Discrete Mathematics 338 (2015) 1845–1855

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Decompositions of triangle-free 5-regular graphs into paths of length five ☆

F. Botler *, G.O. Mota, Y. Wakabayashi

Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão 1010, 05508-090, São Paulo, Brazil

ARTICLE INFO

Article history:
Received 14 April 2014
Received in revised form 15 April 2015
Accepted 16 April 2015
Available online 5 June 2015

Keywords:
Graph decomposition
Paths
Regular graphs
Matching
Triangle-free

ABSTRACT

A P_k-decomposition of a graph G is a set of edge-disjoint paths with k edges that cover the edge set of G. Kotzig (1957) proved that a 3-regular graph admits a P_3-decomposition if and only if it contains a perfect matching. Kotzig also asked what are the necessary and sufficient conditions for a $(2k+1)$-regular graph to admit a decomposition into paths with $2k+1$ edges. We partially answer this question for the case $k = 2$ by proving that the existence of a perfect matching is sufficient for a triangle-free 5-regular graph to admit a P_5-decomposition. This result contributes positively to the conjecture of Favaron et al. (2010) that states that every $(2k+1)$-regular graph with a perfect matching admits a P_{2k+1}-decomposition.

1. Introduction

In this paper, the term decomposition always refer to an edge-decomposition of a graph. Given a graph $G = (V, E)$, a graph decomposition of G is a set of edge-disjoint subgraphs of G that cover E. The problem of finding decompositions of graphs into subgraphs of certain types is a classical problem in graph theory that traces back to the late 19th century. One of the earliest results of this nature is a theorem of Petersen (1891) that states that every $2k$-regular graph can be decomposed into 2-factors. Many surveys and books on this topic have appeared in the literature, among which the reader may refer to [1,4,8,12,17–20].

In general, finding or deciding the existence of some nontrivial graph decomposition is a hard problem, and much effort has been devoted to studying decompositions of particular classes of graphs into some classes of subgraphs. If we restrict our attention to decompositions of arbitrary graphs into cycles or paths, we come across many interesting conjectures and to the following old and elegant result of Lovász [27].

Theorem 1.1 (*Lovász*). *Every n-vertex graph can be decomposed into at most $\lfloor n/2 \rfloor$ paths and cycles.*

In fact, according to Lovász [27], in 1966 Gallai conjectured that every n-vertex connected graph admits a decomposition into at most $\lceil n/2 \rceil$ paths, and Hajós conjectured that any Eulerian graph can be decomposed into at most $\lfloor n/2 \rfloor$ cycles. We also refer to Bondy [3] for these and other conjectures. Looking for asymptotic results, Erdős and Gallai [13,14] conjectured that every n-vertex graph can be decomposed into $O(n)$ cycles and edges. Many researchers have obtained partial results on these conjectures (see [9,10,15]).

☆ This research has been partially supported by CNPq Projects (Proc. 477203/2012-4 and 456792/2014-7), Fapesp Project (Proc. 2013/03447-6) and MaCLinC Project of NUMEC/USP, Brazil.

* Corresponding author.
E-mail addresses: fbotler@ime.usp.br (F. Botler), mota@ime.usp.br (G.O. Mota), yw@ime.usp.br (Y. Wakabayashi).

http://dx.doi.org/10.1016/j.disc.2015.04.018
0012-365X/

Decompositions of regular graphs have been extensively investigated in the last decades, mostly restricted to decompositions into paths of fixed length. We denote by P_k (resp. C_k) a path (resp. cycle) of length k, that is, with k edges. (We observe that this notation is not so standard.) Jacobson, Truszczyński and Tuza [22] proved that every 4-regular bipartite graph admits a P_4-decomposition. For other results concerning $2k$-regular graphs and cartesian products of regular graphs, the reader is referred to [24,29]; and for results on decompositions of regular graphs with large girth, we mention Kouider and Lonc [26].

