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Classification of regular maps with prime number of faces and the asymptotic behaviour of their reflexible to chiral ratio

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ABSTRACT

In this paper we classify the reflexible and chiral regular oriented maps with p faces of valency n, and then we compute the asymptotic behaviour of the reflexible to chiral ratio of the regular oriented maps with p faces. The limit depends on p and for certain primes p we show that the limit can be 1, greater than 1 and less than 1. In contrast, the reflexible to chiral ratio of regular polyhedra (which are regular maps) with Suzuki automorphism groups, computed by Hubard and Leemans (2014), has produced a nill asymptotic ratio.

1. Introduction

A regular oriented map $\mathcal{M}$ is a triple $(G; a, b)$ consisting of a 2-generated finite permutation group G with two distinguished generators a and b satisfying $(ab)^2 = 1$. The pair $\{|a|, |b|\}$, where $|g|$ stands for the order of g, is the classical type of $\mathcal{M}$. If $\mathcal{M} = (G; a, b)$ is a regular oriented map of type $\{m, n\}$, then its dual $D(\mathcal{M}) = (G; b^{-1}, a^{-1})$ is a regular oriented map of type $\{n, m\}$. An isomorphism $(G; a, b) \to (H; a', b')$ is a group isomorphism $\psi : G \longrightarrow H$ that takes a to a' and b to b'. If a regular oriented map $\mathcal{M} = (G; a, b)$ is isomorphic to its mirror image $\overline{\mathcal{M}} = (G; a^{-1}, b^{-1})$, then $\mathcal{M}$ is *reflexible*, otherwise $\mathcal{M}$ is *chiral*.

The work of Drmota and Nedela [10], albeit not addressing regularity, shows that the reflexible to chiral ratio function $\frac{A(n)}{U(n)}$ determined in [7], of oriented reflexible maps with n edges over oriented chiral maps with n edges, goes to zero as $n \to \infty$. Does this nill asymptotic question extend to any restricted reflexible to chiral ratio? A recent work of Hubard and Leemans [13] on Suzuki groups $Sz(q)$, for q an odd power of 2, shows that $O(g(q)) \sim q.O(f(q))$, that is, the reflexible to chiral ratio $\frac{f(q)}{g(q)}$ (computed up to isomorphism and duality) of regular polyhedra (maps corresponding to regular polytopes of rank 3) with automorphism group $Sz(q)$, goes to zero as $q \to \infty$.

Among other things, we compute in this paper a non-nil asymptotic chiral ratio by restricting the ratio to regular oriented maps with prime number of faces. More specifically, we consider the ratio $RC_p(n) = \frac{T_pRM(n)}{T_pCM(n)}$, where $T_pRM(n)$ is the number of reflexible regular oriented maps with p faces up to pn darts and $T_pCM(n)$ ($p > 3$ and $n \geq p - 1$) is the number of chiral

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regular oriented maps with p faces up to pn darts, and compute its limit when $n \to \infty$. For some classes of primes p we show that the limit can be 1, greater than 1 and less than 1. The main theorem (Theorem 10) states:

Theorem. *For any odd prime $p > 3$, the function $RC_p(n) = \frac{T_pRM(n)}{T_pCM(n)}$, $n \geq p-1$, has limit given by*

$$\lim_{n\to\infty} RC_p(n) = \frac{p-1}{2\,\sigma_p},$$

where $\sigma_p = \sum_{k=2}^{p-1} \sum_{\substack{b|\gcd(\frac{p-1}{2},k)\\ b\geq 2,\ \frac{k}{b}\ odd}} \Phi(2b)$. Here Φ is the Euler totient function.

Du, Kwak and Nedela in [11] classified, and enumerated for each order and degree, the orientable regular embeddings of simple graphs of prime order and in [12] those of order a product of two primes. In other words, they classified the regular oriented simple maps of prime order, and of order a product of two primes. We note that in this paper we deal with regular maps, but not necessarily simple. Regular maps of type $\{m, q\}$ are regular hypermaps of type $(q, 2, m)$. Up to a duality, a primer hypermap is a generalisation of a simple map (map with underlying simple graph). In [2] we have classified the primer hypermaps with a prime number of hyperfaces and in [3] we have extended the classification to the regular oriented hypermaps with a prime number of hyperfaces.

In this paper we derive a classification of the regular oriented maps with p (prime) faces by identifying which of the regular oriented hypermaps with p hyperfaces are maps (the classification of the regular oriented maps with a prime number of vertices is obtained by duality), get an enumeration formula for the regular maps with p faces with fixed valency, count the number of reflexible and of chiral up to given valency and then determine the limit of the reflexible to chiral ratio.

This paper has 3 sections. The first is the actual introduction which includes two subsections, one giving a quick overview of the theory of regular oriented hypermaps and the second summarising the classification of the regular hypermaps with a prime number of hyperfaces by writing down the most important results of [3] that are used in the third section. For a complementary reading on these subjects we address the reader to [14,15,9,8,6,2]. In section two we derive a classification of the regular oriented maps with p (prime) faces by determining those hypermaps that are maps. In section three we compute the asymptotic behaviour of the reflexible to chiral ratio $RC_p(n)$. We show that the limit of $RC_p(n)$ does exist for any prime p and that this limit depends on p.

Functions in this paper are read from right to left.

1.1. Regular oriented maps

As mentioned before, a *regular oriented map* is a triple $\mathcal{M} = (G; a, b)$ consisting of a (permutation) group G, called the *monodromy* group of $\mathcal{M}$, and two generators a and b of G that act on G (the set of *darts*) by right multiplication such that $(ab)^2 = 1$. The *faces*, *vertices* and *edges* of $\mathcal{M}$ are, respectively, the left cosets $g\langle a\rangle$, $g\langle b\rangle$ and $g\langle ab\rangle$. This triple describes an embedding of a graph $\mathcal{G}$ in an oriented surface $\mathcal{S}$ (i.e., an orientable surface with a fixed orientation). Graphs in this paper are multi-graphs, that is, they may have multiple edges, loops and free-edges. The *darts* of $\mathcal{M}$ are the half-edges[1] of $\mathcal{G}$. The permutations a and b locally permute the darts counter clockwise (CCW) around faces and vertices respectively (actually it is more common in the literature to see a and b as permutations of darts CCW around vertices and edges instead). The *type* of $\mathcal{M}$ is the triple $(k, 2, n)$, the classical notation being $\{n, k\}$, where the positive integers k, 2 and n are respectively the vertex-, edge- and face- valencies. An extended version of the type is the *M-sequence* $[k, 2, n;\ V, E, F;\ |G|]$ where $(k, 2, n)$ is the type, V, E and F are respectively the number of vertices, edges and faces, and $|G|$ is the size of G (or the number of darts of $\mathcal{M}$). The Euler characteristic of the underlying surface $\mathcal{S}$ is the *characteristic of* $\mathcal{M}$, and it is given by the formula $\chi = V + E + F - |G|$.

