrivial domino-free graph such that all induced cycles are squares;*
(vii) *for fixed $n \geq 3$, at least one component is an even cycle C_{2n} and all components that are not even cycles C_{2n} are nontrivial domino-free graphs such that all induced cycles are squares and for all $n + 1 < m < 2n$, the endpoints of any path in that component with m vertices are at a distance at most $2n - m + 1$ apart.*

After characterizing the finite C-MH graphs, we discuss a new family of C-IH graphs, the *expanded cycle graphs* $\mathrm{EC}(n_1, \ldots, n_k)$, providing examples sought in [11]. These graphs may be regarded as generalizations of even cycles, and they are discussed in more detail in Sections 3 and 5.

In Section 2, we clarify the notation that will be used and provide some background regarding connected-homomorphism–homogeneous graphs. Sections 3 and 4 are devoted to characterizing the finite C-MH graphs, with the connected case handled in Section 3 and the general case in Section 4. In Section 5, we prove that the expanded cycle graphs are C-IH and make note of some open problems.

2. Preliminaries

2.1. Notation

We begin by explaining some notation to be used throughout this paper. Any definition not explicitly mentioned here can be found in any textbook on graph theory, such as [2] or [7]. For the remaining part of the paper, we assume that all graphs are finite. We write $v \sim w$ to indicate that two vertices v and w are adjacent. Likewise, if a vertex v is adjacent to an entire set of vertices U, we write $v \sim U$. Rather than explicitly mentioning the vertex set of a graph, we will engage in a slight abuse of notation, using G to refer both to a graph G and to the vertex set of G.

A *homomorphism* between two graphs is a map preserving edges. Formally, a map $\phi : G_1 \to G_2$ is called a homomorphism if, for all vertices v and w in G_1, $\phi(v) \sim \phi(w)$ whenever $v \sim w$. A *monomorphism* is an injective homomorphism. An *isomorphism* is an invertible homomorphism where the inverse map is also a homomorphism. Equivalently, it is a one-to-one, onto map $\phi : G_1 \to G_2$ such that $v \sim w$ if and only if $\phi(v) \sim \phi(w)$ for all vertices v and w in G_1.

Observe that both a monomorphism and a homomorphism may map non-adjacent vertices to adjacent vertices, while an isomorphism cannot. Additionally, a general homomorphism can map non-adjacent vertices to the same vertex, while monomorphisms and isomorphisms cannot.

When necessary, we will use $[V]$ or $[a_1, \dots, a_n]$ to denote the subgraph induced by a set of vertices V or $\{a_1, \dots, a_n\}$. We will also refer to the induced subgraph $[V]$ as V, when the context is clear. We will use P_n to refer to the *path of length* $n - 1$, meaning a graph with vertices $v_1, \dots, v_n$, where $v_i \sim v_{i+1}$ for $1 \le i \le n-1$ and where there are no other adjacencies. An *induced path* will always refer to an induced subgraph P_n. On the other hand, a *path* refers to *any* subgraph P_n, not necessarily an induced subgraph. We often write both kinds of paths with vertices $v_1, v_2, \dots, v_n$ as the sequence $v_1 v_2 \cdots v_n$. If we have already made it clear that we are dealing with an induced path, we may subsequently refer to it only as a path.

For $n \ge 3$, we will use C_n to refer to the *n-cycle*, with vertices $a_1, \dots, a_n$, and $a_i \sim a_{i+1}$ for $1 \le i \le n-1$, $a_n \sim a_1$, with no other adjacencies. Henceforth, all cycles will have this labelling unless otherwise specified. We may also write an n-cycle as the sequence $a_1 a_2 \cdots a_n a_1$. We will also refer to a 3-cycle as a *triangle* and a 4-cycle as a *square*. The *domino graph* may be constructed by taking a 6-cycle and adding an edge between a_3 and a_6, and will be labelled in this manner unless otherwise specified.

Now, we make the following remarks about the bridge graphs B_n from Definition 1, needed in Sections 3 and 4:

- The only induced cycles in B_n are 6-cycles, involving the common endpoints of the induced 3-paths and two of the induced paths connecting them.
- For $n \ge 2$, any induced path in B_n is part of an induced 6-cycle.

Finally, we make a couple of observations regarding bipartite graphs. In the statement of the next lemma, $\Delta(G)$ refers to the maximum degree of the graph G.

Lemma 1 (*Lockett [11]*)**.** *For a nontrivial bipartite graph G, every k-subset of a part has a common neighbour for each $k \le \Delta(G)$ if and only if G is either the bipartite complement of a perfect matching or each part of G has a common neighbour.*

Lastly, notice that every homomorphism between connected bipartite graphs preserves the bipartition.

2.2. C-XY graphs and C-XY-symmetries

Now, we discuss connected-homomorphism–homogeneous graphs, which are the focus of this paper.

Let X and Y be any of H, M, and I, which represent the prefixes homo-, mono-, and iso-, respectively. A graph G is *(C-)XY-homogeneous*, or (C-)XY, if every X-morphism from any finite (connected) induced subgraph of G into G extends to a Y-morphism from G into itself.

Since the MH graphs are a subclass of the C-MH graphs, the characterization of finite MH graphs provides a starting point for the current project:

Proposition 1 (*Cameron & Nešetřil [1]*)**.** *A finite graph is MH if and only if it is a disjoint union of complete graphs of the same size.*

When comparing this result to the list of finite C-MH graphs (Theorem 4), it is striking to see how relaxing the definition makes for a richer and more interesting characterization. At any rate, the main importance of the above is in conjunction with the next result. Here and elsewhere, $N(v)$ denotes the neighbour set of the vertex v.

Lemma 2 (*Lockett [11]*)**.** *If G is a C-XY graph, then for each $v \in G$, the graph $[N(v)]$ is XY.*

Putting these two results together, we find that the neighbour set of any vertex in a finite C-MH graph is a disjoint union of complete graphs of the same size, a very helpful condition. The following is even more helpful for our characterization:

Lemma 3 (*Lockett [11]*)**.** *For any $n \ge 3$, the cycle C_n is C-MI.*

Since every C-MI graph is C-MH, every cycle is C-MH. In general, a cycle is not C-HH, so this result tells us that we must include cycles on the list of connected C-MH graphs, separate from the C-HH graphs. We will also have the chance to use the result in other contexts throughout the paper.

Following [11], given two graphs G_1 and G_2, we say that G_1 is *C-XY-morphic* to G_2 if every X-morphism from any finite connected induced subgraph of G_1 into G_2 extends to a Y-morphism from G_1 into G_2. Furthermore, the graphs G_1 and G_2 are *C-XY-symmetric* if G_1 is C-XY-morphic to G_2 and G_2 is C-XY-morphic to G_1. Note that a graph is C-XY if and only if it is C-XY-symmetric to itself.

It is interesting to observe that C-XY-symmetry is not generally an equivalence relation on C-XY graphs. While the relation is symmetric by definition, it is observed in [11] that C-HH-symmetry is not transitive. The argument used there also shows that C-MH- and C-IH-symmetries are not transitive.

The following result explains our interest in C-XY-symmetry. It is proved for general relational structures in [11], but is stated here specifically for graphs:

Proposition 2 (*Lockett [11]*). *A graph G is C-XY if and only if all components of G are C-XY and are pairwise C-XY-symmetric.*

Consequently, the easiest way to classify the finite C-MH graphs is to characterize the connected C-MH graphs and then determine which pairs of connected C-MH graphs are C-MH-symmetric.

Two graphs G_1 and G_2 are *homomorphically equivalent* if there exist homomorphisms $\phi_1 : G_1 \to G_2$ and $\phi_2 : G_2 \to G_1$. As the name suggests, homomorphic equivalence is an equivalence relation. Note that if two graphs G_1 and G_2 are C-MH-symmetric, then they are also homomorphically equivalent. Therefore, a useful way to prove that two graphs are not C-MH-symmetric is to prove that they are not homomorphically equivalent.

The study of graph homomorphisms is undertaken in detail in [8], but we focus only on the relevant details here. If H is an induced subgraph of G, a homomorphism $r : G \to H$ is called a *retraction* if the restriction of r to H is the identity map. If such a retraction exists, we say that G *retracts* to H. A *core* is a graph that does not retract to any of its proper induced subgraphs.

In [8], it is shown that each graph G embeds a core that is unique up to isomorphism. This core is characterized by being the smallest induced subgraph of G to which G is homomorphic, and we call this subgraph the *core of* G. A graph is homomorphically equivalent to its core, and from this, it can be shown that any two homomorphically equivalent graphs have the same core. So, if we can determine that two graphs have different cores, they cannot be homomorphically equivalent and therefore are not C-MH-symmetric.

Therefore, the following facts about cores will be useful to us:

- Every complete graph is a core.
- A connected graph G has core K_1 if and only if $G \cong K_1$.
- A nontrivial connected graph is bipartite if and only if its core is K_2.
- Every odd cycle is a core.

In particular, the third remark above tells us that amongst the nontrivial connected graphs, bipartite graphs are C-MH-symmetric only to bipartite graphs and nonbipartite graphs are C-MH-symmetric only to nonbipartite graphs. Hence, either every component of a C-MH graph is bipartite, or every component is nonbipartite.

3. The connected case

This section is devoted to the proof of the connected C-MH characterization (Theorem 3). We have already seen that the list will include cycles and Lockett's list of connected C-HH graphs. Now, we show that the bridge graphs B_n belong to the list. In fact, we prove the stronger claim that for any $n_1, n_2 \geq 2$, the graphs B_{n_1} and B_{n_2} are C-MH-symmetric, which we will need in Section 4.