Kotzig [25] proved that a 3-regular graph G admits a P_3-decomposition if and only if G contains a perfect matching. In fact, Kotzig proved a slightly stronger result (on two P_3-decompositions). The proof used by Kotzig is presented by Bouchet and Fouquet [7]. This result was generalized by Jaeger, Payan, and Kouider [23], who proved that a $(2k+1)$-regular graph that contains a perfect matching can be decomposed into bistars. In another direction, Heinrich, Liu and Yu [21] proved that every $3m$-regular graph without cut-edges admits a P_3-decomposition. Kotzig asked what are the necessary and sufficient conditions for a $(2k+1)$-regular graph G to be decomposable into paths of length $2k+1$. A necessary condition is that G must contain a k-factor. Favaron, Genest, and Kouider [16] proved that this condition is not sufficient. For $k=2$ (that is, for a 5-regular graph), Favaron, Genest, and Kouider [16] proved that it is sufficient that G contains a perfect matching and no cycles of length four to admit a P_5-decomposition. Here we prove that every triangle-free 5-regular graph that contains a perfect matching admits a P_5-decomposition.

This paper is organized as follows. In Section 2 we give some definitions and establish the notation. In Section 3 we show that triangle-free 5-regular graphs containing a perfect matching admit a decomposition into copies of P_5 and some specific trails T_5 with five vertices. Section 4 contains some lemmas which enable us to reduce the number of copies of T_5 and increase the number of copies of P_5, obtaining a decomposition closer to the desired one. In Section 5 we use the results obtained in Sections 3 and 4 to obtain a P_5-decomposition.

2. Basic definitions and notation

The basic terminology and notation used in this paper are standard (see [2,11]). A *mixed graph* is a simple graph in which some edges may receive an orientation. More precisely, it is a triple $\bar{G}=(V,E,A)$ consisting of a vertex set V, an (undirected) edge set E and a directed edge set A, such that (V,E) is a simple graph and (V,A) is a simple directed graph; and furthermore, no two distinct edges in $E\cup A$ have the same endpoints. Given a mixed graph $\bar{G}$, we denote by $V(\bar{G})$, $E(\bar{G})$ and $A(\bar{G})$, the sets of vertices, undirected edges and directed edges of $\bar{G}$, respectively. Let $\hat{A}(\bar{G})$ be the *underlying edge set* of $A(\bar{G})$, i.e., the set of edges obtained by removing the orientation of the directed edges in $A(\bar{G})$. We denote by G the *underlying graph* of $\bar{G}$, i.e., G is the graph such that $V(G)=V(\bar{G})$ and $E(G)=E(\bar{G})\cup\hat{A}(\bar{G})$. We note that, in this paper all (mixed) graphs are simple.

Let $\bar{G}$ be a mixed graph. For ease of notation, we write simply ab to refer to an undirected edge $\{a,b\}\in E$ or a directed edge $(a,b)\in A(\bar{G})$, and use the term *edge* to refer to an element that belongs to $E(\bar{G})\cup A(\bar{G})$. When the orientation of an edge is relevant, we write $ab\in A(\bar{G})$, or specify that ab is a directed edge. A mixed subgraph $\bar{H}$ of $\bar{G}$ is a mixed graph such that $V(\bar{H})\subseteq V(\bar{G})$, $E(\bar{H})\subseteq E(\bar{G})$ and $A(\bar{H})\subseteq A(\bar{G})$. Given a set of mixed subgraphs $\bar{H}_1,\ldots,\bar{H}_k$ of $\bar{G}$, we denote by $\bigcup_{i=1}^k \bar{H}_i$ the mixed subgraph $\bar{H}=\left(\bigcup_{i=1}^k V(\bar{H}_i),\bigcup_{i=1}^k E(\bar{H}_i),\bigcup_{i=1}^k A(\bar{H}_i)\right)$.