If $\mathcal{M} = (G; a, b)$ and $\mathcal{M}' = (G'; a', b')$ are two regular oriented maps, then $\mathcal{M}$ *covers* $\mathcal{M}'$ if the assignment $a \mapsto a'$, $b \mapsto b'$ can be extended to a (*canonical*) epimorphism of monodromy groups $G \to G'$. The map $\mathcal{M}$ is *isomorphic* to $\mathcal{M}'$, $\mathcal{M} \cong \mathcal{M}'$, if the canonical epimorphism $G \mapsto G'$ is an isomorphism. A map is *reflexible* if it is isomorphic to its mirror image $\overline{\mathcal{M}} = (G; a^{-1}, b^{-1})$, otherwise it is *chiral*. The *chirality group* of $\mathcal{M}$ is the smallest normal subgroup $X(\mathcal{M})$ of G such that $\mathcal{M}/X(\mathcal{M})$ is reflexible. This group ranges from $X(\mathcal{M}) = 1$ when $\mathcal{M}$ is reflexible, to $X(\mathcal{M}) = Mon(\mathcal{M})$ when $\mathcal{M}$ is totally chiral [6,5]. The Chirality index of $\mathcal{M}$ is the size $\kappa = \kappa(\mathcal{M}) = |X(\mathcal{M})|$.

Let Δ denote the free product $C_2{*}C_2{*}C_2$ generated by r_0, r_1 and r_2, and Γ be the normal subgroup of index 2 in Δ generated by $a = r_0r_1$ and $b = r_1r_2$, a free group of rank 2. Any regular oriented map $\mathcal{M}$ corresponds to a unique normal subgroup M in Γ, called the *fundamental map subgroup* (or just *map subgroup*), such that $\mathcal{M} \cong (\Gamma/M; Ma, Mb)$. In this context, the chirality group of $\mathcal{M}$ is given by $X(\mathcal{M}) = M\overline{M}/M$, where $\overline{M} = M^{r_1}$. If $\langle a, b\colon R(a, b)\rangle$ is a presentation of the monodromy group G, where $R(a, b)$ denotes a set of relators on a and b, then the chirality group of $\mathcal{M}$ is $X(\mathcal{M}) = \langle R(a^{-1}, b^{-1})\rangle^G$, the normal closure in G of the subgroup generated by $R(a^{-1}, b^{-1})$ [1].

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[1] Each edge, seen as a triple $\{u, m_{u,v}, v\}$ composed of two "black" vertices u and v (vertices of the maps) and a middle "white" vertex $m_{u,v}$, gives rise to two half-edges $\{u, m_{u,v}\}$ and $\{m_{u,v}, v\}$.

Relaxing the condition $(ab)^2 = 1$ in the above definition, we end up with the definition of regular oriented *hypermap*. Everything we said about maps applies equally to hypermaps. The *type* of a hypermap is now a triple (k, m, n) where m is not necessarily equal to 2. M-sequences give rise to H-sequences which are 7-tuples $[k, m, n; V, E, F; |G|]$ with m not necessarily equal to 2.

A regular oriented hypermap $\mathcal{H} = (G; a, b)$ is *(face-)canonical metacyclic* if $\langle a\rangle$ is normal in G and factors G into a cyclic group; this means that a and b are the canonical generators of the metacyclic group $M(n, r, s, t) = \langle a, b\colon a^n = 1,\ b^r = a^s,\ bab^{-1} = a^t\rangle$ where the parameters n, r, s, t satisfy the metacyclic conditions $(t-1)s = 0 \mod n$, $t^r = 1 \mod n$. Similarly, we say that $(G; a, b)$ is *vertex-canonical* if $\langle b\rangle$ is normal in G and $G/\langle b\rangle$ is a cyclic quotient. In this case $G = \langle a, b\colon b^n = 1,\ a^m = b^s,\ aba^{-1} = b^t\rangle$ where $(t-1)s = 0 \mod n$ and $t^m = 1 \mod n$. Both face- and vertex- canonical metacyclic hypermaps have cyclic chirality groups with chirality index $\frac{n}{\gcd(n,\, t^2-1)}$; while the chirality group of a face-canonical hypermap is the cyclic group generated by a^{t^2-1}, the chirality group of a vertex-canonical hypermap is generated by b^{t^2-1} [8]. Therefore a (face- or vertex-) canonical metacyclic hypermap is chiral if and only if $t^2 \neq 1 \mod n$.

The regular oriented hypermaps with 1 and 2 hyperfaces are all reflexible and the chiral hypermaps with 3 and 4 hyperfaces are all canonical metacyclic; in the particular case of 3 and 4 hyperfaces, $r = F$ (F is the number of hyperfaces) and the parameters satisfy the additional conditions $n \geq 13 - 2F$ and $t^{F-2} \neq 1 \mod n$. There are no chiral maps up to 4 faces [8] and with 5 faces all chiral maps have chirality index 5 [4].