Proposition 3. *For any integers $n_1, n_2 \geq 2$, the graphs B_{n_1} and B_{n_2} are C-MH-symmetric.*

Proof. It suffices to show that B_{n_1} is C-MH-morphic to B_{n_2}, because n_1 and n_2 are interchangeable. Let A be the set of common endpoints of the n_1 copies of P_4 in B_{n_1}, with vertices a_1 and a_2. We will refer to these vertices, and the corresponding vertices in B_{n_2}, as *extremal vertices*. Let B denote the set of remaining vertices in B_{n_1}, where $B = \{b_{11}, b_{12}, b_{21}, b_{22}, \ldots, b_{n_1 1}, b_{n_1 2}\}$. Furthermore, assume that $a_1 \sim b_{i1}, b_{i1} \sim b_{i2}, b_{i2} \sim a_2$ for $1 \leq i \leq n_1$. Now, we consider all possible domains for a monomorphism from a connected induced subgraph of B_{n_1} into B_{n_2}. For the remaining part of the proof, let ϕ denote the arbitrary monomorphism and let $\mathcal{D}$ denote the domain of ϕ. We will show how to construct an extension ψ of ϕ to the whole graph in each case.

First, suppose that $|\mathcal{D} \cap A| = 0$, so that $\mathcal{D}$ is isomorphic to K_1 or K_2. The domain and range of this partial map are isomorphic induced paths of length 0 or 1 respectively, and hence part of induced 6-cycles in B_{n_1} and B_{n_2}, as observed in Section 2. By Lemma 3, cycles are C-MI, so the map extends to an isomorphism between the two cycles. Without loss of generality, the cycle in the domain is $a_1 b_{11} b_{12} a_2 b_{22} b_{21} a_1$. By defining $\psi(b_{ij})$ to be $\phi(b_{1j})$ for $i \geq 3$, the extended map ψ is a homomorphism from B_{n_1} into B_{n_2}.

Now, we suppose that $|\mathcal{D} \cap A| = 1$. Without loss of generality, suppose that a_1 is in $\mathcal{D}$ and a_2 is not. If at most two vertices of the form b_{i1} are in $\mathcal{D}$, then the domain is an induced path in B_{n_1} and contained in an induced 6-cycle. The image must be an isomorphic path, also contained in an induced 6-cycle, since extra edges between the vertices would induce a cycle with less than 6 vertices. As in the previous paragraph, we can extend the map to an isomorphism between the two cycles and then to a homomorphism defined on the entire graph B_{n_1}.

If at least three vertices of the form b_{i1} belong to $\mathcal{D}$, then a_1 must be mapped to one of the extremal vertices in B_{n_2}, because there are at least three adjacencies with a_1 to preserve. All vertices b_{i1} in the domain are mapped to distinct vertices adjacent to $\phi(a_1)$. If the map is not defined for the adjacent vertex b_{i2}, we send b_{i2} to the only vertex other than $\phi(a_1)$ adjacent to

$\phi(b_{i1})$, and if ϕ was already defined for b_{i2}, it must already have mapped to that vertex. If neither b_{i1} nor b_{i2} are in the domain for some i, define $\psi(b_{i1})$ to be an arbitrary neighbour of $\phi(a_1)$ and define $\psi(b_{i2})$ to be the only neighbour of $\psi(b_{i1})$ aside from $\phi(a_1)$. Lastly, we map a_2 to the extremal vertex in B_{n_2} different from $\phi(a_1)$. In this way, we preserve all adjacencies.

Finally, suppose that $|\mathcal{D} \cap A| = 2$. Now, some path connecting a_1 and a_2 belongs to $\mathcal{D}$ since it is connected, which without loss of generality is $a_1b_{11}b_{12}a_2$. If exactly one vertex in some edge $b_{i1}b_{i2}$ is included in the domain, suppose that it is b_{21}, by relabelling if necessary. The vertex b_{22} needs to map to a common neighbour of $\phi(b_{21})$ and $\phi(a_2)$. However, the induced path $b_{21}a_1b_{11}b_{12}a_2$ contained in the domain must map to an isomorphic path under a monomorphism into B_{n_2}, so that there is always a way to extend the map for b_{22} by completing a 6-cycle in the domain and the range. If no vertices in a given edge $b_{i1}b_{i2}$ are included in the domain, we let $\psi(b_{i1}) = \phi(b_{11})$ and $\psi(b_{i2}) = \phi(b_{12})$ for any applicable i. If we perform the above two steps for every vertex not in the domain, ψ is a homomorphism defined on all of B_{n_1}.

We have now accounted for all possible connected domains, showing that B_{n_1} is C-MH-morphic to B_{n_2}. $\square$

By setting $n_1 = n_2$, we immediately get that B_n is C-MH for $n \geq 2$. Observing that $B_1 \cong P_4$ is a bipartite graph such that each part has a common neighbour, we see that B_1 is on the list of connected C-HH graphs, and is therefore also C-MH. This gives us the following.

Proposition 4. *For any positive integer n, the bridge graph B_n is C-MH.*

Furthermore, for $n \geq 3$, B_n is not C-HH, so this family is truly a source of new C-MH graphs. While this can be shown by checking the list of connected C-HH graphs, we will show it directly. Labelling the vertices $a_1, a_2, b_{11}, b_{12}, \dots, b_{n1}, b_{n2}$ as in Proposition 3, the homomorphism $a_1 \mapsto a_1, b_{11} \mapsto b_{11}, b_{12} \mapsto b_{12}, b_{21} \mapsto b_{21}, b_{22} \mapsto b_{22}, b_{31} \mapsto b_{31}, b_{32} \mapsto a_1$ from a connected induced subgraph of B_n into B_n cannot be extended to a homomorphism defined on all of B_n. We would need a_2 to map to a common neighbour of a_1, b_{12}, and b_{22}, but such a vertex does not exist.

Now, we show that the list of connected C-MH graphs is complete. The plan is to consider arbitrary finite connected C-MH graphs that are neither a cycle nor C-HH, which we will suggestively call "B-graphs". We will eventually show that every B-graph is a graph B_n for some $n \geq 3$.

First, we show that every B-graph is bipartite. Next, we use this to show that every B-graph not isomorphic to B_n for $n \geq 3$ must embed the domino graph. Finally, we show that every connected bipartite C-MH graph embedding the domino graph is actually a C-HH graph. In particular, there cannot be a B-graph embedding the domino graph, so the only possible B-graphs are bridge graphs B_n for $n \geq 3$, completing the proof of Theorem 3.

The proof that every B-graph is bipartite has three parts. The proof of Part 1 is largely inspired by an argument used in [11], but the other two parts use new arguments.

Proposition 5. *Every B-graph is bipartite.*

Part 1. Any finite connected graph that is C-MH but not C-HH is triangle-free.

Proof. Let G be a finite connected C-MH graph that is not C-HH. Suppose to the contrary that G embeds a 3-cycle $a_1a_2a_3a_1$. It follows that for any two adjacent vertices v and w in G, the monomorphism $a_1 \mapsto v, a_2 \mapsto w$ can be extended to a homomorphism where the image of a_3 is a common neighbour of v and w, so that every edge must be part of a triangle.

We now claim that every induced cycle in G is a triangle. If there were a longer induced cycle $a_1a_2 \cdots a_ka_1$ for minimal $k \geq 4$, then there is a vertex b outside the cycle that is adjacent to both a_1 and a_2, because every edge is part of a triangle. We observe that neither a_3 nor a_k is adjacent to b, because if either vertex were adjacent to b, then $[N(b)]$ would contain an induced path $a_1a_2a_3$ or $a_ka_1a_2$. By Proposition 1, such an induced path does not embed in any finite MH graph, in contradiction to Lemma 2. Furthermore, if b is adjacent to some a_i where $3 < i < k$ is maximal, then $ba_i \cdots a_ka_1b$ is an induced cycle that is not a triangle and has fewer than the minimal number of vertices.

Therefore, b is only adjacent to a_1 and a_2. Hence, the map taking b to a_1 and fixing $a_2, \dots, a_k$ is a monomorphism defined on a connected induced subgraph of G. If this map extends to a homomorphism, then a_1 maps to a common neighbour of a_k, a_1, and a_2. But, we have just seen that no common neighbour of a_1 and a_2 can also be a neighbour of a_k, so we have a contradiction. Therefore, if G contains a 3-cycle and is C-MH, then all cycles are 3-cycles. Furthermore, by Lemma 2 and Proposition 1, the neighbour set of every vertex is a disjoint union of complete graphs of the same size.

Recall that K_n-treelike graphs are characterized as the connected graphs where all induced cycles are triangles and the neighbour set of each vertex is a disjoint union of K_{n-1} graphs *for fixed n*. If we can show that the complete graphs in different neighbour sets have the same size, we can conclude that G is a K_n-treelike graph for some n and is on the list of C-HH graphs, which is a contradiction. This is immediate from the observation that all maximal cliques in a C-MH graph have the same size. If $[v_1, \dots, v_{n_1}]$ and $[w_1, \dots, w_{n_2}]$ are maximal cliques in G with $n_1 < n_2$, the map $w_i \mapsto v_i$ for $1 \leq i \leq n_1$ is a monomorphism between two connected induced subgraphs of G that cannot be extended to a homomorphism, since w_{n_1+1} needs to map to a nonexistent common neighbour of $v_1, \dots, v_{n_1}$. $\square$

Part 2. For any $n \geq 4$, a finite connected C-MH graph that is not C-HH cannot embed both C_n and C_{n+1}.

Proof. Suppose to the contrary that G is a C-MH graph that is not C-HH while embedding C_n and C_{n+1} for some $n \geq 4$. Let $a_1a_2 \cdots a_na_1$ denote an induced cycle C_n and $b_1b_2 \dots b_{n+1}b_1$ denote an induced cycle C_{n+1} in G. The map $b_i \mapsto a_i$, $1 \leq i \leq n$ is a monomorphism between connected induced subgraphs of G, so by the C-MH property, the map extends to a homomorphism from G into itself where b_{n+1} maps to a common neighbour of a_n and a_1. This induces a triangle in G, contradicting Part 1. □

Part 3. Every B-graph embeds only even cycles.

