

## Note

## Combinatorics of reductions between equivalence relations



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## ARTICLE INFO

## Article history:

Received 27 October 2014

Received in revised form 5 May 2015

Accepted 12 May 2015

Available online 6 June 2015

## ABSTRACT

We discuss combinatorial conditions for the existence of various types of reductions between equivalence relations, and in particular identify necessary and sufficient conditions for the existence of injective reductions.

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## Keywords:

Equivalence relation

Reduction

## 1. Introduction

Let  $E$  and  $F$  be equivalence relations on sets  $X$  and  $Y$ , respectively. A *homomorphism* from  $E$  to  $F$  is a function  $\phi : X \rightarrow Y$  such that  $x E x'$  implies  $\phi(x) F \phi(x')$  for all  $x, x' \in X$ . A homomorphism  $\phi$  from  $E$  to  $F$  induces a map  $\tilde{\phi} : X/E \rightarrow Y/F$  between the quotients defined by  $\tilde{\phi}([x]_E) = [\phi(x)]_F$ . We obtain special kinds of homomorphisms by requiring  $\phi$  or  $\tilde{\phi}$  to have certain properties such as being one-to-one or onto. For instance, if  $\tilde{\phi}$  is one-to-one then  $\phi$  is called a *reduction*. In this note we study the combinatorics of reductions between equivalence relations, and attempt to identify necessary and sufficient conditions for the existence of reductions of various natural types. We will see that certain types admit simple combinatorial characterizations while others do not. Our main results are a necessary and sufficient condition for the existence of an injective reduction from  $E$  to  $F$  and a complete diagram of implications between the various types of reducibility that we consider. While reductions between equivalence relations are often studied in the context of descriptive set theory, we work in the purely combinatorial context without making any definability assumptions on equivalence relations or reductions.

Many of the combinatorial problems we consider may be viewed as special instances of the general matching problem addressed in [1]. However, an application of the abstract framework of [1] to our context would be lengthier and more difficult than the self-contained proof we give in [Theorem 3.2](#). The matching problem has a long history, originating with [3]; other notable works include [2,6,5], and [4].

## 2. Reductions of equivalence relations

We now define the various types of homomorphisms that we will consider. Let  $E$  and  $F$  be equivalence relations on sets  $X$  and  $Y$ , respectively, let  $\phi : X \rightarrow Y$  be a homomorphism from  $E$  to  $F$ , and let  $\tilde{\phi}$  be the induced map on classes. We consider the following properties of the maps  $\phi$  and  $\tilde{\phi}$ :

- (i)  $\phi$  is one-to-one;
- (ii)  $\phi$  is onto;

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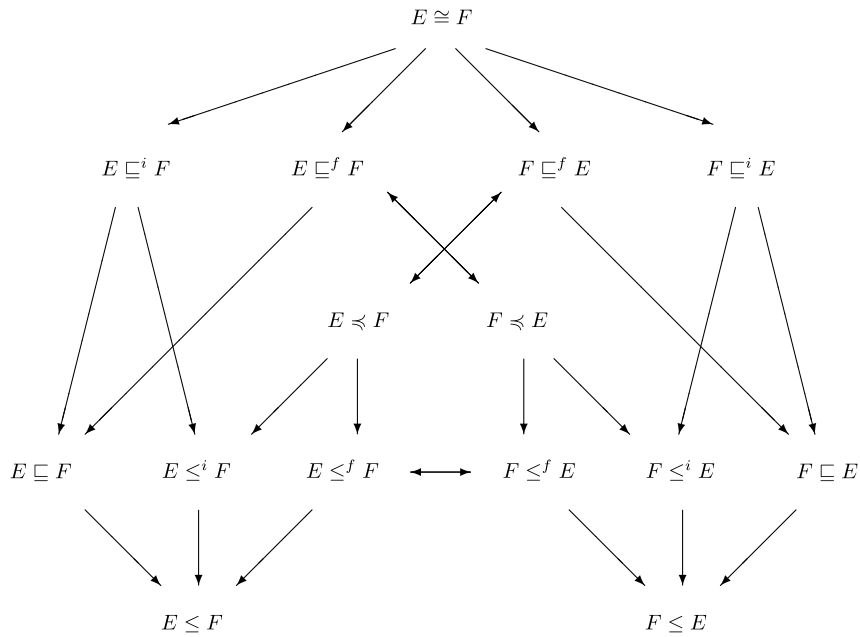


Fig. 1. Implications between types of reducibility.