1.2. Regular hypermaps with prime number of hyperfaces

In this section we summarise the main results of [3] that are relevant to this paper. The classification of regular oriented hypermaps with p prime hyperfaces is given in the following theorem, where

$$M(n, p, u, t) = \langle a, b\colon a^n = 1,\ b^p = a^u,\ b^{-1}ab = a^t\rangle$$

is the metacyclic group with parameters n, p, u, t, and

$$G^{p,\ell,t}_{n,u,v} = \langle a, b\colon a^n = 1,\ b^p = a^u,\ [a^\ell, b] = 1, bab^{-t} = a^v\rangle.$$

Proposition 1 (*[3, Theor. 6]*). *Let p be a prime number. If $\mathcal{H} = (G; a, b)$ is a regular oriented hypermap with p hyperfaces, each of valency n, then $\mathcal{H}$ is isomorphic to one of the following hypermaps:*

(1) $\mathcal{CM}_{n,p,u,t} = (M(n, p, u, t);\ a,\ b)$, *for some , $u,\ t \in \{0, 1, \ldots, n-1\}$ such that*

$$(t-1)u = 0 \mod n \quad \text{and} \quad t^p = 1 \mod n;$$

(2) $\mathcal{H}^{p,\ell,t,k}_{n,u,v} = (G^{p,\ell,t}_{n,u,v};\ a,\ ba^k)$ *(p odd prime), for some $\ell \in \{2, \ldots, n\}$, $u, v \in \{0, \ldots, n-1\}$, $k \in \{0, \ldots, \ell-1\}$ and $t \in \{2, \ldots, p-1\}$ such that*
- (H1) $\gcd(p-1, n) = 0 \mod \ell$,
- (H2) $t^\ell = 1 \mod p$ *and* $t^i \neq 1 \mod p$ *for* $i \in \{1, 2, \ldots, \ell-1\}$ *(that is, t has order ℓ in $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus \{0\}$),*
- (H3) $u = 0 \mod \ell,\ v = 1 \mod \ell$ *and*
- (H4) $(t-1)u + p(v-1) = 0 \mod n$.

Moreover, all these hypermaps $\mathcal{H}^{p,\ell,t,k}_{n,u,v}$ for ℓ, t, k, n, u, v satisfying the above conditions, have p hyperfaces of valency n, and different parameters (ℓ, t, k, u, v) correspond to non-isomorphic hypermaps with p hyperfaces of valency n.

Corollary 2 (*[3, Cor. 7]*). *$G^{p,\ell,t}_{n,u,v}$ is a metacyclic group isomorphic to $G^{p,\ell,t}_{n,0,1} = M(p, n, 0, t) = \langle\beta, \alpha\colon \beta^p = 1,\ \alpha^n = 1,\ \alpha^{-1}\beta\alpha = \beta^t\rangle$ under the isomorphism $\psi : a \mapsto \alpha, b \mapsto \beta\alpha^\theta$, where $\theta = c(1-v) + du$, for some c, d satisfying $c(t-1) + dp = 1 = \gcd(t-1, p)$. Moreover, $\mathcal{H}^{p,\ell,t,k}_{n,u,v} \cong R_{\theta+k}(\mathcal{H}^{p,\ell,t}_n)$, where $\mathcal{H}^{p,\ell,t}_n$ is the canonical metacyclic hypermap $(G^{p,\ell,t}_{n,0,1}; \alpha, \beta)$.*

The following propositions give the chirality groups and the chirality index of these hypermaps.

Proposition 3 (*[3, Theor. 9]*). *The chirality groups of $\mathcal{CM}_{n,p,u,t}$ and $\mathcal{H}^{p,\ell,t,k}_{n,u,v}$ are the cyclic groups $\langle a^{t^2-1}\rangle$ and $\langle b^{t^2-1}\rangle$ respectively. The chirality index of $\mathcal{CM}_{n,p,u,t}$ is $\frac{n}{(n,t^2-1)}$ while the chirality index of $\mathcal{H}^{p,\ell,t,k}_{n,u,v}$ is*

$$\frac{p}{\gcd(p, t^2-1)} = \begin{cases} 1, & t = -1 \mod p \\ p, & t \in \{2, \ldots, p-2\}. \end{cases}$$

2. Regular maps with prime number of faces

In this section we identify the regular oriented hypermaps $\mathcal{H} = (G; a, b)$ with prime number of hyperfaces that are maps, and enumerate them for fixed prime p and valency n. For it we need to find those hypermaps that satisfy $|ab| = 2$.

The primer map of a map

Let $\mathcal{M} = (G; a, b)$ be a regular oriented map. The monodromy elements a and b acting on the left induce automorphisms ϕ_a and ϕ_b. The primer map of $\mathcal{M}$ is the map $\mathcal{P}(\mathcal{M}) = (P; A, B)$, where $P = \langle A, B\rangle$ and $A = {\pi_a}^{-1}, B = {\pi_b}^{-1}$, where π_a and π_b are the permutations induced by the action of the automorphism ϕ_a and ϕ_b on the faces of $\mathcal{M}$. Being covered by $\mathcal{M}$ the primer map $\mathcal{P}(\mathcal{M})$ is of course a map, though it may be degenerated, that is $AB = 1$; and this happens if and only if $A = B = 1$, that is, if and only if the map $\mathcal{M}$ has one face. So if $\mathcal{M}$ is a regular oriented map with a prime number of faces, then its primer map $\mathcal{P} = \mathcal{P}(\mathcal{M})$ is necessarily non-degenerated and has the same number of faces. According to the Classification Theorem 16 and Corollary 17 of [2], $\mathcal{P}$ is a primer map with a prime number p of faces (of valency ℓ) if and only if (1) $p = 2$ and $\mathcal{P}$ is the spherical map $\mathcal{P}_0^{2,1,1}$ ($k = 0, \ell = 1, t = 1$), or (2) $p > 2$ and $\mathcal{P} = \mathcal{P}_k^{p,\ell,t} = (P; y, yx^k)$, with $k = \frac{\ell}{2} - 1$ and ℓ even, where $P = M(p, \ell, 0, t) = \langle x, y \colon x^p = 1, y^\ell = 1, x^y = x^t\rangle$ and the parameter $t \in \{1, 2, \dots, p-1\}$ satisfies $|t| = \ell$ in the multiplicative group $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus \{0\}$. According to Corollary 17 of [2], the H-sequence (H-seq) of $\mathcal{P}$ when $p > 2$ is one of the following:

(II) If $k = 0$ ($\Rightarrow \ell = 2$), then H-seq($\mathcal{P}$) $= [p, 2, 2\,;\ 2, p, p\,;\ 2p]$;
(IV) If $0 < k < \ell - 1$ ($\Rightarrow \ell \geq 4$), then

$$\text{H-seq}(\mathcal{P}) = \begin{cases} [\ell, 2, \ell\,;\ p, p\frac{\ell}{2}, p\,;\ \ell p], & \text{if } \ell = 0 \mod 4,\\ [\frac{\ell}{2}, 2, \ell\,;\ 2p, p\frac{\ell}{2}, p\,;\ \ell p], & \text{if } \ell = 2 \mod 4.\end{cases}$$

As before, let $\mathcal{P}_{II}^p = \{\mathcal{P}_0^{p,2,t}\}_t$ and $\mathcal{P}_{IV}^p = \{\mathcal{P}_k^{p,\ell,t}\}_{\ell,t}$, where $0 < k < \ell - 1$, be the families of p-primer maps with H-sequences (II) and (IV) respectively.