Proof. Consider an arbitrary B-graph G, and suppose to the contrary that G embeds an odd cycle. Let $a_1a_2\cdots a_{2k+1}a_1$ be a minimal induced odd cycle. By Part 1, $k \geq 2$. Without loss of generality, since G is not a cycle and is also connected, there must be a vertex b from outside the cycle that is adjacent to a_1. The monomorphism $a_{2k+1} \mapsto b, a_1 \mapsto a_1, a_2 \mapsto a_2, \ldots, a_{2k-1} \mapsto a_{2k-1}$ must extend to a homomorphism from G into itself by the C-MH property, where a_{2k} must map to a common neighbour b' of b and a_{2k-1}. Now, if b' belongs to the $(2k+1)$-cycle, then it must be a_{2k-2} or a_{2k}, because these are the only vertices adjacent to a_{2k-1}. We claim that $b' \neq a_{2k-2}$. If the two vertices were the same, then $ba_1a_2\cdots a_{2k-2}b$ would be an induced $(2k-1)$-cycle or would embed a smaller odd cycle if other edges existed between b and the cycle, contradicting the minimality of k.

From here, we proceed in two cases. First suppose that $k = 2$. By Part 2, it follows that neither C_4 nor C_6 embeds in G. If $b' = a_{2k} = a_4$ holds, then $a_4a_5a_1ba_4$ is an induced copy of C_4, which is a contradiction. Now, we can conclude that b' does not belong to the cycle. It follows that $a_1bb'a_3a_4a_5a_1$ is an induced copy of C_6, or embeds a triangle or square. This contradicts Parts 1 and 2, so $k = 2$ is impossible.

Now, we handle the case $k \geq 3$. Here, we observe that the monomorphism $a_2 \mapsto b, a_1 \mapsto a_1, a_{2k+1} \mapsto a_{2k+1}, \ldots, a_4 \mapsto a_4$ must extend to a homomorphism from G into itself, where a_3 maps to a common neighbour b'' of b and a_4. If b'' belongs to the cycle, then it must be either a_3 or a_5, because it is adjacent to a_4. If $b'' = a_5$, then $ba_1a_{2k+1}a_{2k}\cdots a_5b$ is a $(2k-1)$-cycle or embeds a smaller odd cycle, contradicting minimality of k.

Next, we observe that $b'' \neq b'$. If the two vertices were equal, then $a_4a_5\cdots a_{2k-1}b'a_4$ would be a $(2k-3)$-cycle or embed a smaller odd cycle, contradicting the minimality of k. Now, since $b' \neq a_{2k-2}$, $b'' \neq a_5$ and $b' \neq b''$ all hold, the induced subgraph $bb''a_4a_5\cdots a_{2k-2}a_{2k-1}b'b$ is either a $(2k-1)$-cycle or embeds a smaller odd cycle, even if $b' = a_{2k}$ or $b'' = a_3$. This contradicts the minimality of k once more. Therefore, G embeds only even cycles and is consequently bipartite. □

Now that we know every B-graph must be bipartite, we show, for a given B-graph G, if $G \ncong B_n$ for $n \geq 3$, then G embeds the domino graph. We show this with the goal of proving later that there cannot be a B-graph embedding the domino graph, leaving bridge graphs as the only possibility.

To prove this, our plan is as follows: first, we use the fact that every B-graph is bipartite to show that every such graph embeds either C_6 or the domino graph. Using this property, we will show that every two vertices in the same part of the bipartition of any B-graph have a common neighbour. Finally, we use these tools to prove our desired result.

Lemma 4. *Every B-graph embeds either C_6 or the domino graph.*

Proof. Let G be a B-graph. We know that G is bipartite by Proposition 5, and therefore embeds only even cycles. We know G must embed a cycle, because trees are vacuously domino-free graphs where all induced cycles are squares, and all such graphs are on the list of connected C-HH graphs. Let n be the largest integer such that a $2n$-cycle embeds in G. If $n = 2$, then all induced cycles in G are squares. If G also embeds the domino graph, we are done, and if G does not embed the domino graph, then G is on the list of connected C-HH graphs, a contradiction. If $n = 3$, then G embeds C_6, and we are done.

Now, suppose that $n \geq 4$, and let $a_1a_2\cdots a_{2n}a_1$ be an induced $2n$-cycle in G. Without loss of generality, since G is connected and not a cycle, there must be another vertex b from outside the cycle that is adjacent to a_1. Note that b cannot be adjacent to vertices of the form a_{2k}, because G is bipartite. Next, we observe that if $b \sim a_3$ and $b \sim a_{2n-1}$, then $[b, a_{2n-1}, a_{2n}, a_1, a_2, a_3]$ is isomorphic to the domino graph, so without loss of generality, we may suppose that $b \not\sim a_3$.

Now, the monomorphism that fixes $a_4, a_5, \ldots, a_{2n}, a_1$ and maps a_2 to b indicates that b and a_4 have a common neighbour b', by the C-MH property. In particular, $b' \neq a_3$, since $b \not\sim a_3$. Also, b' cannot be a_2 or a_1, because neither vertex is adjacent to a_4. In addition, since G is bipartite, $a_1 \not\sim b'$ and $a_3 \not\sim b'$. It follows that $[b, a_1, a_2, a_3, a_4, b']$ is an embedded 6-cycle if $b' \not\sim a_2$ and is isomorphic to the domino graph if $b' \sim a_2$. □

Note that in the latter part of the proof, no mention was made that G was not C-HH. This observation leads to the following corollary of the proof above, which will be useful when we consider the disconnected case:

Corollary 1. *For $n \geq 4$, any connected bipartite C-MH graph that is not a cycle and embeds C_{2n} also embeds either C_6 or the domino graph.*

Now that we know B-graphs must embed either C_6 or the domino graph, we derive another important fact about these graphs. The result that we prove is more general than is needed here, so that it may be employed in Section 4. However, it has a corollary for B-graphs that will be very useful in the current section.

Lemma 5. *Let G_1 and G_2 be connected bipartite graphs, where G_1 embeds C_6 or the domino graph. If G_1 is C-MH-morphic to G_2, then any two vertices in the same part of G_2 have a common neighbour.*

Proof. Let $a_1a_2\cdots a_6a_1$ be a copy of C_6 or the domino graph in G_1, with adjacencies as given in Section 2, and let b and c be two arbitrary vertices in the same part of G_2. Since G_2 is connected and bipartite, there is a path of length $2k$, for minimal $k \geq 1$, between b and c. We show that $k = 1$.

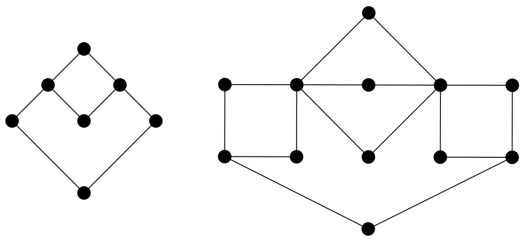

Fig. 4. The expanded cycles EC(2, 1, 1) and EC(3, 2, 1, 2).

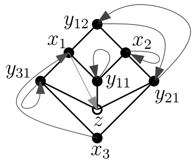

Fig. 5. The monomorphism and its extension from Lemma 6.

Suppose to the contrary that $k \geq 2$. It follows that at least five vertices are in any shortest path from b to c, so let $d_1 = b$, $d_{2k+1} = c$, and $d_1d_2d_3d_4d_5$ be the path consisting of the first five vertices in some shortest path $d_1d_2 \cdots d_{2k}d_{2k+1}$ connecting b and c. Because G_1 is C-MH-morphic to G_2, the monomorphism $a_i \mapsto d_i$, $1 \leq i \leq 5$ extends to a homomorphism where a_6 must map to some common neighbour d' of d_1 and d_5. Since the path $d_1 \cdots d_{2k+1}$ is shortest possible, the path is induced, so d' does not belong to the path. However, it follows that $d_1d'd_5 \cdots d_{2k+1}$ is a path of length less than $2k$ between b and c, contradicting minimality of k. Therefore, we must have $k = 1$, so any two vertices in the same part of G_2 have a common neighbour. □

Now, since each B-graph is a connected bipartite graph that is C-MH-morphic to itself and embeds either C_6 or the domino graph, we apply the above, where G_1 is a B-graph and $G_1 = G_2$, to get the following corollary:

Corollary 2. *Any two vertices in the same part of any B-graph have a common neighbour.*

We now have the tools we need in order to show that every B-graph that is not a bridge graph must embed the domino graph. In order to simplify the proof, we first identify a special circumstance where a B-graph is forced to embed the domino graph. To do so, we must now define the expanded cycle graphs alluded to in the Introduction. These graphs will be the primary source of our attention in Section 5.

Definition 2. For positive integers $n_1, \ldots, n_k$, the *expanded cycle graph* $\mathrm{EC}(n_1, \ldots, n_k)$ is bipartite, with parts $X = \{x_1, \ldots, x_k\}$ and $Y = Y_1 \cup \cdots \cup Y_k$, where $Y_1, \ldots, Y_k$ are pairwise disjoint and $|Y_i| = n_i$ for each i. In this graph, the only adjacencies are $x_i \sim Y_i$ and $Y_i \sim x_{i+1}$ for each i, taking subscripts modulo k.

For examples, see Fig. 4. These graphs may be seen as a generalization of even cycles because, in the case $k \geq 2$ and $n_1 = \cdots = n_k = 1$, we have $\mathrm{EC}(n_1, \ldots, n_k) \cong C_{2k}$. Intuitively, an expanded cycle is obtained by starting with an even cycle and "expanding" certain vertices in the cycle into several vertices.

We observe that for $k \geq 3$, every induced path of length exceeding 2 is part of an induced $2k$-cycle in $\mathrm{EC}(n_1, \ldots, n_k)$, one that includes every vertex in X and one vertex from each cell Y_i. This will be important in Section 5.