- (iii)  $\tilde{\phi}$  is one-to-one;
- (iv)  $\tilde{\phi}$  is onto;
- (v)  $\text{ran}(\phi)$  is  $F$ -invariant; i.e., if  $y \in \text{ran}(\phi)$  and  $y F y'$ , then  $y' \in \text{ran}(\phi)$ .

It is straightforward to check that the only implications holding between these properties are those following from the fact that  $\phi$  is onto if and only if  $\tilde{\phi}$  is onto and  $\text{ran}(\phi)$  is  $F$ -invariant. It follows that there are sixteen distinct Boolean combinations of these properties. Since we will always take  $\phi$  to be a reduction (i.e., we assume (iii) holds), this reduces the number of distinct combinations to eight. We now introduce terminology and notation for these eight types of reductions.

**Definition 2.1.** Let  $E, F, \phi$ , and  $\tilde{\phi}$  be as above.

- (1)  $\phi$  is a *reduction* if (iii) holds;
- (2)  $\phi$  is an *embedding* if (i) and (iii) hold;
- (3)  $\phi$  is a *surjectively reducible* if (ii)–(v) hold;
- (4)  $\phi$  is an *isomorphism* if (i)–(v) hold;
- (5)  $\phi$  is an *invariantly reducible* if (iii) and (v) hold;
- (6)  $\phi$  is a *fully reducible* if (iii) and (iv) hold;
- (7)  $\phi$  is an *invariantly embeddable* if (i), (iii), and (v) hold;
- (8)  $\phi$  is a *fully embeddable* if (i), (iii), and (iv) hold.

**Definition 2.2.** If  $E$  and  $F$  are equivalence relations on sets  $X$  and  $Y$ , respectively, we say that  $E$  is *reducible* to  $F$  and write  $E \leq F$  if there is a reduction from  $E$  to  $F$ , and we say that  $E$  and  $F$  are *bireducible* and write  $E \sim F$  if  $E \leq F$  and  $F \leq E$ . We introduce analogous terminology and notation for the other types of reductions as follows:

(1) reducible	$\leq$	$\sim$
(2) embeddable	$\sqsubseteq$	$\approx$
(3) surjectively reducible	$\preceq$	$\approx^s$
(4) isomorphic	$\cong$	$\cong$
(5) invariantly reducible	$\leq^i$	$\sim^i$
(6) fully reducible	$\leq^f$	$\sim^f$
(7) invariantly embeddable	$\sqsubseteq^i$	$\approx^i$
(8) fully embeddable	$\sqsubseteq^f$	$\approx^f$

We display all the direct implications between these relations in Figs. 1 and 2, and we include a proof of Proposition 2.3 in Section 5.

**Proposition 2.3.** The diagrams in Figs. 1 and 2 are complete; that is, in each diagram, for every pair of nodes  $A$  and  $B$ , the implication  $A \Rightarrow B$  holds if and only if it is implied by the arrows in the diagram.

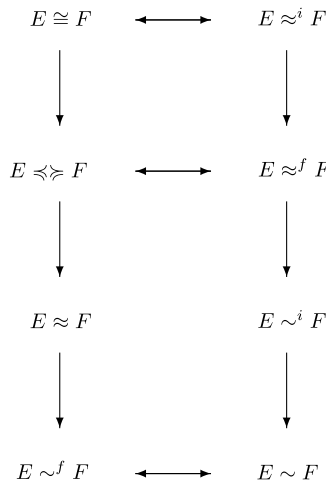


Fig. 2. Implications between equivalences on the class of equivalence relations.

Note, however, that certain implications involving more than two relations may not be displayed in the diagrams; for instance, the fact that the conjunction of  $E \leq F$  and  $F \leq E$  implies  $E \leq^f F$  is not displayed in Fig. 1.

### 3. The main theorem

Now we consider the problem of finding necessary and sufficient combinatorial conditions for the existence of reductions of the various types between equivalence relations.

**Definition 3.1.** Given an equivalence relation  $E$  and a (possibly infinite) cardinal  $\kappa$ , let  $n_\kappa(E)$  be the number of  $E$ -classes of cardinality  $\kappa$ . Similarly, let  $n_{\geq\kappa}(E)$  be the number of  $E$ -classes of size at least  $\kappa$  and  $n_{\leq\kappa}(E)$  the number of  $E$ -classes of size at most  $\kappa$ .