The classification

Theorem 4. *If $\mathcal{M} = (G; a, b)$ is a regular oriented map with p (prime) faces, of valency n, then $\mathcal{M}$ is isomorphic to one of the following maps:*

(1) $\mathcal{CM}_{n,t} = (M(n, 2, -(t+1), t);\ a,\ b)$,
a map with $p = 2$ faces, for some $t \in \{1, \dots, n-1\}$ such that $t^2 = 1 \mod n$. These maps are all reflexible.

(2i) $\mathcal{M}_{n,u,n-u-1}^{p,2,p-1,0} = (G_{n,u,n-u-1}^{p,2,p-1};\ a,\ b)$,
(p odd prime, and $n = 2 \mod 4$), where $u = p^{\frac{n-2}{2}} \mod n$.
$\mathcal{M}_{n,u,n-u-1}^{p,2,p-1,0}$ is reflexible and its primer map $\mathcal{P} \in \mathcal{P}_{II}^p$.

(2ii) $\mathcal{M}_{n,u,v}^{p,\ell,t,k} = (G_{n,u,v}^{p,\ell,t};\ a,\ ba^k)$,
(p odd prime > 3, and n even), $k = \frac{\ell}{2} - 1 > 0$, for some even $\ell \in \{4, \dots, n\}$, $u, v \in \{0, \dots, n-1\}$, and $t \in \{2, \dots, p-1\}$, such that
(M1) *$\gcd(p-1, n) = 0 \mod \ell$ and $\frac{n}{\ell} = 1 \mod 2$,*
(M2) *$t^\ell = 1 \mod p$ and $t^i \neq 1 \mod p$ for $i \in \{1, 2, \dots, \ell - 1\}$,*
(that is, t has order ℓ in $\mathbb{Z}_p^* = \mathbb{Z} \setminus \{0\}$),
(M3) *$u = 0 \mod \ell$, $v = 1 \mod \ell$ and*
(M4) *$(1-t)u = p(v-1) \mod n$.*
$\mathcal{M}_{n,u,v}^{p,\ell,t,k}$ is chiral, with chirality index p, and its primer map $\mathcal{P} \in \mathcal{P}_{IV}^p$.

Moreover, all these maps $\mathcal{M}_{n,u,v}^{p,\ell,t,k}$ with ℓ, t, k, n, u, v satisfying the above conditions, have p hyperfaces of valency n, and different parameters (ℓ, t, k, u, v) correspond to non-isomorphic maps with p faces of valency n.

Furthermore, denoting by $NM_{(j)}(p, n)$ the number of regular oriented maps with p faces of valency n in each item (j), $j = 1$, 2i and 2ii, we have:

- $NM_{(1)}(p, n) = \begin{cases} 0, & \text{if } p > 2,\\ |U_2(n)|, & \text{if } p = 2,\end{cases}$
  where $U_2(n)$ is the subgroup of the units of $\mathbb{Z}_n$ whose elements t satisfy $t^2 = 1 \mod n$, that is, $U_2(n)$ is the set of square roots of unity modulo n. In this case, writing $n = 2^{e_0} p_1^{e_1} \dots p_k^{e_k}$, where $e_i \geq 0$ and the p_i's are distinct odd primes dividing n, then

$$NM_{(1)}(2, n) = 2^{\lambda(e_0)+k} = \begin{cases} 2^k, & \text{if } e_0 = 0, 1;\\ 2^{1+k}, & \text{if } e_0 = 2;\\ 2^{2+k}, & \text{if } e_0 > 2;\end{cases}$$

  where $\lambda(0) = \lambda(1) = 0$, $\lambda(2) = 1$, $\lambda(e) = 2$, for $e > 2$.
- $NM_{(2i)}(p, n) = \begin{cases} 0, & \text{if } n \neq 2 \mod 4,\\ 1, & \text{if } n = 2 \mod 4.\end{cases}$

- *$NM_{(2ii)}(p,n)=0$, if n odd, and $NM_{(2ii)}(p,n)=\sum_{\substack{\ell|\gcd(p-1,n)\\ \ell\geq 4,\ \ell\text{ even}\\ \frac{n}{\ell}\text{ odd}}}\Phi(\ell)$, if n even.*

Proof. (1) If $\mathcal{M}=(M(n,2,u,t);\ a,\ b)$ is canonical metacyclic map (case 1), since $p\geq 2$, $2=|ab|=\frac{pn}{\gcd(n,t^{p-1}+\cdots+1+u)}$ implies that

$$n\leq\frac{pn}{2}=\gcd(n,t^{p-1}+\cdots+1+u)\leq n,$$

so $p=2$. This implies that $t+1+u=0\mod n$. Conversely, if $p=2$ and $t+1+u=0\mod n$, then $\mathcal{M}=(M(n,2,u,t);\ a,\ b)$ is a map. Thus u is a function of t. By the metacyclic condition $t^2=1\mod n$ and Proposition 3, $\mathcal{M}$ has trivial chirality group, so $\mathcal{M}$ is reflexible.

As u is determined by t,

$$NM_{(1)}(p,n)=0,\text{ if }p>2,\quad\text{ and }\quad NM_{(1)}(2,n)=|U_2(n)|.$$

Let $\tau(n)=|U_2(n)|$. By the Chinese Remainder Theorem this function is multiplicative: $\tau(nm)=\tau(n)\tau(m)$ for any positive integers n,m such that $\gcd(n,m)=1$. Having in account that $\gcd(t-1,t+1)$ is 1 if t is even and 2 if t is odd, we have $\tau(p^e)=2$ if p is an odd prime, and $\tau(2^e)=1$, if $e=1$, $\tau(2^e)=2$, if $e=2$, and $\tau(2^e)=4$, if $e>2$. Combining and making the convention $\tau(1)=1$ and writing $n=2^{e_0}p_1^{e_1}\ldots p_k^{e_k}$, where $e_i\geq 0$ and p_i's are the k distinct odd primes dividing n, then we get the well known formula

$$NM_{(1)}(2,n)=\tau(n)=\tau(2^{e_0})\tau(p_1^{e_1})\ldots\tau(p_k^{e_k})=2^{\lambda(e_0)}2^{\delta(e_1)+\cdots+\delta(e_k)}=2^{\lambda(e_0)+k},$$

where $\lambda(0)=\lambda(1)=0$, $\lambda(2)=1$, $\lambda(e)=2$, for $e>2$, and $\delta(0)=0$ and $\delta(e)=1$, for $e>0$.