We say that a mixed graph $\bar{H}$ is a *copy* of a mixed graph $\bar{G}$ if H is isomorphic to G. A *path* P in $\bar{G}$ is a sequence of distinct vertices $P=v_0v_1\cdots v_k$ such that v_iv_{i+1} is an edge in $\bar{G}$, for $i=0,1,\ldots,k-1$. (Note that, possibly $v_{i+1}v_i$ is a directed edge, for some i in $\{0,\ldots,k-1\}$). For convenience, we will also consider that such a path P is a mixed graph with $V(P)=\{v_0,v_1,\ldots,v_k\}$ and $E(P)\cup A(P)=\{v_0v_1,v_1v_2,\ldots,v_{k-1}v_k\}$. The *length* of P is the number of edges in P. We denote by P_k any path of length k, and we denote by T_k the graph (trail) that is obtained from a path $v_0v_1\cdots v_{k-1}$ by the addition of the edge $v_{k-1}v_1$. We refer to T_k simply as $v_0v_1\cdots v_{k-1}v_1$. If a mixed graph $\bar{H}$ is a copy of P_k or T_k we also write $\bar{H}=v_0\cdots v_k$ or $\bar{H}=v_0\cdots v_{k-1}v_1$, respectively.

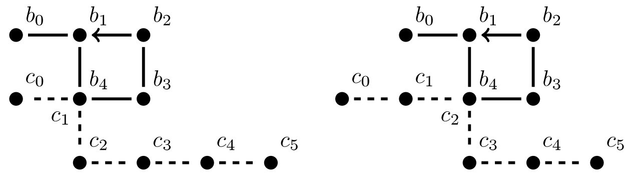
We say that a set $\{\bar{H}_1,\ldots,\bar{H}_k\}$ of mixed graphs is a *decomposition* of a mixed graph $\bar{G}$ if $\bigcup_{i=1}^k E(\bar{H}_i)=E(\bar{G})$, $\bigcup_{i=1}^k A(\bar{H}_i)=A(\bar{G})$, and furthermore $E(\bar{H}_i)\cap E(\bar{H}_j)=\emptyset$ and $A(\bar{H}_i)\cap A(\bar{H}_j)=\emptyset$ for all $1\le i<j\le k$. Let $\mathcal{H}$ be a family of graphs. An *$\mathcal{H}$-decomposition* $\mathcal{D}$ of $\bar{G}$ is a decomposition of $\bar{G}$ such that each element of $\mathcal{D}$ is isomorphic to an element of $\mathcal{H}$. If $\mathcal{H}=\{H\}$ we say that $\mathcal{D}$ is an *H-decomposition*.

In the next section we present a result that will allow us to explain the idea behind the proof of the main result, and will also motivate the definitions given thereafter.

3. Canonical $\{P_5,T_5\}$-decomposition

In this section we show that a triangle-free 5-regular graph G that contains a perfect matching is the underlying graph of a mixed graph $\bar{G}$ that admits a $\{P_5,T_5\}$-decomposition that has some special properties. The mixed graph $\bar{G}$ we shall deal with is one obtained from G by assigning an orientation to the edges of each cycle of a given 2-factor F of G, obtaining a set of directed cycles. We shall refer to such an orientation as an *Eulerian orientation* of F. In such a mixed graph, we say that a copy $v_0v_1\cdots v_5$ of P_5 (resp. a copy $v_0v_1\cdots v_4v_1$ of T_5) is *canonical* if its directed edges are precisely v_1v_0 and v_4v_5 (resp. v_1v_0 and v_2v_1). If the Eulerian orientation of a 2-factor of G is called $\mathcal{E}$, then we say that a $\{P_5,T_5\}$-decomposition $\mathcal{D}_{\mathcal{E}}$ of $\bar{G}$ is *$\mathcal{E}$-canonical*, or simply *canonical*, if each element of $\mathcal{D}_{\mathcal{E}}$ is canonical.

We will need the following two well-known results.



(a) A TP-couple in position 1. (b) A TP-couple in position 2.

Fig. 1. Examples of TP-couples.

Theorem 3.1 (*Petersen [28]*)**.** *Every* $2k$*-regular graph contains a* 2*-factor.*

Theorem 3.2 (*Kotzig [25]*)**.** *Every* 3*-regular graph containing a perfect matching admits a* P_3*-decomposition.*

Given a triangle-free 5-regular graph G containing a perfect matching, we use Theorem 3.1 to obtain a mixed graph $\bar{G}$, then we use Theorem 3.2 to show that $\bar{G}$ has a canonical $\{P_5, T_5\}$-decomposition.