**Theorem 3.2.** Let  $E$  and  $F$  be equivalence relations on sets  $X$  and  $Y$ , respectively. Then

- (1)  $E \leq F \iff |X/E| \leq |Y/F|$ ;
- (2)  $E \sqsubseteq F \iff (\forall\kappa) n_{\geq\kappa}(E) \leq n_{\geq\kappa}(F)$ ;
- (3)  $E \succ F \implies (\forall\kappa) [n_{\leq\kappa}(E) \leq n_{\leq\kappa}(F) \wedge n_{\geq\kappa}(E) \geq n_{\geq\kappa}(F)]$ ;
- (4)  $E \cong F \iff (\forall\kappa) n_\kappa(E) = n_\kappa(F)$ ;
- (5)  $E \leq^i F \implies (\forall\kappa) n_{\leq\kappa}(E) \leq n_{\leq\kappa}(F)$ ;
- (6)  $E \leq^f F \iff |X/E| = |Y/F|$ ;
- (7)  $E \sqsubseteq^i F \iff (\forall\kappa) n_\kappa(E) \leq n_\kappa(F)$ ;
- (8)  $E \sqsubseteq^f F \iff F \preceq E$ .

The bi-implications (1), (4), (6), and (7) are trivial to prove, as are the forward implications in (2), (3), and (5). The backward direction of (2) appears to be somewhat harder, and is our main result. Additionally, we will show that the necessary conditions given in (3) and (5) are not sufficient, and we argue that there are no simple combinatorial conditions characterizing the surjective or invariant reducibility of  $E$  to  $F$ .

Now we present our proof of Theorem 3.2(2), which will make use of the following lemma. Recall that a class  $A$  of ordinals is closed if  $\sup B \in A$  for every subset  $B$  of  $A$ , and that a subset  $A$  of an ordinal  $\kappa$  is cofinal in  $\kappa$  if for each  $\gamma < \kappa$  there is some  $\gamma' \in A$  such that  $\gamma \leq \gamma' < \kappa$ .

**Lemma 3.3.** Let  $\kappa$  be an infinite cardinal, and let  $A$  be the class of ordinals that can be partitioned into  $\kappa$  many cofinal subsets. Then  $A$  is closed.

**Proof.** Let  $\gamma$  be a limit point of  $A$ , and let  $\langle \gamma_\alpha : \alpha < \text{cf}(\gamma) \rangle$  be a continuous increasing sequence of elements of  $A$  with limit  $\gamma$ . For each  $\alpha < \text{cf}(\gamma)$ , let  $\{P_v^\alpha : v < \kappa\}$  be a partition of  $\gamma_\alpha$  into  $\kappa$  many cofinal subsets. For each  $v < \kappa$ , define

$$P_v = \bigcup_{\alpha < \text{cf}(\gamma)} (P_v^{\alpha+1} - \gamma_\alpha).$$

The set  $\{P_v : v < \kappa\}$  is a partition of  $\gamma$  into  $\kappa$  many cofinal subsets.

Note that for an ordinal  $\gamma$  and an infinite cardinal  $\kappa$ ,  $\gamma$  may be partitioned into  $\kappa$  many cofinal subsets if and only if  $\gamma = \kappa \cdot \alpha$  for some ordinal  $\alpha$ .

**Proof of Theorem 3.2(2).** The forward direction is immediate. For the backward direction, we must show that assuming

$$(\forall\kappa) n_{\geq\kappa}(E) \leq n_{\geq\kappa}(F),$$

there exists an injective function  $\phi : X \rightarrow Y$  satisfying

$$(\forall x, x' \in X) \quad x E x' \Leftrightarrow \phi(x) F \phi(x').$$

Let us begin by fixing an enumeration  $\langle C_\xi : \xi < \alpha \rangle$  of the  $E$ -classes such that  $|C_\xi| \leq |C_\eta|$  whenever  $\xi < \eta < \alpha$ , as well as an enumeration  $\langle D_\xi : \xi < \beta \rangle$  of the  $F$ -classes such that  $|D_\xi| \leq |D_\eta|$  whenever  $\xi < \eta < \beta$ . Notice that since  $n_{\geq 1}(E) \leq n_{\geq 1}(F)$ , we have  $|\alpha| \leq |\beta|$ .