(2) If $\mathcal{M}=\mathcal{M}^{p,\ell,t,k}_{n,u,v}=(G^{p,\ell,t}_{n,u,v};\ a,\ ba^k)$, as $G=G^{p,\ell,t}_{n,u,v}$ has order pn and p is odd, then $|ab|=2$ implies n even. We now distinguish two cases according as $\mathcal{P}\in\mathcal{P}^p_{II}$ or $\mathcal{P}^p_{IV}$.

(2i) $\mathcal{P}\in\mathcal{P}^p_{II}$. Then $\ell=2$ and $k=\frac{\ell}{2}-1=0$. Since $t^2=1\mod p$ and $t\neq 1\mod p$, $t=-1\mod p$, so $t=p-1$ and $P=G/\langle a^2\rangle=D_p$ is a dihedral group of order $2p$. Since $bab^{-t}=a^v\Leftrightarrow bab^{1-p}=a^v\Leftrightarrow bab=a^vb^p\Leftrightarrow bab=a^{u+v}$,

$$G=\langle a,b\colon a^n=1,b^p=a^u,[a^2,b]=1,bab=a^{u+v}\rangle$$

and $\mathcal{M}$ is a map if and only if $(ab)^2=1\Leftrightarrow a^{u+v+1}=1\Leftrightarrow u+v+1=0\mod n$. Note that $u+v\in\{0,1,\ldots,n-1\}$. If $\mathcal{M}$ is a map, then $n=2\mod 4$; in fact, replacing $t=p-1$ in Eq. (H4) of part (2) of Proposition 1 we get

$$(2-p)u=p(v-1)\mod n.$$

Then

$$\begin{aligned}u+v+1=0\mod n&\Leftrightarrow u+v-1+2=0\mod n\\&\Rightarrow pu+p(v-1)+2p=0\mod n\\&\Leftrightarrow pu+(2-p)u+2p=0\mod n\\&\Rightarrow u+p=0\mod\tfrac{n}{2}.\end{aligned}$$

As $u+p$ is odd, then $n=2\mod 4$. Since $t^2=1\mod p$, by Proposition 3, $\mathcal{M}$ has chirality index 1, that is, $\mathcal{M}$ is reflexible. Let $\mathcal{M}^{p,n}_{II}$ be the family of regular maps $\mathcal{M}$ with p faces of valency n such that its primer $\mathcal{P}(\mathcal{M})\in\mathcal{P}^p_{II}$. Since n and $p-1$ are both even, and $t=-1\mod p$, the conditions $(H1)$ and (H2) are satisfied. Condition (H3) is equivalent to u even and v odd. Now as $u+v=-1\mod n$ and $u+v<n$, then $u+v=n-1$ and this implies that $v=n-1-u$, which is odd if u is even. Condition (H4) translates to

$$\begin{aligned}(t-1)u+p(v-1)=0\mod n&\Leftrightarrow(p-2)u+p(v-1)=0\mod n\\&\Leftrightarrow p(u+v)-p=2u\mod n\\&\Leftrightarrow p(n-1)-p=2u\mod n\\&\Leftrightarrow u=p\tfrac{n-2}{2}\mod n\quad(\text{since }\tfrac{n}{2}\text{ is odd})\end{aligned}$$

and this determines uniquely $u\in\{0,1,\ldots,n-1\}$. Hence for each odd prime p and each $n=2\mod 4$, there is a unique regular map with p faces of valency n, that is,

$$NM_{(2i)}(p,n)=|\mathcal{M}^{p,n}_{II}|=1.$$

(2ii) $\mathcal{P}\in\mathcal{P}^p_{IV}$. Then $\ell\geq 4$, ℓ is even and $k=\frac{\ell}{2}-1>0$. Now $\mathcal{M}=(G;a,ba^k)$, where $G=G^{p,\ell,t}_{n,u,v}=\langle a,b\colon a^n=1,\ b^p=a^u,\ [a^\ell,b]=1,bab^{-t}=a^v\rangle$, which is a group of order $|G|=pn$. Then $\mathcal{M}$ is a map if and only if $|b\,a^{k+1}|=2\Leftrightarrow|b\,a^{\frac{\ell}{2}}|=2$. Since ℓ divides $p-1$, we must have $p>3$. Consider the isomorphism $\psi: a\mapsto\alpha, b\mapsto\beta\alpha^\theta$, of Corollary 2, where $\theta=c(1-v)+du$ and c,d are integers satisfying $c(t-1)+dp=1$. This isomorphism maps $G=G^{p,\ell,t}_{n,u,v}$ to

$G^{p,\ell,t}_{n,0,1} = M(p,n,0,t) = \langle \beta,\alpha\colon \beta^p = 1,\ \alpha^n = 1,\ \alpha^{-1}\beta\alpha = \beta^t\rangle$. The image by ψ of $b\,a^{\frac{\ell}{2}}$ is $\beta\,\alpha^{\theta+\frac{\ell}{2}}$. So $\mathcal{M}$ is a map if and only if

$$\begin{aligned}
\beta\,\alpha^{\theta+\frac{\ell}{2}}\,\beta\,\alpha^{\theta+\frac{\ell}{2}} = 1 &\Leftrightarrow \beta\,\alpha^{2\theta+\ell}\,\alpha^{-\theta-\frac{\ell}{2}}\,\beta\,\alpha^{\theta+\frac{\ell}{2}} = 1\\
&\Leftrightarrow \beta\,\alpha^{2\theta+\ell}\,\beta^{t^{\theta+\frac{\ell}{2}}} = 1\\
&\Leftrightarrow \alpha^{2\theta+\ell} = \beta^{-(t^{\theta+\frac{\ell}{2}}+1)} \in \langle\alpha\rangle\cap\langle\beta\rangle = 1\\
&\Leftrightarrow 2\theta+\ell = 0 \mod n \quad\wedge\quad t^{\frac{\ell}{2}}+1 = 0 \mod p;
\end{aligned}$$