Lemma 3.3. *Let G be a triangle-free* 5*-regular graph containing a perfect matching. Then G is the underlying graph of a mixed graph $\bar{G}$ such that $A(\bar{G})$ induces a* 2*-factor that has an Eulerian orientation $\mathcal{E}$, and $\bar{G}$ admits an $\mathcal{E}$-canonical $\{P_5, T_5\}$-decomposition.*

Proof. Let G be a triangle-free 5-regular graph containing a perfect matching M. Let $G_4 = G - M$ be the 4-regular graph obtained from G by removing the edges of M. By Theorem 3.1, the graph G_4 contains a 2-factor F. Let G_3 be the 3-regular graph $G - E(F)$. Note that $M \subseteq E(G_3)$, and thus, by Theorem 3.2, the graph G_3 admits a P_3-decomposition, say $\mathcal{D}_3$.

Let $\mathcal{E}$ be an Eulerian orientation of F, and let $\vec{F}$ be the directed graph induced by such an orientation. For each path $P \in \mathcal{D}_3$, let x_P and y_P be the end vertices of P, and let $x_P x_{\vec{F}}$ and $y_P y_{\vec{F}}$ be the directed edges of $\vec{F}$ that leave x_P and y_P, respectively.

Consider the mixed graph $\bar{G} = \big(V(G), E(G_3), A(\vec{F})\big)$ (note that G is the underlying graph of $\bar{G}$) and let $P = x_P v_P w_P y_P$ be an element of $\mathcal{D}_3$, i.e., a path of length three in G_3. We claim that $Q_P = x_{\vec{F}} x_P v_P w_P y_P y_{\vec{F}}$ is either isomorphic to P_5 or to T_5. Note that, since $\bar{G}$ contains no multiple edges, we have $y_{\vec{F}} \neq w_P$, and if $y_{\vec{F}} = x_P$ then $x_{\vec{F}} \neq y_P$. Moreover, since G is triangle-free, we have $y_{\vec{F}} \neq v_P$. By symmetry, analogous arguments hold for $x_{\vec{F}}$. Furthermore, since $\vec{F}$ is induced by the Eulerian orientation $\mathcal{E}$, we have $y_{\vec{F}} \neq x_{\vec{F}}$. Then, one of the following three cases occurs: (a) all vertices of Q_P are distinct, (b) $y_{\vec{F}} = x_P$, or (c) $x_{\vec{F}} = y_P$. In case (a), Q_P is a canonical copy of P_5, and in case (b), Q_P is a canonical copy of T_5. In these two cases, let $\bar{Q}_P = Q_P$. In case (c), Q_P is isomorphic to the trail $T = y_{\vec{F}} y_P w_P v_P x_P y_P$, which is a canonical copy of T_5, so in this case let $\bar{Q}_P = T$.

Let $\mathcal{D}_{\mathcal{E}} = \{\bar{Q}_P : P \in \mathcal{D}_3\}$. Note that each vertex $v \in V(G)$ is the endpoint of exactly one path $P \in \mathcal{D}_3$. This follows from the fact that G_3 is a 3-regular graph and therefore the set formed by all the intermediate edges of the paths in $\mathcal{D}_3$ is a perfect matching of G_3. Thus, the set $\mathcal{D}_{\mathcal{E}}$ decomposes $\bar{G}$. Thus, by construction, $\mathcal{D}_{\mathcal{E}}$ is an $\mathcal{E}$-canonical $\{P_5, T_5\}$-decomposition of $\bar{G}$, and the proof is complete. $\square$

The idea behind the proof of our main result is to show that the canonical decomposition given by Lemma 3.3 has other special properties, which will be used to "disentangle" pairs (or sequences) of T_5 and P_5, obtaining only P_5s. For that, we have to define some concepts to show how this disentanglement can be performed.