With the formalities out of the way, we can now prove the following lemma:

Lemma 6. *Any B-graph that embeds EC*$(2, 1, 1)$ *also embeds the domino graph.*

Proof. Let G be a B-graph embedding EC(2, 1, 1). In the copy of EC(2, 1, 1), let $Y_i = \{y_{i1}, \ldots, y_{in_i}\}$ for each i. The map that fixes y_{11} and x_2, interchanges y_{12} and y_{21}, sends x_3 to x_1, and fixes y_{31} is a monomorphism defined on a connected induced subgraph of G. By the C-MH property, this map extends to a homomorphism from G into itself, where x_1 must map to a common neighbour z of y_{11}, y_{21}, and y_{31}, which does not belong to EC(2, 1, 1). The vertex z cannot be adjacent to x_1 or x_2 because G is bipartite. Therefore, $[x_1, y_{11}, x_2, y_{21}, z, y_{31}]$ is isomorphic to the domino graph. The mapping is shown in Fig. 5. □

We are now ready to show that aside from the bridge graphs, every B-graph must embed the domino graph.

Proposition 6. *Every B-graph not isomorphic to B_n for $n \geq 3$ embeds the domino graph.*

Proof. From Lemma 4, we know that any B-graph G must embed C_6 or the domino graph. Hence, it suffices to show that if $G \ncong B_n$ for $n \geq 3$, then G embeds the domino graph whenever G embeds C_6.

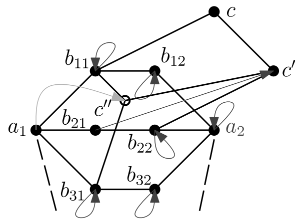

Fig. 6. The monomorphism and its extension from Proposition 6.

So, we assume that G embeds $C_6 \cong B_2$, and choose a positive integer $k \geq 2$ such that B_k embeds in G, but B_l does not embed for any $l > k$. Fix a copy of B_k in G and denote it by B_k. As in Proposition 3, let a_1 and a_2 denote the extremal vertices and $b_{11}, b_{12}, b_{21}, b_{22}, \ldots, b_{k1}, b_{k2}$ denote the remaining vertices in B_k, with $a_1 \sim b_{i1}, b_{i1} \sim b_{i2}, b_{i2} \sim a_2$ for each i. The graph G is connected and is not isomorphic to C_6 or B_n for $n \geq 3$, so there must be a vertex c from $G \setminus B_k$ adjacent to a vertex in B_k. The vertex c is adjacent to an extremal vertex or to a vertex b_{ij}.

Suppose that the former option holds. Without loss of generality, $c \sim a_1$. The only other vertices in B_k that c may be adjacent to are of the form b_{i2}, since G is bipartite. If $c \sim b_{i2}$ and $c \sim b_{j2}$ for $i \neq j$, then $[a_1, c, b_{i2}, a_2, b_{j2}, b_{j1}]$ is isomorphic to the domino graph. If c is adjacent to b_{i2} for exactly one i, then $[a_1, a_2, b_{i1}, b_{i2}, b_{j1}, b_{j2}, c] \cong \mathrm{EC}(2, 1, 1)$ for any $j \neq i$, and by Lemma 6, G embeds the domino graph. Hence, we can suppose that c is not adjacent to any vertex in B_k except a_1.

By Corollary 2, c and a_2 must have a common neighbour c'. We know that $c' \neq a_1$, because $a_1 \not\sim a_2$. Since c is not adjacent to any vertex in B_k but a_1, we may conclude that $c' \notin B_k$. Using an argument identical to the one in the previous paragraph, with a_2 in place of a_1 and c' in place of c, we may assume that c' is not adjacent to any vertex in B_k except a_2. Now, $[B_k \cup \{c, c'\}] \cong B_{k+1}$, a contradiction to the maximality of k.

Now, suppose instead that c is adjacent to a vertex b_{ij}. Without loss of generality, $c \sim b_{11}$. If c is adjacent to both a_2 and a vertex of the form b_{i1} where $i \neq 1$, then $[a_1, b_{i1}, b_{i2}, a_2, c, b_{11}]$ is isomorphic to the domino graph. If c is adjacent to a_2 and not adjacent to any vertices of the form b_{i1} for $i \neq 1$, then $[a_1, a_2, b_{11}, b_{12}, b_{i1}, b_{i2}, c] \cong \mathrm{EC}(2, 1, 1)$ for any such i and G must embed the domino graph by Lemma 6. Likewise, if c is adjacent to a vertex of the form b_{i1} for $i \neq 1$ but is not adjacent to a_2, we still have $[a_1, a_2, b_{11}, b_{12}, b_{i1}, b_{i2}, c] \cong \mathrm{EC}(2, 1, 1)$, so that G embeds the domino graph. There are no other possible adjacencies, as G is bipartite. Therefore, we may suppose that c is adjacent only to b_{11} in B_k.

Now, by Corollary 2, c and b_{22} have a common neighbour c'. Of course, $c' \neq b_{11}$, because b_{11} and b_{22} are nonadjacent. We may therefore suppose that $c' \notin B_k$, by the previous paragraph. We now have a situation with c' symmetric to the one with c just discussed, so we may assume that c' is adjacent only to b_{22} in B_k. Note that $[a_1, a_2, b_{11}, b_{12}, b_{21}, b_{22}, c, c'] \cong B_3$, so in the case $k = 2$, we have a contradiction.

If $k \geq 3$, the map that fixes $b_{11}, b_{12}, b_{22}, b_{31}, b_{32}$, and a_2, while mapping b_{21} to c', as shown in Fig. 6, is a monomorphism defined on a connected induced subgraph of G. By the C-MH property, this map must extend to a homomorphism where a_1 is mapped to a common neighbour c'' of b_{11}, b_{31}, and c'. We can suppose that c'' is distinct from a_1 and c, because we have already seen that the domino graph embeds when $a_1 \sim c'$ or $c \sim b_{31}$. Now, notice that $[a_1, b_{11}, c, c', c'', b_{31}]$ is isomorphic to the domino graph. $\square$

At this point, all that remains to be shown is that there cannot be a B-graph embedding the domino graph. Before showing this, we will need one more tool, which is a partial strengthening of Lemma 5. Again, we prove a more general result than is necessary for this section, but we will need the more general form in Section 4.

Lemma 7. *Let G_1 and G_2 be connected bipartite graphs, where G_1 embeds the domino graph. If G_1 is C-MH-morphic to G_2, then any three vertices in the same part of G_2 have a common neighbour.*

Proof. Let G_1 be a connected bipartite graph embedding the domino graph. Let $a_1a_2a_3a_4a_5a_6a_1$ denote an embedded domino graph in G_1, with the extra edge between a_3 and a_6. By Lemma 5, any two vertices in the same part of G_2 have a common neighbour. Hence, for any three vertices b_1, b_2, and b_3 in the same part of G_2, the vertices b_1 and b_2 have a common neighbour c_1, and b_2 and b_3 have a common neighbour c_2. If $c_1 = c_2$, then we are done. Otherwise, since G_1 is C-MH-morphic to G_2, the map $a_1 \mapsto b_1, a_2 \mapsto c_1, a_3 \mapsto b_2, a_4 \mapsto c_2, a_5 \mapsto b_3$ is a monomorphism from a connected induced subgraph of G_1 into G_2 that can be extended to a homomorphism where a_6 is mapped to a common neighbour of b_1, b_2, and b_3. $\square$

Taking G_1 to be a connected bipartite C-MH graph embedding the domino graph, we apply the above with $G_1 = G_2$ to give us the following corollary for use in the final result of the section:

Corollary 3. *Any three vertices in the same part of a connected bipartite C-MH graph embedding the domino graph have a common neighbour.*

Now, we may complete the proof of the connected C-MH characterization using the following result, which is an adaptation of the proof of Lemma 21 in [11].

Proposition 7. *Any connected bipartite C-MH graph embedding the domino graph is C-HH.*

Proof. Let G be a connected bipartite C-MH graph embedding the domino graph, with parts X and Y. Since G embeds the domino graph, a glance at the list of connected C-HH graphs shows that G is C-HH if and only if each part of G has a common neighbour or $G \cong \overline{L(K_{2,n})}$ for some $n \geq 4$. Using Lemma 1, we deduce that G will be C-HH if and only if every k-subset of a part has a common neighbour for every $k \leq \Delta(G)$. So, we suppose that this condition does not hold and derive a contradiction. Choose k to be the smallest integer such that some k-subset of a part does not have a common neighbour. Without loss of generality, suppose that $\{x_1, \ldots, x_k\}$ is a k-subset of X without a common neighbour. By Corollary 3, we know that $k \geq 4$.

By minimality of k, every $(k-1)$-subset of a part has a common neighbour, so we can find vertices $y_1, \ldots, y_k$ in Y such that $y_i \sim \{x_1, \ldots, x_k\} \setminus \{x_i\}$ for each i. Furthermore, the vertices $y_1, \ldots, y_k$ must be distinct because the vertices $x_1, \ldots, x_k$ do not have a common neighbour. Thus, $[x_1, \ldots, x_k, y_1, \ldots, y_k] \cong \overline{L(K_{2,k})}$. Since no vertex in this induced subgraph has degree $\Delta(G)$, this subgraph is proper. Therefore, without loss of generality, since G is connected, there must be a vertex $x' \in X \setminus \{x_1, \ldots, x_k\}$ such that $x' \sim y_k$. If x' is adjacent to at least one of $y_1, \ldots, y_{k-1}$, we consider the monomorphism that fixes $x_1, \ldots, x_{k-1}, y_1, \ldots, y_{k-1}$ and sends x' to x_k. Since $k \geq 4$, the domain is connected. On the other hand, if x' is not adjacent to any of $y_1, \ldots, y_{k-1}$, we certainly know that x_{k-1}, x_k, and x' have a common neighbour y' because every 3-subset of a part has a common neighbour. Note that y' cannot be any of $y_1, \ldots, y_k$. In this case, by defining the map as above, but also fixing y', the map is again a monomorphism on a connected domain. In either case, the k-subset $\{x_1, \ldots, x_{k-1}, x'\}$ with common neighbour y_k is mapped to the k-subset $\{x_1, \ldots, x_k\}$ with no common neighbour. It follows that G cannot be C-MH, a contradiction. $\square$

We conclude that a B-graph embedding the domino graph cannot possibly exist. Therefore, in conjunction with Proposition 6, we know that the only possible B-graphs are bridge graphs B_n for some $n \geq 3$, completing the proof of the connected C-MH characterization.