note that $\theta = 0 \mod \ell$, so $t^\theta = 1 \mod p$. Now since t has order ℓ in the cyclic group $\mathbb{Z}_p^*$, $t^{\frac{\ell}{2}} \neq 1 \mod p$ and so,

$$t^{\frac{\ell}{2}}+1 = 0 \mod p \;\Leftrightarrow\; (t^{\frac{\ell}{2}}-1)(t^{\frac{\ell}{2}}+1) = 0 \mod p \;\Leftrightarrow\; t^\ell - 1 = 0 \mod p,$$

that is, the condition $t^{\frac{\ell}{2}}+1 = 0 \mod p$ is redundant. Thus, $\mathcal{M}$ is a map if and only if

$$2\theta+\ell = 0 \mod n. \tag{1}$$

By manipulating the four equations $\theta p = u$, $\theta(1-t) = v-1$, $\theta = c(1-v)+du$ and $c(t-1)+dp = 1$, Eq. (1) is equivalent to the following pair of equations

$$\begin{cases} 2u + p\ell = 0 \mod n\\ 2(v-1) + \ell(1-t) = 0 \mod n.\end{cases} \tag{2}$$

In fact, multiplying (1) by p and using $\theta p = u$ we get the first equation of (2), and multiplying (1) by $(1-t)$ and using $\theta(1-t) = v-1$ we get the second equation of (2). Conversely, multiplying the first equation of (2) by d and the second by c and subtracting we get:

$$\begin{aligned}
&2du + dp\ell \;-\; (2c(v-1) + c\ell(1-t)) = 0 \mod n\\
\Leftrightarrow\;& 2(du + c(1-v)) + \ell(dp + c(t-1)) = 0 \mod n\\
\Leftrightarrow\;& 2\theta + \ell = 0 \mod n.
\end{aligned}$$

Since t has order ℓ in $\mathbb{Z}_p^*$ and $\ell \geq 4$, $t^2 \neq 1 \mod p$, and so, by Proposition 3, $\mathcal{H}$ is chiral with chirality index p.

Let $\mathcal{M}^{p,n}_{IV}$ be the set of regular maps $\mathcal{M}$ with p faces of valency n such that its primer $\mathcal{P}(\mathcal{M}) \in \mathcal{P}^p_{IV}$. Let $\vartheta(u,v)$ be the number of pairs (u,v) such that $u, v \in \{0,1,\ldots,n-1\}$, u and $v-1$ are multiples of ℓ (condition (M3)), and u, v satisfy the system of two equations (2) and the condition $(M4)$. Then

$$NM_{(2ii)}(p,n) = |\mathcal{M}^{p,n}_{IV}| = \sum_{\substack{\ell|\gcd(p-1,n)\\ \ell\geq 4,\ \ell\text{ even}}}\ \sum_{t\in G_\ell}\sum_k \vartheta(u,v) = \sum_{\substack{\ell|\gcd(p-1,n)\\ \ell\geq 4,\ \ell\text{ even}}}\ \sum_{t\in G_\ell}\vartheta(u,v),$$

since k is uniquely determined. We recall that G_ℓ is the set of elements of order ℓ in the cyclic group $\mathbb{Z}_p{}^* = C_{p-1}$.

Computing the solutions u that are multiples of ℓ of the first equation of (2). Let $u = \mu\ell$. Then

$$2u = -p\ell \mod n \;\Leftrightarrow\; 2\mu = -p \mod \tfrac{n}{\ell}.$$

This has solutions if and only if $\gcd(2, \frac{n}{\ell}) = 1$, that is, if and only if $\frac{n}{\ell}$ is odd. The number of solutions that are multiples of ℓ is then 1; the solution is $u = \mu\ell$ where $\mu = -p\frac{1+\frac{n}{\ell}}{2} \mod \frac{n}{\ell}$.

Analogously, the second equation of (2) also has only one solution $v-1$ which is a multiple of ℓ. The solution is $v = 1+\gamma\ell$, where $\gamma = (t-1)\frac{1+\frac{n}{\ell}}{2} \mod \frac{n}{\ell}$.

One easily sees that the solution pair (u,v), just found, also satisfies $(M4)$. Hence,

$$NM_{(2ii)}(p,n) = \sum_{\substack{\ell|\gcd(p-1,n)\\ \ell\geq 4,\ \ell\text{ even}}}\ \sum_{t\in G_\ell}\vartheta(u,v) = \sum_{\substack{\ell|\gcd(p-1,n)\\ \ell\geq 4,\ \ell\text{ even}\\ \frac{n}{\ell}\text{ odd}}}\ \sum_{t\in G_\ell} 1 = \sum_{\substack{\ell|\gcd(p-1,n)\\ \ell\geq 4,\ \ell\text{ even}\\ \frac{n}{\ell}\text{ odd}}} \Phi(\ell). \quad\square$$

Corollary 5. *Regular oriented maps with* 2 *or* 3 *faces are reflexible.*

Corollary 6. *Denoting by $NM(p,n)$ the number of regular oriented maps with p (prime) faces of valency n, then for odd prime p we have:*

$$NM(p,n) = \begin{cases} 0, & \text{if } n \text{ odd},\\ NM_{(2ii)}(p,n), & \text{if } n = 0 \mod 4,\\ 1 + NM_{(2ii)}(p,n), & \text{if } n = 2 \mod 4.\end{cases}$$

This corollary says that for primes $p > 2$, there are no regular oriented maps with p faces of odd valency.

Corollary 7. *Regular oriented maps with an odd prime number* $p > 3$ *of faces of valency* $n \neq 2 \mod 4$, *are chiral with chirality index* p.

Theorem 11 of [3] gives the H-sequences of the regular oriented hypermaps with p (prime) hyperfaces. Now we adapt the H-sequences for the regular oriented maps with p faces.