3.1. Couples in a $\{P_5, T_5\}$-decomposition

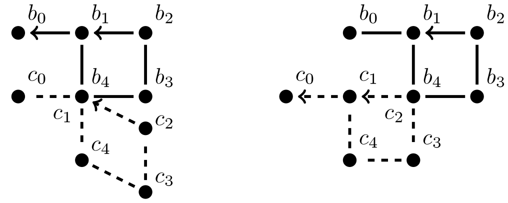
We say that a copy $\bar{B}$ of T_5 is *well-oriented* if we can label the vertices of $\bar{B}$ such that $\bar{B} = b_0 b_1 b_2 b_3 b_4 b_1$ and either $A(\bar{B}) = \{b_2 b_1\}$ or $A(\bar{B}) = \{b_2 b_1, b_1 b_0\}$. In this case, the vertex b_4 is called the *connection-vertex* of $\bar{B}$, and is denoted by $\text{cv}(\bar{B})$.

Let $\bar{P} = v_0 v_1 v_2 v_3 v_4 v_5$ be a copy of P_5 in a mixed graph $\bar{G}$. We say that $\bar{P}$ is a *roofed path* in $\bar{G}$ if $v_4 v_1$ is an edge of $E(\bar{G}) \cup \hat{A}(\bar{G})$. Furthermore, we say that $v_4 v_1$ is the *roof* of $\bar{P}$.

If $\bar{B}$ is a subgraph of $\bar{G}$ and $ab \in A(\bar{B})$ is such that the degree of vertex a in the underlying graph of $\bar{B}$ is at least 2, then we say that ab is an *internal directed edge* of $\bar{B}$. Moreover, if $\bar{B}$ is an element of a decomposition $\mathcal{D}$, then we also say that ab is *internal to* $\mathcal{D}$.

Let $\bar{G}$ be a 5-regular mixed graph and let $\mathcal{D}$ be a $\{P_5, T_5\}$-decomposition of $\bar{G}$. Given such a decomposition, it is very important to understand how pairs of P_5 and T_5, or pairs of two T_5's appear in the decomposition, so that we can disentangle them obtaining only P_5s. The following definitions play special roles in this process.

Let $(\bar{B}, \bar{C})$ be a pair of elements of $\mathcal{D}$, where $\bar{B}$ is a well-oriented copy of T_5, say $b_0 b_1 b_2 b_3 b_4 b_1$, and $\bar{C}$ is a copy of P_5, say $c_0 c_1 c_2 c_3 c_4 c_5$. We say that $(\bar{B}, \bar{C})$ is a TP-*couple* of $\mathcal{D}$ if $\text{cv}(\bar{B}) \in \{c_1, c_2, c_3, c_4\}$ (possibly, $\bar{B}$ and $\bar{C}$ may have more common vertices). If $\text{cv}(\bar{B}) \in \{c_1, c_4\}$, then we say that $(\bar{B}, \bar{C})$ is a TP-*couple in position 1*, and if $\text{cv}(\bar{B}) \in \{c_2, c_3\}$, then we say that $(\bar{B}, \bar{C})$ is a TP-*couple in position 2* (see Fig. 1). Now let $\bar{D}$ be a well-oriented copy of T_5, say $w_0 w_1 w_2 w_3 w_4 w_1$, such that $\text{cv}(\bar{D}) = w_4$. We say that $(\bar{B}, \bar{D})$ is a TT-*couple* of $\mathcal{D}$ if $\text{cv}(\bar{B}) \in \{w_1, w_2\}$ (possibly, $\bar{B}$ and $\bar{D}$ may have more common vertices). If $\text{cv}(\bar{B}) = w_1$, then we say that $(\bar{B}, \bar{D})$ is a TT-*couple in position 1*, and if $\text{cv}(\bar{B}) = w_2$, then we say that $(\bar{B}, \bar{D})$ is a TT-*couple in position 2* (see



(a) A TT-couple in position 1. (b) A TT-couple in position 2.

Fig. 2. Examples of TT-couples.