4. The general case

4.1. Overview and nonbipartite case

We now turn our attention to the general case. By Proposition 2, we only need to determine which pairs of connected C-MH graphs are C-MH-symmetric. Since C-HH-symmetric graphs are also C-MH-symmetric, we will be able to employ results and arguments from Lockett's C-HH characterization in [11] throughout this section.

First, we treat the trivial case. Evidently, K_1 is C-MH-symmetric to itself and has core K_1, and in Section 2, we observed that no other connected graph has core K_1. Hence, K_1 is C-MH-symmetric to a connected graph G if and only if $G \cong K_1$. For the rest of this section, we assume that all components are nontrivial.

In Section 2, we observed that two graphs with different cores cannot be C-MH-symmetric. In particular, if some component in a C-MH graph is bipartite, then all components are bipartite, and if some component is nonbipartite, then all components are nonbipartite.

We first consider the case where all components are nonbipartite. On the list of connected C-MH graphs, the only nonbipartite graphs are odd cycles and K_n-treelike graphs, where $n \geq 3$. In Section 2, we noted that an odd cycle is its own core, and it is not hard to see that the core of any K_n-treelike graph is K_n.

We start by handling the case where some component is a K_n-treelike graph, for $n \geq 3$. Lockett was able to establish the following in her work with C-HH graphs:

Lemma 8 (*Lockett [11]*). *For fixed n, any two K_n-treelike graphs are C-HH-symmetric.*

It should be noted that Lockett proves the result explicitly only for $n \geq 3$. However, the only K_1-treelike graph is K_1, which is clearly C-HH-symmetric to itself. Additionally, K_2-treelike graphs are trees, which are domino-free graphs such that all induced cycles are squares. Lockett proves that any two nontrivial connected graphs of the latter kind are C-HH-symmetric, so the result above is indeed contained in Lockett's work.

Therefore, for a fixed n, any two K_n-treelike graphs are also C-MH-symmetric. Additionally, for fixed $n \geq 3$, the only connected C-MH graphs that are C-MH-symmetric to K_n-treelike graphs are other K_n-treelike graphs. This is immediate from the observation that the only connected C-MH graphs with core K_n for $n \geq 3$ are K_n-treelike graphs, as one can see by checking the list. Hence, we get the following:

Corollary 4. *For a fixed $n \geq 3$, if G_1 is a K_n-treelike graph and G_2 is a connected C-MH graph, then G_1 is C-MH-symmetric to G_2 if and only if G_2 is K_n-treelike.*

We can also easily handle the odd cycles. First, we note that C_3 is a K_3-treelike graph, which we have already discussed. Hence, we now consider cycles C_{2n+1} where $n \geq 2$.

Lemma 9. *If G is a connected C-MH graph, then for each $n \geq 2$, C_{2n+1} is C-MH-symmetric to G if and only if $G \cong C_{2n+1}$.*

Proof. Let $n \geq 2$ be given. Since C_{2n+1} is C-MH, it is clearly C-MH-symmetric to itself. On the list of connected C-MH graphs, the only graph with a core of C_{2n+1} is C_{2n+1}, so amongst the connected C-MH graphs, C_{2n+1} is only C-MH-symmetric to C_{2n+1}. $\square$

We have now determined all C-MH-symmetries between nonbipartite connected C-MH graphs. Next, we discuss the plan for the more interesting bipartite case. Looking at the list, any connected bipartite C-MH graph must be one of the following:

(a) A domino-free graph such that all induced cycles are squares.
(b) A bipartite graph such that each part has a common neighbour.
(c) A bipartite complement of a perfect matching $\overline{L(K_{2,n})}$ $(n \geq 3)$.
(d) A bridge graph B_n.
(e) An even cycle C_{2n}.

As indicated in the list, we will subsequently refer to the first two types of graphs on the list as type (a) and type (b) graphs, respectively. We will include K_2-treelike graphs as type (a) graphs in the list above.

The task of determining which graphs G with bipartite components are C-MH divides naturally into three cases, which cover all possibilities:

Case1: Some component of G embeds the domino graph.
Case2: Each component of G is domino-free and some component embeds C_6.
Case3: Each component of G is domino-free and C_6-free.

We treat each case in turn.

4.2. Bipartite case 1: Domino embeds

To begin, we must recall Lemma 7, from which we may conclude that whenever a C-MH graph with bipartite components has a component embedding the domino graph, as in the current case, any three vertices in each part of each component have a common neighbour.

By looking at the list of connected bipartite C-MH graphs given above, the only graphs where any three vertices in each part have a common neighbour are graphs of type (b), graphs $\overline{L(K_{2,n})}$ for $n \geq 4$, and some graphs of type (a). However, the next result implies that every nontrivial graph of type (a) where every three vertices in each part have a common neighbour is also of type (b), so we need not discuss type (a) graphs separately. That is, each component is either of type (b) or isomorphic to $\overline{L(K_{2,n})}$ for some $n \geq 4$.

Lemma 10. *A nontrivial graph of type* (a) *where every two vertices in each part have a common neighbour is also of type* (b).

Proof. We prove the contrapositive. Let G be a nontrivial graph of type (a) that is not of type (b). Since every graph of type (a) is connected and does not embed C_6, we conclude $G \ncong \overline{L(K_{2,n})}$ for any $n \geq 1$. Therefore, Lemma 1 tells us that some k-subset of a part of G does not have a common neighbour, for some $k \leq \Delta(G)$. Hence, we may choose the least such integer k. Since G is nontrivial, $k \neq 1$, and we claim that $k = 2$. If not, let $\{x_1, x_2, \dots, x_k\}$ be a k-subset of a part without a common neighbour, where $k \geq 3$. Now, each $(k-1)$-subset of a part has a common neighbour, so there are distinct vertices $y_1, \dots, y_k$ such that y_i is a common neighbour of $\{x_1, \dots, x_k\} \setminus \{x_i\}$ for $1 \leq i \leq k$. As such, $[x_1, \dots, x_k, y_1, \dots, y_k] \cong \overline{L(K_{2,k})}$, which embeds a 6-cycle when $k \geq 3$, contradicting the fact that all induced cycles in G are squares. It follows that $k = 2$, as desired, so there are two vertices in a part of G without a common neighbour. □

To complete this case, we only need to determine when graphs of type (b) and graphs $\overline{L(K_{2,n})}$ are C-MH-symmetric to each other. The conditions under which two bipartite complements of perfect matchings are C-MH-symmetric was discussed in Lemma 25 of [11]. While the argument was developed in the context of C-HH-symmetry, it applies equally well to C-MH-symmetry.

Lemma 11 (*Lockett [11]*)**.** *For $n_1, n_2 \geq 3$, the graphs $\overline{L(K_{2,n_1})}$ and $\overline{L(K_{2,n_2})}$ are C-MH-symmetric if and only if $n_1 = n_2$.*

Now, we determine which pairs of type (b) graphs are C-MH-symmetric. As it turns out, the next lemma implies that any two type (b) graphs are C-MH-symmetric, while establishing a stronger result that will be useful later.

Lemma 12. *Every connected bipartite graph is C-MH-morphic to any graph of type* (b).

Proof. Let G_1 be a connected bipartite graph, G_2 be a graph of type (b), and ϕ be a monomorphism from a connected induced subgraph of G_1 into G_2. Denote the parts of the bipartition of G_1 by X_1 and Y_1, and the parts of G_2 by X_2 and Y_2. By assumption, there is a vertex $x \in X_2$ serving as a common neighbour for the set Y_2 and a vertex $y \in Y_2$ acting as a common neighbour for the set X_2. Now, without loss of generality, we may suppose that ϕ maps a vertex of X_1 to a vertex of X_2.

Since any homomorphism between bipartite graphs must preserve the bipartition, we may extend ϕ to every vertex in G_1 by mapping every vertex in X_1 for which ϕ is not yet defined to x and every vertex in Y_1 for which ϕ is not yet defined to y. Since x and y are each adjacent to every vertex in the other part of the bipartition, all adjacencies are preserved. □

Every type (b) graph is in particular connected and bipartite, so each type (b) graph is C-MH-morphic to every type (b) graph. It follows that any two graphs of type (b) are C-MH-symmetric.

Now, we must ask when a type (b) graph and a graph $\overline{L(K_{2,n})}$ are C-MH-symmetric. This question has been completely answered by Lockett in her work on C-HH graphs. We will reproduce some of the arguments here for the purpose of clarity. We begin with the following result, which combines portions of Lemmas 26 and 27 in [11].

Lemma 13 (*Lockett [11]*)**.** *Let G_1 be a graph of type (b) and $G_2 \cong \overline{L(K_{2,n})}$, where $n \geq 3$. If $\Delta(G_1) \leq n-1$, then G_1 is C-HH-morphic to G_2. If $\Delta(G_1) \geq n$ and G_1 is PCM(n)-free, then G_1 is C-HH-symmetric to G_2.*

Using this result, we can complete our investigation of possible C-MH-symmetries in the case where the domino graph embeds:

Lemma 14. *If G_1 is a graph of type* (b) *and $G_2 \cong \overline{L(K_{2,n})}$, where $n \geq 3$, then G_1 and G_2 are C-MH-symmetric if and only if G_1 is PCM(n)-free.*

Proof. First, suppose that G_1 is PCM(n)-free. If $\Delta(G_1) \geq n$, then by Lemma 13, G_1 and G_2 are C-HH-symmetric and hence C-MH-symmetric. If $\Delta(G_1) \leq n-1$, then by Lemma 13, G_1 is C-HH-morphic to G_2 and therefore C-MH-morphic to G_2. We already know that G_2 must be C-MH-morphic to G_1 by Lemma 12, so in fact G_1 and G_2 are C-MH-symmetric.