Theorem 8. *Let* $\mathcal{M} = (G; a, b)$ *be a regular oriented map with* p *(prime) faces, of valency* n*. Then:*

(1) *If* $\mathcal{M}$ *is* $\mathcal{CM}_{n,t}$, *then* $p = 2$, $u = n-(t+1)$ *and* $t \in \{1, \ldots, n-1\}$ *such that* $t^2 = 1 \mod n$. *In this case* $\mathcal{M}$ *has M-sequence:*

$$M\text{-}seq(\mathcal{M}) = \left[\frac{2n}{(n,u)}\ ,\ 2\ ,\ n\ ;\ (n,u)\ ,\ n\ ,\ 2\ ;\ 2n\right].$$

(2i) *If* $\mathcal{M}$ *is* $\mathcal{M}^{p,2,p-1,0}_{n,u,n-u-1}$, *then* p *is odd,* $k = 0$, $t = p-1$, $u + v + 1 = 0 \mod n$ *and* $n = 2 \mod 4$ *and* $u = p^{\frac{n-2}{2}} \mod n$. *Then*

$$M\text{-}seq(\mathcal{M}) = \left[\frac{pn}{(n,u)}\ ,\ 2\ ,\ n\ ;\ (n,u)\ ,\ \frac{pn}{2}\ ,\ p\ ;\ pn\right],$$

where $(n,u) = 2$ *if* $p \nmid \frac{n}{2}$ *and* $(n,u) = 2p$ *if* $p \mid \frac{n}{2}$.

(2ii) *If* $\mathcal{M}$ *is* $\mathcal{M}^{p,\ell,t,k}_{n,u,v} = (G^{p,\ell,t}_{n,u,v};\ a,\ ba^k)$, *then*

$$M\text{-}seq(\mathcal{M}) = \left[\frac{n}{(n,\theta+k)}, 2, n; p(n,\theta+k), \frac{pn}{2}, p; pn\right],$$

where $\theta = c(1-v) + du$ *and* c, d *are integers satisfying* $c(t-1) + dp = 1$.

3. Asymptotic behaviour of the reflexible–chiral ratio

Let $T_pRM(n)$ and $T_pCM(n)$ be, respectively, the total number of reflexible and chiral regular oriented maps with p faces up to pn darts:

$$T_pRM(n) = \sum_{m=2}^{n} NM_{(2i)}(p,m),$$

$$T_pCM(n) = \sum_{m=4}^{n} NM_{(2ii)}(p,m).$$

Notice that duals are not counted in either of the formulae, because the number of faces in duals is not p; but in the second formula the two chiral enantiomers are counted. The function $T_pCM(n)$ is not zero when $n \geq p-1$. Now let $RC_p(n) = \frac{T_pRM(n)}{T_pCM(n)}$ for $p > 3$ and $n \geq p-1$. For each prime $p > 3$ we wish to know what is the limit of RC_p (if it exists) when $n \to \infty$.

Theorem 9. *For any odd prime* $p > 3$, *the function* $RC_p(n) = \frac{T_pRM(n)}{T_pCM(n)}$, $n \geq p-1$, *has limit given by*

$$\lim_{n\to\infty} RC_p(n) = \frac{p-1}{2\,\sigma_p},$$

where $\sigma_p = \sum_{k=2}^{p-1} NM_{(2ii)}(p,2k) = \sum_{k=2}^{p-1} \sum_{\substack{b \mid \gcd(\frac{p-1}{2},k) \\ b\geq 2,\ \frac{k}{b}\ odd}} \Phi(2b)$.

Proof. (1) Calculus of $T_pCM(n)$:

Let $\Psi_p(m)$ denote the function $NM_{(2ii)}(p,m)$ for fixed odd prime p. Since $\Psi_p(m) = 0$ for $m < 4$,

$$T_pCM(n) = \sum_{m=1}^{n} \Psi_p(m).$$

The function $\Psi_p(n)$ *is periodic with period* $2(p-1)$. In fact, since $\gcd(p-1, n+k(p-1)) = \gcd(p-1, n)$ for any positive integer k, the function $\Psi_p(n)$ is periodic and seems to have period $p-1$, however the restriction $\frac{n}{\ell}$ odd implies the period to be $2(p-1)$ instead.

Dividing n by $2(p-1)$, say $n = 2k(p-1) + r$, for some $0 \leq r < 2(p-1)$, then we can write

$$T_pCM(n) = T_pCM(2k(p-1)+r) = k \sum_{m=1}^{2(p-1)} \Psi_p(m) + R_p = k\sigma_p + R_p,$$

where $R_p = \sum_{m=1}^{r} \Psi(m)$ and $\sigma_p = \sum_{m=1}^{2(p-1)} \Psi_p(m) = \sum_{k=2}^{p-1} \Psi_p(2k)$, since $\Psi_p(m) = 0$ for m odd or $m < 4$.
(2) Calculus of $T_pRM(n)$.

Let r' be $(n-2) \mod 4$, that is, let $n = 2 + 4k' + r'$ for some k' and some $r' \in \{0, 1, 2, 3\}$. Since $NM_{(2i)}(p, m) = 0$ for $m \neq 2 \mod 4$, and 1 otherwise, then

$$T_pRM(2 + 4k' + r') = \sum_{m=2}^{2+4k'+r'} NM_{(2i)}(p, m) = \sum_{k''=0}^{k'} NM_{(2i)}(p, 2 + 4k'') = k' + 1. \tag{3}$$

But $n = 2k(p-1) + r = 2 + 2k(p-1) + r - 2 = 2 + 4k\frac{p-1}{2} + r - 2$, with $r < 2(p-1)$. Dividing $r-2$ by 4 we get $r - 2 = 4q + r'$ for some $r' < 4$ and $q \leq r - 2 < 2(p-2)$. Then $n = 2 + 4(k\frac{p-1}{2} + q) + r'$ and so,

$$T_pRM(n) = k\frac{p-1}{2} + q + 1.$$

Therefore,

$$RC_p(n) = \frac{k\frac{p-1}{2} + q + 1}{k\sigma_p + R_p}$$

and thus,

$$\lim_{n\to\infty} RC_p(n) = \frac{p-1}{2\,\sigma_p}. \quad \square$$

The above formula proves the existence of the limit and shows that the limit is not null. However it does not show if the limit is smaller, equal or greater than one. A prime number p is called *safe prime* if $\frac{p-1}{2}$ is also prime. Define p to be a *safe* 2*-prime* if $q = \frac{p-1}{2}$ is a product of two distinct primes p_1 and p_2 (let $p_1 < p_2$). If $p_1 = 2$ we say that p is an *even safe* 2*-prime* and if $p_1 > 2$ we say that p is an *odd safe* 2*-prime*.