We will prove the result by induction on  $|\alpha|$ . For the base case where  $\alpha$  is finite, we have  $|C_n| \leq |D_n|$  for all  $n < \alpha$ , so the greedy approach of mapping each  $C_n$  into  $D_n$  yields the desired embedding. For the inductive step, suppose that  $\alpha$  is infinite and that we have proven the theorem for every pair of equivalence relations  $(E', F')$  satisfying

$$(\forall \kappa) \quad n_{\geq \kappa}(E') \leq n_{\geq \kappa}(F')$$

such that the number of  $E'$ -classes is strictly less than  $|\alpha|$ .

Since  $|\alpha| \leq |\beta| \leq \beta$ , there is at least one ordinal  $\gamma \leq \beta$  that can be partitioned into  $|\alpha|$  many cofinal subsets. By [Lemma 3.3](#), there is a largest such  $\gamma \leq \beta$ , which we fix. We first claim that  $|\beta - \gamma| < |\alpha|$ . If not, let  $\delta$  be the least ordinal such that  $\gamma + \delta = \beta$ , so that  $|\delta| = |\beta - \gamma|$ . Then

$$\gamma + |\alpha| \leq \gamma + |\beta - \gamma| = \gamma + |\delta| \leq \gamma + \delta = \beta,$$

contradicting the choice of  $\gamma$ .

Let  $\sigma$  be the least ordinal less than  $\alpha$  such that  $|C_\sigma| > |D_\xi|$  for all  $\xi < \gamma$  if such an ordinal exists, and let  $\sigma = \alpha$  otherwise. Then for each  $\nu < \sigma$  there is some  $\xi' < \gamma$  such that  $|C_\nu| \leq |D_{\xi'}|$ . Let  $\{P_\nu : \nu < \sigma\}$  be a partition of  $\gamma$  into cofinal subsets (such a partition exists because  $\gamma$  can be partitioned into  $|\alpha|$  many cofinal subsets and  $\sigma \leq \alpha$ ). Given any  $\nu < \sigma$ , we may pick  $\xi' < \gamma$  such that  $|C_\nu| \leq |D_{\xi'}|$ , and then we may pick  $\xi \in P_\nu$  such that  $\xi' \leq \xi$  (and hence  $|D_{\xi'}| \leq |D_\xi|$ ). Therefore we have

$$(\forall \nu < \sigma)(\exists \xi \in P_\nu) \quad |C_\nu| \leq |D_\xi|.$$

Because of this, we may define an injection  $\phi_0$  from  $\bigcup_{\nu < \sigma} C_\nu$  to  $\bigcup_{\xi < \gamma} D_\xi$  such that

$$(\forall x, x' \in \bigcup_{\nu < \sigma} C_\nu) \quad x E x' \Leftrightarrow \phi_0(x) F \phi_0(x').$$

If  $\sigma = \alpha$  we are done, so assume  $\sigma < \alpha$ . Let  $X_1 = \bigcup_{\sigma \leq \nu < \alpha} C_\nu$  and  $Y_1 = \bigcup_{\gamma \leq \xi < \beta} D_\xi$ , and let  $E' = E \upharpoonright X_1$  and  $F' = F \upharpoonright Y_1$ . Since  $|\beta - \gamma| < |\alpha|$ , by the definition of  $\sigma$  and the hypothesis that  $n_{\geq |C_\sigma|}(E) \leq n_{\geq |C_\sigma|}(F)$  we have that  $|\alpha - \sigma| < |\alpha|$ . That is, there are strictly fewer than  $|\alpha|$  many  $E'$ -classes. Also notice that  $(\forall \kappa) n_{\geq \kappa}(E') \leq n_{\geq \kappa}(F')$ . We may now apply the inductive hypothesis to obtain an injective reduction  $\phi_1$  from  $E'$  to  $F'$ . At this point we are finished, since the function  $\phi_0 \cup \phi_1$  is an injective reduction from  $E$  to  $F$ .

#### 4. Counterexamples

In this section we present some examples to show that the necessary conditions given in [Theorem 3.2](#) for the existence of invariant and surjective reductions are not sufficient, and indeed we argue that for these types of reducibility, no “nice” necessary and sufficient conditions exist.

**Example 4.1.** Let  $E$  and  $F$  be equivalence relations each having exactly one equivalence class of size  $n$  for  $1 \leq n < \omega$  and no additional classes except that  $E$  has exactly one class of size  $\aleph_0$ . Then for all cardinals  $\kappa$  we have  $n_{\leq \kappa}(E) \leq n_{\leq \kappa}(F)$  and  $n_{\geq \kappa}(E) \geq n_{\geq \kappa}(F)$ , but it is not difficult to check there can be no invariant reduction from  $E$  to  $F$ .