Conversely, suppose that G_1 embeds a PCM(n) graph H. Let the parts of G_2 be X and Y, where $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$. Also, let the parts of H be W and Z, where $W = \{w_1, \ldots, w_m\}$ and $Z = \{z_1, \ldots, z_n\}$ for some m such that $2 \leq m \leq n$, and let $w_i \not\sim z_i$ for $1 \leq i \leq m$. Note that the vertices in Z have a common neighbour because each part of G_1 has a common neighbour. Thus, the monomorphism sending w_i to x_i for $1 \leq i \leq m$ and z_j to y_j for $1 \leq j \leq n$ cannot be extended to a homomorphism from G_1 into G_2, since the set Z with a common neighbour has been mapped onto the set Y without a common neighbour. Therefore, G_1 and G_2 are not C-MH-symmetric. $\square$

4.3. Bipartite case 2: Domino-free and C_6 embeds

We now consider the case where C_6 embeds in some component and the domino graph does not embed in any component. Because each component is domino-free, none of the components are isomorphic to $\overline{L(K_{2,n})}$ for $n \geq 4$.

Since C_6 embeds in some component, Lemma 5 tells us that any two vertices in a part of each component have a common neighbour. By looking at the list of connected bipartite C-MH graphs, the only such graphs not embedding the domino graph where any two vertices in a part have a common neighbour are certain graphs of type (b) and of type (a), together with all bridge graphs, including $C_6 \cong \overline{L(K_{2,3})} \cong B_2$. However, Lemma 10 tells us that every nontrivial graph of type (a) where every two vertices in a part have a common neighbour is also of type (b), so we do not need to handle them separately.

In other words, in the current case, each component must be either a type (b) graph or a bridge graph. Hence, we only need to determine when two graphs from these families are C-MH-symmetric. By Lemma 12, we already know that any two graphs of type (b) are C-MH-symmetric, and Proposition 3 establishes that for $n_1, n_2 \geq 2$, any two bridge graphs B_{n_1} and B_{n_2} are C-MH-symmetric.

Now, we investigate the conditions under which a graph $B_n (n \geq 2)$ and a graph of type (b) are C-MH-symmetric.

Lemma 15. *If G_1 is a graph of type* (b) *and $G_2 \cong B_n$ for some $n \geq 2$, then G_1 and G_2 are C-MH-symmetric if and only if G_1 is P_5-free.*

Proof. By Lemma 12, we know that G_2 will always be C-MH-morphic to G_1, so what we must show is that G_1 is C-MH-morphic to G_2 if and only if G_1 does not embed P_5. First, suppose that G_1 embeds P_5, with vertices $v_1, \ldots, v_5$. Labelling the vertices of G_2 as in Proposition 3, the map $v_1 \mapsto b_{12}$, $v_2 \mapsto b_{11}$, $v_3 \mapsto a_1$, $v_4 \mapsto b_{21}$, $v_5 \mapsto b_{22}$ is a monomorphism from a connected induced subgraph of G_1 into G_2. However, v_1, v_3, and v_5 have a common neighbour, since they all belong to the same part of G_1, while b_{12}, b_{22}, and a_1 do not have a common neighbour, so the map cannot be extended to a homomorphism from G_1 into G_2.

Conversely, suppose that G_1 does not embed P_5 and let ϕ be a monomorphism from a connected induced subgraph of G_1 into G_2. First, suppose that the domain of ϕ is an induced path. Of course, the path contains at most four vertices. The image of each such induced path must be an induced path in G_2, since the only induced cycles in G_2 are 6-cycles. However, since the path is of length at most three, this means the range of ϕ is contained in a bipartite induced subgraph of G_2 where each part has a common neighbour. Therefore, by Lemma 12, the map extends to a homomorphism into this subgraph, and hence into G_2.

Let the parts of G_1 be X and Y. If the domain is not an induced path, then without loss of generality, it must include a vertex $x_1 \in X$ together with neighbours $y_1, y_2, \ldots, y_k$ in Y, where $k \geq 3$, because the domain cannot contain any cycles, which can be seen as follows. An induced even cycle with more than 4 vertices in the domain is impossible, since G_1 is P_5-free. Moreover, a domain containing a 4-cycle is not possible because a 4-cycle maps to a 4-cycle under a monomorphism between bipartite graphs, and G_2 embeds no such cycle. So, if the domain is neither a cycle nor an induced path, it must contain a vertex with at least three neighbours.

Now, without loss of generality, $\phi(x_1) = a_1$ because at least three neighbours of x_1 are in the domain, and $\phi(y_i) = b_{i1}$ for $i \in \{1, \ldots, k\}$. No more than one additional neighbour of any of $y_1, \ldots, y_k$ can be in the domain, because the vertices b_{i1} each have only one neighbour aside from a_1. Also, no more than one of $y_1, \ldots, y_k$ has another neighbour in the domain, for otherwise the domain would embed P_5 or a 4-cycle. The former possibility is forbidden by assumption, and the latter is impossible because G_2 contains no 4-cycle.

Therefore, if the domain contains another neighbour of any of $y_1, y_2, \ldots, y_k$, we may suppose without loss of generality that this neighbour x_2 is adjacent to y_1 in the domain, and maps to b_{12}. Additionally, none of $y_1, \ldots, y_k$ have any other neighbours in the domain. In particular, x_2 is not adjacent to any of $y_2, \ldots, y_k$. It follows that no neighbour of x_2 except y_1 can belong to the domain, because the domain must be P_5-free. Hence, regardless of whether such a vertex x_2 is in the domain or not, the range is a bipartite graph where each part has a common neighbour, with the common neighbours respectively being a_1 and b_{11}, and therefore the map extends to a homomorphism that is onto the range by Lemma 12. □

The next result simplifies the conditions of Lemma 15.

Lemma 16. *A graph of type* (b) *is of type* (a) *if and only if it is P_5-free.*

Proof. Let G be a graph of type (b). If G is not of type (a), then it embeds either the domino graph or an even cycle with more than four vertices, both of which embed P_5. Conversely, if G embeds P_5, then let $v_1v_2v_3v_4v_5$ denote a copy of P_5 in G. Since G is of type (b), v_1, v_3, and v_5 must have a common neighbour v', distinct from v_2 and v_4. It follows that $[v_1, v_2, v_3, v_4, v_5, v']$ is isomorphic to the domino graph, so G is not a graph of type (a). □

Putting the last two results together, we get the following corollary, which gives us the condition used in the final classification result:

Corollary 5. *If G_1 is a graph of type* (b) *and $G_2 \cong B_n$ for some $n \geq 2$, then G_1 and G_2 are C-MH-symmetric if and only if G_1 is also of type* (a).

To conclude, we observe what the above tells us about B_1. The graph B_1 is of type (a) and type (b), so it is C-MH-symmetric to B_n for $n \geq 2$. Additionally, since any two graphs of type (b) are C-MH-symmetric, B_1 is C-MH-symmetric to any graph that is both type (a) and type (b), including itself. Together with Proposition 3, we can now conclude that *any* two bridge graphs are C-MH-symmetric. Also, we may conclude that *every* bridge graph is C-MH-symmetric to every graph that is both type (a) and type (b).

4.4. Bipartite case 3: Domino-free and C_6-free

Lastly, we handle the case where every component of a given bipartite C-MH graph is C_6-free and domino-free. We know from Corollary 1 that for $n \geq 4$, a connected bipartite C-MH graph properly containing an induced $2n$-cycle will embed either C_6 or the domino graph. So, we see that the only connected bipartite C-MH graphs neither embedding C_6 nor the domino graph are graphs of type (a) and cycles C_{2n}, $n \geq 4$. Yet again, we need to determine which pairs of these graphs are C-MH-symmetric. The next two results consider each family by itself. Though we focus only on even cycles with at least eight vertices, the result below also applies to 6-cycles.

Lemma 17. *For $n_1, n_2 \geq 3$, two even cycles C_{2n_1} and C_{2n_2} are C-MH-symmetric if and only if $n_1 = n_2$.*

Proof. Since even cycles are C-MH, two even cycles will obviously be C-MH-symmetric when they have the same size. Conversely, suppose without loss of generality that $n_1 < n_2$. Let $C_{2n_1} = a_1 \cdots a_{2n_1}a_1$ and $C_{2n_2} = b_1 \cdots b_{2n_2}b_1$. The monomorphism that sends a_i to b_i for $1 \leq i \leq 2n_1 - 1$ is defined on a connected domain, but it cannot be extended to a homomorphism of the two graphs. While a_{2n_1} is a common neighbour of a_{2n_1-1} and a_1, there is no common neighbour of b_1 and b_{2n_1-1}, so there is no way to define a homomorphic extension for the vertex a_{2n_1}. □

The next result not only shows that any two nontrivial graphs of type (a) are C-MH-symmetric, but also proves a stronger claim that will be useful when we consider type (a) graphs together with even cycles. The argument in the proof was used in the context of C-HH graphs by Lockett in the proof of Lemma 15 in [11], and we now apply it to our discussion of C-MH-symmetry.

Lemma 18. *Any graph of type* (a) *is C-HH-morphic, and therefore C-MH-morphic, to every nontrivial connected graph.*

Proof. Let G_1 be a graph of type (a), G_2 be an arbitrary nontrivial connected graph and let ϕ be a homomorphism from a connected induced subgraph A of G_1 into G_2. Since G_1 is connected, if A is a proper induced subgraph, we can find a vertex v in $G_1 \backslash A$ with a neighbour in A. We will show how to define the mapping for v such that the extension remains a homomorphism, from which it will follow that we can repeat the argument until the domain is all of G_1. Define A_v as $N(v) \cap A$. Since $N(v)$ is an independent set for any vertex v in a bipartite graph, A_v is independent. If $A_v = \{a\}$, then certainly $\phi(a)$ has a neighbour v' in G_2, because G_2 is connected and nontrivial. By defining $\phi(v) = v'$, adjacencies are preserved.