Fig. 2). Furthermore, if $(\bar{B}, \bar{X})$ is a couple, then we say that $\bar{B}$ and $\bar{X}$ are, respectively, the *top* and the *base* of $(\bar{B}, \bar{X})$. Hereafter, whenever we refer to a *couple*, we mean either a TP-couple or a TT-couple.

Let $\bar{G}$ be a mixed 5-regular graph and $\mathcal{D}$ a $\{P_5, T_5\}$-decomposition of $\bar{G}$. Let $\bar{B} = b_0b_1b_2b_3b_4b_1$ and $\bar{C} = c_0c_1c_2c_3c_4c_5$ be elements of $\mathcal{D}$ such that $\text{cv}(\bar{B}) = b_4 = c_1$. We say that the TP-couple $(\bar{B}, \bar{C})$ is *solvable* if $\bar{B}^+ = b_0b_1b_2b_3b_4c_0$ and $\bar{C}^+ = b_1c_1c_2c_3c_4c_5$ are edge-disjoint paths (of length five) in $\bar{G}$. (Note that $\bar{B}^+$ is a roofed path in $\bar{G}$ and $(\bar{B}, \bar{C})$ is a TP-couple in position 1.)

As a solvable TP-couple can be decomposed into two paths of length 5, it is of our interest to obtain $\{P_5, T_5\}$-decompositions in which every TP-couple is solvable. Besides that, we need that other additional properties hold so that we can disentangle all TT-couples. The next concept captures the properties that we need.

Definition 3.4. A $\{P_5, T_5\}$-decomposition $\mathcal{D}$ of a mixed graph $\bar{G}$ is called complete if the following three conditions hold.
(i) Every copy of T_5 in $\mathcal{D}$ is well-oriented;
(ii) Every directed edge of $\bar{G}$ is internal to $\mathcal{D}$;
(iii) Every TP-couple of $\mathcal{D}$ is solvable.

Let G be a graph with maximum degree 5, and suppose G is the underlying graph of a mixed graph $\bar{G}$ such that $A(\bar{G})$ induces a 2-factor with an Eulerian orientation. We remark that no element of a complete $\{P_5, T_5\}$-decomposition of $\bar{G}$ is the top of more than one couple, or the base of more than two couples, otherwise G would contain vertices of degree at least 6.

Lemma 3.5. *Let G be a triangle-free 5-regular graph containing a perfect matching. If G is the underlying graph of a mixed graph $\bar{G}$ such that $A(\bar{G})$ induces a 2-factor with an Eulerian orientation $\mathcal{E}$, and $\bar{G}$ admits an $\mathcal{E}$-canonical $\{P_5, T_5\}$-decomposition $\mathcal{D}_{\mathcal{E}}$, then $\mathcal{D}_{\mathcal{E}}$ is complete.*

Proof. Let G be a triangle-free 5-regular graph containing a perfect matching. Suppose that G is the underlying graph of a mixed graph $\bar{G}$ such that $A(\bar{G})$ induces a 2-factor with an Eulerian orientation $\mathcal{E}$, and $\bar{G}$ admits an $\mathcal{E}$-canonical $\{P_5, T_5\}$-decomposition $\mathcal{D}_{\mathcal{E}}$. We will prove that $\mathcal{D}_{\mathcal{E}}$ is a complete decomposition.

Items (i) and (ii) of Definition 3.4 follow from the fact that $\mathcal{D}_{\mathcal{E}}$ is $\mathcal{E}$-canonical. Thus, we only have to prove that every TP-couple of $\mathcal{D}_{\mathcal{E}}$ is solvable. Let $(\bar{B}, \bar{C})$ be a TP-couple of $\mathcal{D}_{\mathcal{E}}$, where $\bar{B} = b_0b_1b_2b_3b_4b_1$, $\bar{C} = c_0c_1c_2c_3c_4c_5$. Note that we can assume that $\text{cv}(\bar{B}) = b_4 = c_1$ (or c_4, by symmetry). In fact, suppose that b_4 is c_2 (or c_3, by symmetry) and let e be the directed edge leaving b_4. Since $\bar{C}$ is canonical, e is not in $\bar{C}$. Thus, let $\bar{X}$ be the element of $\mathcal{D}_{\mathcal{E}}$ that contains e. Since, by item (ii), e is internal, the vertex b_4 would have degree six, a contradiction.