To dispel the impression that the finiteness of the cardinals  $n_{\geq \kappa}(E)$  is the sole source of the problem, we give another counterexample where this time  $n_\kappa(E)$  and  $n_\kappa(F)$  are either 0 or infinite for all  $\kappa$ . Our construction uses Fodor’s Lemma, which is typical for the uncountable case of the matching problem (see, for instance, [1, Lemma 4.9]).

**Example 4.2.** There exist equivalence relations  $E$  and  $F$  such that

- (1) for all cardinals  $\kappa$ ,  $n_\kappa(E)$  and  $n_\kappa(F)$  are either 0 or  $\aleph_0$ ;
- (2)  $(\forall \kappa) n_{\leq \kappa}(E) = n_{\leq \kappa}(F)$ ;
- (3)  $(\forall \kappa) n_{\geq \kappa}(E) = n_{\geq \kappa}(F)$ ;
- (4)  $E \not\leq^i F$ , and hence also  $E \not\leq F$ .

**Proof.** It suffices to specify  $n_\kappa(E)$  and  $n_\kappa(F)$  for each cardinal  $\kappa$ . Let  $n_1(E) = \aleph_0$  and  $n_{\aleph_\alpha}(E) = \aleph_0$  for every countable limit ordinal  $\alpha$ , and let  $n_\kappa(E) = 0$  for every other cardinal  $\kappa$ . Let  $n_1(F) = \aleph_0$  and  $n_{\aleph_{\alpha+1}}(F) = \aleph_0$  for every countable limit ordinal  $\alpha$ , and let  $n_\kappa(F) = 0$  for every other cardinal  $\kappa$ .

It is clear that conditions (1) through (3) are satisfied. Suppose, towards a contradiction, that  $\phi$  is an invariant reduction from  $E$  to  $F$ . For every countable limit ordinal  $\alpha$ ,  $\phi$  maps each  $E$ -class of size  $\aleph_\alpha$  onto an  $F$ -class of size less than  $\aleph_\alpha$ . For each countable limit ordinal  $\alpha$ , arbitrarily pick some  $E$ -class  $C_\alpha$  of size  $\aleph_\alpha$ . Hence, the function  $\phi$  maps each class  $C_\alpha$  onto some

$F$ -class of size  $\aleph_{g(\alpha)}$  for some  $g(\alpha) < \alpha$ . We have now defined a regressive function  $g$  from the (stationary) set of countable limit ordinals to  $\omega_1$ . By Fodor’s Lemma,  $g$  is constant on some stationary set. This means that there is some  $\beta < \omega_1$  such that  $\phi$  maps  $\omega_1$  many  $E$ -classes onto  $F$ -classes of size  $\aleph_\beta$ . Since there are at most  $\aleph_0$  many  $F$ -classes of size  $\aleph_\beta$ , this is a contradiction.

Examples 4.1 and 4.2 suggest that in general there is no “nice” combinatorial characterization of the existence of an invariant or surjective reduction from one equivalence relation to another, and we now describe one way of making this precise. Define a *nice condition* to be a conjunction of statements of the form “for all cardinals  $\kappa, a R b$ ”, where  $a$  is one of the four terms

$$n_\kappa(E), n_{\leq \kappa}(E), n_{\geq \kappa}(E), |X/E|,$$

$b$  is one of the four terms

$$n_\kappa(F), n_{\leq \kappa}(F), n_{\geq \kappa}(F), |Y/F|,$$

and  $R$  is one of the six relations

$$\leq, \geq, =, \neq, <, > .$$

The proof of the following proposition is straightforward but tedious, and we omit it.

**Proposition 4.3.** *Every nice condition which is implied by  $E \leq^i F$  follows from the condition*

$$(\forall \kappa) \quad n_{\leq \kappa}(E) \leq n_{\leq \kappa}(F),$$

and every nice condition which is implied by  $E \approx F$  follows from the condition

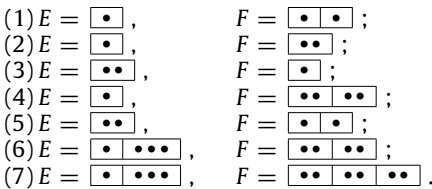
$$(\forall \kappa) \quad [n_{\leq \kappa}(E) \leq n_{\leq \kappa}(F) \wedge n_{\geq \kappa}(E) \geq n_{\geq \kappa}(F)].$$

In this sense parts (3) and (5) of Theorem 3.2 are optimal, and Examples 4.1 and 4.2 show that none of the relations  $E \approx F$ ,  $E \leq^i F$ , and  $E \sqsubseteq^f F$  can be characterized by a nice condition.