We may therefore suppose that $|A_v| \geq 2$. First, we show that every pair of vertices in A_v has a common neighbour in A. Consider two vertices a_1 and a_2 in A_v. Since A is connected, there is a shortest path P between a_1 and a_2 in A. Since P is shortest possible, it must be an induced path. Suppose that $P \cap A_v \neq \{a_1, a_2\}$, and consider the first two vertices a_1 and a_1' from A_v in P. That is, all other vertices in the path from a_1 to a_1' belong to $A \backslash A_v$. It follows that the path from a_1 to a_1', together with v, is an induced cycle and therefore a 4-cycle. Hence, the path from a_1 to a_1' is $a_1b_1a_1'$ for some $b_1 \in A \backslash A_v$. Similarly, if a_2' is the next vertex from A_v in P, then the path from a_1' to a_2' is $a_1'b_2a_2'$ for some $b_2 \in A \backslash A_v$. Consequently, $[a_1, b_1, a_1', b_2, a_2', v]$ is isomorphic to the domino graph, a contradiction. Hence $P = a_1b_1a_2$, so b_1 is a common neighbour of a_1 and a_2.

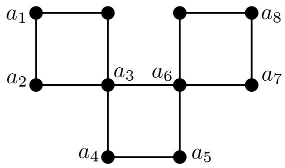

Fig. 7. A graph of type (a) not meeting the conditions of Lemma 19.

We will now show that if $2 \leq k < |A_v|$ and every k-subset of A_v has a common neighbour in A, then so does every $(k+1)$-subset. Suppose to the contrary that some $(k+1)$-subset does not have a common neighbour in A. Certainly, each of the $k+1$ k-subsets of this $(k+1)$-subset have a common neighbour, each of which is necessarily distinct from the rest. Hence, this $(k+1)$-subset and the $k+1$ common neighbours of the k-subsets induce $\overline{L(K_{2,k+1})}$, which embeds C_6 when $k \geq 2$, a contradiction.

The conclusion we can derive is that A_v has a common neighbour $a \in A$, and by setting $\phi(v) = \phi(a)$, all adjacencies are preserved, so ϕ can be extended for v and remain a homomorphism. □

Since every graph of type (a) is connected, the result above implies that any two nontrivial graphs of type (a) are C-MH-symmetric. The above will also aid us in our final lemma, which specifies the exact circumstances in which a graph of type (a) and a cycle C_{2n}, $n \geq 4$, are C-MH-symmetric. Again, the result applies equally well to 6-cycles, so we include them in the result. Importantly, also note that the paths in the lemma do not necessarily refer to *induced* paths. Instead, they refer to any sequence of distinct vertices, each adjacent to its predecessor and successor.

Lemma 19. *For a fixed $n \geq 3$, if G is a nontrivial type* (a) *graph, then G is C-MH-symmetric to C_{2n} if and only if, for all $n+1 < m < 2n$, the endpoints of any path in G with m vertices are at a distance at most $2n - m + 1$ apart.*

Proof. All even cycles are nontrivial connected graphs, so by Lemma 18, G is always C-MH-morphic to C_{2n}. What we must show is that C_{2n} is C-MH-morphic to G if and only if the condition on G holds.

Let $n \geq 3$ be fixed and suppose that there is a path $v_1 v_2 \cdots v_m$ in G where $d(v_1, v_m) > 2n - m + 1$ for some $n+1 < m < 2n$. If $a_1, \ldots, a_{2n}$ are the vertices in C_{2n}, then the monomorphism mapping a_i to v_i for $1 \leq i \leq m$ is defined on a connected induced subgraph of C_{2n}, yet cannot be extended to a homomorphism. It would be necessary for a_{2n} to be mapped to a neighbour of v_1, but the closest neighbours of v_1 to v_m are at distance at least $2n - m + 1$ from v_m. Since the remaining cycle vertices $a_{m+1}, \ldots, a_{2n}$ must map to adjacent vertices and only $2n - m$ vertices remain to be mapped, the image of a_{2n} cannot be at a greater distance than $2n - m$ from v_m, so the map cannot extend.

Conversely, assume that for all $n+1 < m < 2n$, the endpoints of every path in G with m vertices are at a distance at most $2n - m + 1$ apart, and consider an arbitrary monomorphism from a connected induced subgraph of C_{2n} onto a connected induced subgraph of G. The image of such a monomorphism is a path with m vertices, where $1 \leq m \leq 2n$. However, if the path has $2n$ vertices, there is nothing to do, because the map is already defined on all of C_{2n}, so we suppose $m < 2n$. Let $v_1 v_2 \cdots v_m$ denote the image path, and without loss of generality, suppose that $a_i \mapsto v_i$ for $1 \leq i \leq m$.

Next, let $k = d(v_1, v_m)$. If $1 \leq m \leq n+1$, then $k \leq n = 2n - (n+1) + 1 \leq 2n - m + 1$. Otherwise, $n+1 < m < 2n$, so by assumption, $k \leq 2n - m + 1$. Either way, there are $2n - m$ vertices not in the initial domain. If $k \geq 1$, the closest neighbours of v_1 to v_m are at distance $k - 1 \leq 2n - m$ from v_m. Hence, we may map the next $k - 1$ of the vertices $a_{m+1}, \ldots, a_{2n}$ to successive vertices on a shortest path from v_m back to v_1, ending at some neighbour v' of v_1. Note that the extension up to this point is a homomorphism. Since a homomorphism preserves the bipartition, a cycle vertex with even index has been mapped to v' at the end of this stage. If $k = 0$, then we must have $m = 1$, and the initial monomorphism is just the map $a_1 \mapsto v_1$. In this case, we select an arbitrary neighbour v' of v_1 for the next step. Now, for all values of k, if we map all remaining cycle vertices with even index to v' and all remaining vertices with odd index to v_1, we have extended the map to a homomorphism. □

While the condition in the above lemma may be hard to verify in practice, we can at least extract a necessary condition and a sufficient condition from the above that may be easier to check. It is certainly necessary that the distance between any two vertices in G be at most n. Otherwise there is, in particular, a path with $n+2$ vertices where the endpoints are at distance $n+1$ from each other, which prevents the graphs from being C-MH-symmetric. However, this condition is not sufficient. A graph of type (a) failing to meet the conditions of the lemma for $n = 5$ is given in Fig. 7. The graph in the figure has a path $a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8$ with eight vertices for which $d(a_1, a_8) = 5$, larger than the allowable distance of 3, even though the distance between any two vertices is at most 5.

On the other hand, to meet the condition of the lemma, it is sufficient that the longest path between any two vertices in G consist of at most $n+1$ vertices, for then there are no paths with m vertices for $n+1 < m < 2n$ at all, so the condition on G vacuously holds.

4.5. Proof of the characterization

By way of summary, we will give a formal proof of our classification result.

Proof of General C-MH Characterization. We make use of Proposition 2, which tells us that a graph is C-MH if and only if each of its components is C-MH and all components are pairwise C-MH-symmetric.

Using this, we can easily verify that each graph on the list is C-MH. By the connected C-MH characterization, all components of each graph in the list are C-MH. By Proposition 3, Corollary 5, Lemmas 8, 9, 11, 12, 14 and 17–19, and the accompanying remarks, all components of the graphs on the list are pairwise C-MH-symmetric, so every graph on the list is C-MH.

Now, we prove the converse. If G is a C-MH graph and some component of G is isomorphic to K_1, then all components are, because the only connected graph homomorphically equivalent to K_1 is K_1 itself. In this case, every component of G is K_1-treelike, and we have a graph of type (i). We henceforth assume that all components of G are nontrivial. If some component of G is a K_n-treelike graph for a given $n \geq 3$, then by Corollary 4, all components of G must be K_n-treelike graphs, and we have a graph of type (i). For fixed $n \geq 2$, if some component of G is a cycle C_{2n+1}, then by Lemma 9, all components of G must be cycles C_{2n+1}, and the graph is of type (ii). Recall that either every component of G is nonbipartite, or every component of G is bipartite, since nonbipartite graphs are not homomorphically equivalent to bipartite graphs. We have now discussed all possible nonbipartite components, so we may henceforth assume that all components of G are bipartite.

We consider three subcases: (1) some component of G embeds the domino graph, (2) no component of G embeds the domino graph but some component embeds C_6, and (3) all components of G are C_6-free and domino-free.

If some component of G embeds the domino graph, then by Lemma 7, all components of G must satisfy the property that any three vertices in a part have a common neighbour. Checking the list of connected C-MH graphs and using Lemma 10, the only connected bipartite C-MH graphs with this property are graphs where each part has a common neighbour and graphs $\overline{L(K_{2,n})}$ where $n \geq 4$, so these are the only possible components. If one of the components is $\overline{L(K_{2,n})}$, then all other components of G that are bipartite complements of perfect matchings are isomorphic to $\overline{L(K_{2,n})}$ by Lemma 11. Furthermore, any components where each part has a common neighbour must be PCM(n)-free by Lemma 14. Hence, the graph is of type (iv). If none of the components are bipartite complements of perfect matchings, then each part of every component has a common neighbour, and the graph is of type (iii).

Next, we suppose that no component of G embeds the domino graph but some component embeds C_6. All graphs $\overline{L(K_{2,n})}$ embed the domino graph for $n \geq 4$, so such a graph cannot be a component of G. Furthermore, by Lemma 5, each component of G has the property that any two vertices in a part have a common neighbour. This rules out cycles C_{2n} as possible components for $n \geq 4$. Using Lemma 10, a nontrivial domino-free graph where all induced cycles are squares with this property is also a bipartite graph where each part has a common neighbour. Therefore, all components are bridge graphs B_n or bipartite graphs where each part has a common neighbour. If one of the components is B_n for some $n \geq 2$, all components that are bipartite where each part has a common neighbour must also be domino-free graphs where all induced cycles are squares by Corollary 5. Therefore, the graph is of type (v). If none of the components are graphs B_n for $n \geq 2$, then all components must be bipartite where each part has a common neighbour, and G is of type (iii).