Theorem 10. *For safe primes p, the function* $RC_p(n) = \frac{T_pRM(n)}{T_pCM(n)}$, $n \geq p-1$, *has limit*

$$\lim_{n\to\infty} RC_p(n) = \frac{p-1}{2\Phi(p-1)} = \begin{cases} 1, & p = 5; \\ \dfrac{p-1}{p-3} > 1, & p > 5. \end{cases}$$

Proof. For safe primes p, $p - 1 = 2q$ for some prime q. Since $\Psi_p(2k) = 0$ for $k \neq 0 \mod q$ and $\Psi_p(4q) = 0$,

$$\sigma_p = \sum_{k=2}^{p-1} \Psi_p(2k) = \sum_{k'=1}^{2} \Psi_p(2k'q) = \Psi_p(2q) = \Phi(2q) = \begin{cases} 2, & \text{if } q = 2; \\ q - 1 = \dfrac{p-3}{2}, & \text{if } q \text{ odd prime.} \end{cases} \quad \square$$

The above theorem says that, for large enough n, the number of reflexible regular oriented maps with 5 faces of valency n is about the same as the number of chiral regular oriented maps with 5 faces of valency n, but for safe primes $p > 5$, there are slightly more reflexible maps with p faces than chiral maps with p faces. With p faces the number of reflexible maps is not always greater than the number of chiral ones as we can see next.

Theorem 11. *For safe* 2*-primes p, the function* $RC_p(n) = \frac{T_pRM(n)}{T_pCM(n)}$, $n \geq p-1$, *has limit*

$$\lim_{n\to\infty} (RC_p(n)) = \begin{cases} \dfrac{p_2}{3p_2 - 2} < 1, & p = \textit{even safe } 2\textit{-prime}; \\ \dfrac{p_1p_2}{3p_1p_2 - 2(p_1 + p_2) + 1} < 1, & p = \textit{odd safe } 2\textit{-prime}. \end{cases}$$

Thus for safe 2*-primes p, if n is large enough, there are slightly more chiral regular oriented maps with p faces than reflexible regular oriented maps with p faces.*

Proof. Let p be a safe 2-prime, and let $q = \frac{p-1}{2} = p_1p_2$, where p_1 and p_2 are distinct primes. Assume $p_1 < p_2$. The non-trivial divisors of p_1p_2 are p_1, p_2 and p_1p_2. Since $\Psi_p(2k) = 0$ for any k not divisible either by p_1, or by p_2 and or by p_1p_2,

$$\begin{aligned}
\sigma_p &= \sum_{\substack{k=2\\ k=0 \mod p_1\\ k\neq 0 \mod p_2}}^{p-1} \Psi_p(2k) + \sum_{\substack{k=2\\ k=0 \mod p_2\\ k\neq 0 \mod p_1}}^{p-1} \Psi_p(2k) + \sum_{\substack{k=2\\ k=0 \mod p_1p_2}}^{p-1} \Psi_p(2k)\\
&= \underbrace{\sum_{\substack{k'=1\\ k'\neq p_2}}^{2p_2-1} \Psi_p(2k'p_1)}_{(\mathrm{I})} + \underbrace{\sum_{\substack{k'=1\\ k'\neq p_1}}^{2p_1-1} \Psi_p(2k'p_2)}_{(\mathrm{II})} + \underbrace{\sum_{k'=1}^{2} \Psi_p(2k'p_1p_2)}_{(\mathrm{III})}.
\end{aligned}$$

Now $\gcd(p_1p_2, k') = p_1$, $b \mid p_1$ and $b > 1 \Rightarrow b = p_1$, and $\frac{k'p_1}{b}$ =odd $\Leftrightarrow k'$ odd. Then $\Psi_p(2k'p_1) = 0$ for k' even, and for k' odd, $\Psi_p(2k'p_1) = \Phi(2p_1)$. Hence

$$(\mathrm{I}) = \sum_{\substack{k''=0\\ k''\neq \frac{p_2-1}{2}}}^{p_2-1} \Psi_p(2(2k''+1)p_1) = \sum_{\substack{k''=0\\ k''\neq \frac{p_2-1}{2}}}^{p_2-1} \Phi(2p_1) = (p_2-1)\Phi(2p_1).$$

Analogously we have,

$$(\mathrm{II}) = \begin{cases} \displaystyle\sum_{k''=0}^{p_1-1} \Psi_p(2(2k''+1)p_2) = \sum_{k''=0}^{p_1-1} \Phi(2p_2) = p_1\Phi(2p_2), & \text{if } p_1 = 2;\\ \displaystyle\sum_{\substack{k''=0\\ k''\neq \frac{p_1-1}{2}}}^{p_1-1} \Psi_p(2(2k''+1)p_2) = \sum_{\substack{k''=0\\ k''\neq \frac{p_1-1}{2}}}^{p_1-1} \Phi(2p_2) = (p_1-1)\Phi(2p_2), & \text{if } p_1 > 2. \end{cases}$$

For (III) we have $\gcd(p_1p_2, k') = p_1p_2$, $b \mid p_1p_2$ and $b > 1 \Rightarrow b = p_1$, p_2, or p_1p_2; $\frac{k'p_1p_2}{b} = k'p_2, k'p_1$, or k' is odd $\Leftrightarrow k'$ odd. If $p_1 = 2$, then $b \neq p_2$. So $k' = 1$ and

$$\Psi_p(2p_1p_2) = \begin{cases} \Phi(2p_1) + \Phi(2p_1p_2) = \Phi(4) + \Phi(4p_2), & \text{if } p_1 = 2;\\ \Phi(2p_1) + \Phi(2p_2) + \Phi(2p_1p_2), & \text{if } p_1 > 2. \end{cases}$$

Thus, for p even safe 2-prime ($p_1 = 2$) we have:

$$\begin{aligned}
\sigma_p &= (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III})\\
&= (p_2-1)\Phi(4) + 2\Phi(2p_2) + \Phi(4) + \Phi(4p_2)\\
&= 6(p_2-1) + 2,
\end{aligned}$$

and for p odd safe 2g-prime ($p_1 > 2$) we have:

$$\begin{aligned}
\sigma_p &= (p_2-1)\Phi(2p_1) + (p_1-1)\Phi(2p_2) + \Phi(2p_1) + \Phi(2p_2) + \Phi(2p_1p_2)\\
&= 3p_1p_2 - 2(p_1+p_2) + 1.
\end{aligned}$$

The limit now follows. □

We end the paper by leaving the following conjecture:

Conjecture 1. *If p is not a safe prime, then* $\lim_{n\to\infty} RC_p(n) < 1$.

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