### 5. Completeness of the diagrams

In this final section we prove Proposition 2.3.

*Proof that the diagram in Fig. 1 is correct and complete.* All displayed implications follow immediately from the definitions, so we need only show that there are no additional implications. We will show that for every node  $A$  in the diagram, there is no implication of the form  $A \Rightarrow B$  that is not displayed. For the top node  $E \cong F$ , this is vacuous. By symmetry, it will suffice to consider the seven nodes on the left half of the diagram. We will accomplish this using the seven pairs of equivalence relations pictured below, where dots represent elements and boxes equivalence classes.



(1) shows that  $E \sqsubseteq^i F$  does not imply  $F \leq E$ . (2) shows that  $E \sqsubseteq^f F$  implies neither  $E \leq^i F$  nor  $F \sqsubseteq E$ . (3) shows that  $E \approx F$  implies neither  $E \sqsubseteq F$  nor  $F \leq^i E$ . (4) shows that  $E \sqsubseteq F$  implies neither  $E \leq^i F$  nor  $F \leq E$ . (5) shows that  $E \leq^i F$  implies neither  $E \sqsubseteq F$  nor  $F \leq E$ . (6) shows that  $E \leq^f F$  implies none of  $E \sqsubseteq F, E \leq^i F, F \sqsubseteq E$ , and  $F \leq^i E$ . Finally, (7) shows that  $E \leq F$  implies none of  $E \sqsubseteq F, E \leq^i F$ , and  $F \leq E$ . These observations suffice to establish the completeness of the diagram in Fig. 1.

*Proof that the diagram in Fig. 2 is correct and complete.* The implication  $E \sim F \Rightarrow E \sim^f F$  is clear, and the implication  $E \approx^i F \Rightarrow E \cong F$  is well-known and follows from the standard Schröder–Bernstein argument. The remaining displayed implications follow immediately from the implications in Fig. 1, so it is only left to show that there are no additional implications. For this it suffices to show the following:

- (1)  $E \approx^f F \not\Rightarrow E \cong F$ ;
- (2)  $E \approx F \not\Rightarrow E \sim^i F$ ;
- (3)  $E \sim^i F \not\Rightarrow E \approx F$ ;
- (4)  $E \sim F \not\Rightarrow E \sim^i F$ ;
- (5)  $E \sim F \not\Rightarrow E \approx F$ .

This may be done using the following equivalence relations, which have no classes other than those described.

1.  $E$  has one class of size  $n$  for each even integer  $n$ ,  $F$  has one class of size  $n$  for each odd integer  $n \geq 3$ , and both  $E$  and  $F$  have  $\aleph_0$  many classes of size 1.
2.  $E$  has  $\aleph_0$  many classes of size  $\aleph_0$  and one class of size 1;  $F$  has  $\aleph_0$  many classes of size  $\aleph_0$  and one class of size 2.

3. Both  $E$  and  $F$  have  $\aleph_0$  many classes of size 1, and  $E$  has one class of size 2.
4.  $E$  has one class of size 1,  $F$  has one class of size 2.
5. Same as (4).

### Acknowledgment

The first author was partially supported by a Rackham Dissertation Fellowship provided by the Rackham Graduate School at the University of Michigan.

### References

- [1] R. Aharoni, C.St.J.A. Nash-Williams, S. Shelah, A general criterion for the existence of transversals, *Proc. Lond. Math. Soc.* 47 (3) (1983) 43–68.
- [2] J. Folkman, Transversals of infinite families with finitely many infinite members, Rand Corporation, Memorandum, RM-5676-RP, 1968.
- [3] P. Hall, On representatives of subsets, *J. Lond. Math. Soc.* 10 (1935) 26–30.
- [4] M. Hall Jr., Distinct representatives of subsets, *Bull. Amer. Math. Soc.* 54 (1948) 922–926.
- [5] L. Mirsky, *Transversal Theory*, Academic Press, New York, 1971.
- [6] S. Shelah, A substitute for Hall's theorem for families with infinite sets, *J. Combin. Theory. Ser. A* 16 (1974) 199–208.