Lastly, suppose that every component of G is C_6-free and domino-free. By Corollary 1, the only bipartite C-MH graphs that have this property are cycles C_{2n} for $n \geq 4$, and domino-free graphs where all induced cycles are squares. Hence, all components are one of these two types. If some component is C_{2n} for $n \geq 4$, then all other components that are even cycles but not squares are also copies of C_{2n} by Lemma 17. Furthermore, all components that are domino-free and where all induced cycles are squares must have the additional property that for each $n+1 < m < 2n$, the endpoints of any path in that component with m vertices are at a distance at most $2n-m+1$ apart by Lemma 19. As such, we have a graph of type (vii). Otherwise, all components are nontrivial domino-free graphs where the only induced cycles are squares, so the graph is of type (vi). □

5. Further work

While the results of this paper bring us closer to characterizing all the classes of homomorphism–homogeneous and connected-homomorphism-homogeneous graphs, many of the classes await a full characterization, in both the finite and countable cases. One possible avenue for research is to try and characterize the countable C-HH and C-MH graphs, with the aid of the characterizations of the finite C-HH and C-MH graphs that have now been obtained.

Additionally, it is observed in [11] that IH and C-IH graphs still lack characterizations, even in the finite case. The first step in characterizing these two classes of graphs is to find examples of IH and C-IH graphs that do not belong to any of the other classes. For instance, we seek examples of C-IH graphs that are not C-MH, not IH, and not C-II. The expanded cycles EC($n_1, \ldots, n_k$) from Definition 2 provide such examples. Before proving this, we note what the existing characterizations tell us about certain expanded cycles:

- The graphs EC(n_1) and EC(n_1, n_2) are bipartite graphs where each part has a common neighbour, and hence are C-HH graphs by Theorem 1.
- As observed in Section 3, if $k \geq 2$ and $n_1 = \cdots = n_k = 1$, then EC$(n_1, \ldots, n_k) \cong C_{2k}$, which is C-MH.

Therefore, in these special cases, EC($n_1, \ldots, n_k$) is C-IH. The next proposition will demonstrate that the rest of the family is also C-IH. Furthermore, aside from the special cases above, the proposition shows that the graphs in the family are not C-MH, not C-II, and not IH.

In the proof of the proposition, we will frequently employ a useful fact. Namely, whenever we have an isomorphism ϕ from a connected bipartite induced subgraph H_1 of EC($n_1, \ldots, n_k$), where each part of H_1 has a common neighbour, onto another subgraph H_2 of EC($n_1, \ldots, n_k$), H_2 is also a bipartite induced subgraph where each part has a common neighbour. Since we showed in Lemma 12 that every connected bipartite graph is C-MH-morphic, and hence C-IH-morphic, to any bipartite graph where each part has a common neighbour, the isomorphism ϕ can be extended to a homomorphism from $\mathrm{EC}(n_1, \ldots, n_k)$ into H_2.

Proposition 8. *If $k \geq 3$ and $n_i > 1$ for some $i \in \{1, 2, \ldots, k\}$, then EC $(n_1, \ldots, n_k)$ is C-IH, but not C-MH, not C-II, and not IH.*

Proof. Without loss of generality, we assume $n_1 > 1$ for the first part of this proof. As in Lemma 6, we let $Y_i = \{y_{i1}, \ldots, y_{in_i}\}$ for each i. To see that $\mathrm{EC}(n_1, \ldots, n_k)$ is not C-II, we consider the isomorphism $x_1 \mapsto y_{11}$. Since $n_1 > 1$, the vertex x_1 has at least three neighbours, but y_{11} has only two, so there is no way for all the neighbours of x_1 to map to distinct neighbours of y_{11}. Hence, the map cannot be extended to an isomorphism defined on all of $\mathrm{EC}(n_1, \ldots, n_k)$. The graph $\mathrm{EC}(n_1, \ldots, n_k)$ is not IH, because the isomorphism $x_1 \mapsto x_1, x_2 \mapsto y_{21}$ between induced subgraphs of $\mathrm{EC}(n_1, \ldots, n_k)$ cannot be extended to a homomorphism. The vertex y_{11} would have to map to a common neighbour of x_1 and y_{21}, but such a neighbour does not exist because x_1 and y_{21} belong to different parts of the bipartition.

Now, we show that $\mathrm{EC}(n_1, \ldots, n_k)$ is not C-MH. First, suppose $k \geq 4$. Again assuming that $n_1 > 1$ without loss of generality, we consider the monomorphism that fixes $x_1, x_2, \ldots, x_{k-1}$ and also fixes $y_{11}, y_{21}, y_{31}, \ldots, y_{(k-1)1}$, but maps y_{k1} to y_{12}. This map cannot extend to a homomorphism, because we need x_k to map to a vertex adjacent to both $y_{(k-1)1}$ and y_{12}, which does not exist when $k \geq 4$. When $k = 3$, the map used in the proof of Lemma 6 shows that $\mathrm{EC}(n_1, n_2, n_3)$ is not C-MH, since y_{11}, y_{21}, and y_{31} do not have a common neighbour.

From this point forward, we will consider all possible domains for an isomorphism between connected induced subgraphs and show that each isomorphism can be extended to a homomorphism from the graph into itself. Let ϕ denote an arbitrary isomorphism and $\mathcal{D}$ denote the initial connected domain of ϕ. All subscripts will be taken modulo k for the remaining part of the proof.

First, suppose that $\mathcal{D}$ contains elements either from at most one cell Y_i, in which case $\mathcal{D} \cap X$ may have 0, 1, or 2 elements, or from two cells Y_i and Y_{i+1}, in which case we also suppose $|\mathcal{D} \cap X| = 1$. It follows that $\mathcal{D}$ must, without loss of generality, be one of the following: a single vertex, a vertex x_{i+1} together with vertices from Y_i, vertices x_i and x_{i+1} together with vertices from Y_i, or a vertex x_{i+1} together with vertices in Y_i and Y_{i+1}. All of the nontrivial domains in the list are bipartite where each part has a common neighbour, and if $\mathcal{D}$ is a single vertex, it is contained in an edge, which is also bipartite where each part has a common neighbour. The range must be isomorphic to $\mathcal{D}$, and hence, in all of these cases, is contained in a bipartite induced subgraph of $\mathrm{EC}(n_1, \ldots, n_k)$ where each part has a common neighbour. Since $\mathrm{EC}(n_1, \ldots, n_k)$ must be C-IH-morphic to any such graph by Lemma 12, the map always extends to a homomorphism from $\mathrm{EC}(n_1, \ldots, n_k)$ into itself.

For the remaining part of the proof, we may suppose there are at least two sets Y_i and Y_j with elements in the domain and that $|\mathcal{D} \cap X| > 1$ holds. First, suppose that $|\mathcal{D} \cap Y_i| = 1$ for all Y_i with vertices in $\mathcal{D}$. It follows that either the domain is an induced path and is mapped to another induced path, or the domain is an induced $2k$-cycle. In the former case, the path has at least four vertices in it under the conditions imposed, so in both cases, the domain and range must each be part of an induced $2k$-cycle. All cycles are C-MI, and hence C-II, so the map extends to an isomorphism of the two cycles. The domain and range of this isomorphism each involve every element of X and one element from each Y_i. Without loss of generality, the element of Y_i in the domain is y_{i1}. By extending the map so that $\phi(Y_i) = \{\phi(y_{i1})\}$ for $1 \leq i \leq k$, the extension is a homomorphism defined on the whole graph.

Finally, suppose that $|\mathcal{D} \cap Y_i| > 1$ for some Y_i, so that some member of X in the domain has at least three neighbours in $\mathcal{D}$ and both cells of Y adjacent to this vertex have vertices in $\mathcal{D}$. For this case, it will be convenient to consider ϕ as a map from an induced subgraph of a copy G_1 of $\mathrm{EC}(n_1, \ldots, n_k)$ into an isomorphic copy G_2 of $\mathrm{EC}(n_1, \ldots, n_k)$, rather than a map from $\mathrm{EC}(n_1, \ldots, n_k)$ into itself. We do this so that we may label vertices in G_2 independently of the labelling of G_1. To make the labelling of G_1 and G_2 unambiguous, we let the parts of G_1 be X^1 and Y^1 and the parts of G_2 be X^2 and Y^2, with the cells of Y^i labelled $Y^i_1, \ldots, Y^i_k$ and vertices in the two graphs receiving similar superscripts.

Without loss of generality, suppose that x^1_1 is a vertex in X^1 with three neighbours in $\mathcal{D}$, where both Y^1_1 and Y^1_k have vertices in $\mathcal{D}$. Note that x^1_1 must be mapped to a vertex in X^2, since at least three adjacencies must be preserved. We label the vertices in G_2 such that $\phi(x^1_1) = x^2_1$. An isomorphism preserves the bipartition, so all elements of $X^1 \cap \mathcal{D}$ must map to elements of X^2, and elements of $Y^1 \cap \mathcal{D}$ must map to elements of Y^2. Since at least two elements of X^1 are in the domain, all elements from a given cell Y^1_i in the domain map to the same cell of Y^2, distinct from the images of all other cells, because non-adjacencies must be preserved, as well as adjacencies. Without loss of generality, we label the vertices in G_2 such that $\phi(Y^1_1 \cap \mathcal{D}) \subseteq Y^2_1$. It follows that all vertices of the form x^1_i in $\mathcal{D}$ map to x^2_i and vertices from Y^1_i in $\mathcal{D}$ map into Y^2_i. We extend the map to a homomorphism by sending all vertices of the form x^1_i to x^2_i and sending all vertices in $Y^1 \setminus \mathcal{D}$ belonging to a given cell Y^1_i to y^2_{i1}.

We have now discussed all possible connected domains, so the proof is complete. □

Only special families of IH and C-IH graphs, such as this one, are known at this point. Aside from the expanded cycles, finite graphs that are C-IH, but not IH, not C-MH, and not C-II have yet to be found. Furthermore, apart from a family of IH graphs called "generalized multiclaws" mentioned in [11], there are no known examples of finite graphs that are IH but not II. If new examples are discovered, they could lead to characterizations of these two types of graphs, completing the characterization of all the classes in the finite case.

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