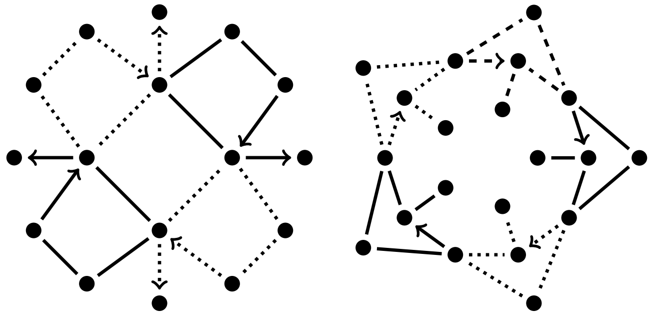
Note that, by (ii), $\bar{C}$ is the element of $\mathcal{D}$ that contains the directed edge leaving b_4. Thus, b_4c_0 is a directed edge of $\bar{C}$. (Note that there is no TP-couple $(\bar{B}, \bar{D})$ with $\bar{D} \neq \bar{C}$, otherwise we would have $d_G(b_4) > 5$. Thus, the outgoing edge of b_4 has to be a directed edge of $\bar{C}$, implying that there is no loss of generality assuming that $b_4 = c_1$). Hence, $(\bar{B}, \bar{C})$ is a TP-couple in position 1. Since $\mathcal{D}_{\mathcal{E}}$ is $\mathcal{E}$-canonical, $\bar{B}$ contains precisely the directed edges b_2b_1 and b_1b_0, and $\bar{C}$ contains precisely the directed edges c_1c_0 and c_4c_5.

To conclude the proof that $(\bar{B}, \bar{C})$ is solvable, we have to show that $\bar{B}' = b_0b_1b_2b_3b_4c_0$ and $\bar{C}' = b_1c_1c_2c_3c_4c_5$ are edge-disjoint paths in $\bar{G}$. For that, it suffices to prove that $c_0 \notin \{b_0, b_1, b_2, b_3, b_4\}$ and $b_1 \notin \{c_1, c_2, c_3, c_4, c_5\}$. Let us start by proving that $c_0 \notin \{b_0, b_1, b_2, b_3, b_4\}$. Since $b_4c_0 \in A(\bar{C})$, we know that $c_0 \notin \{b_1, b_3, b_4\}$, because $\bar{B}$ and $\bar{C}$ are edge-disjoint (and G is simple). If $c_0 \in \{b_0, b_2\}$, then the set $\{c_0, b_1, b_4\}$ induces a triangle, a contradiction.

Since $b_1c_1 \in E(\bar{B})$, we know that $b_1 \notin \{c_1, c_2\}$. If $b_1 = c_3$, then the set $\{b_1, c_1, c_2\}$ induces a triangle in G, a contradiction. Since $A(\bar{G})$ induces a 2-factor with the Eulerian orientation $\mathcal{E}$, every vertex of $\bar{G}$ is incident to exactly one ingoing edge and one outgoing edge. Then, since the ingoing edge incident to b_1 is an edge of $\bar{B}$ (namely, b_2b_1), we know that $b_1 \neq c_5$, otherwise b_1 would have two ingoing edges (c_4b_1 and b_2b_1). Finally, note that $b_1 \neq c_4$, otherwise, b_1 would have two outgoing edges (b_1b_0 and b_1c_5). Therefore, we conclude that $(\bar{B}, \bar{C})$ is solvable, and hence the decomposition $\mathcal{D}_{\mathcal{E}}$ is complete. □

The following corollary, the main result of this section, follows directly from Lemmas 3.3 and 3.5.

Corollary 3.6. *Let G be a triangle-free 5-regular graph containing a perfect matching. Then G is the underlying graph of a mixed graph $\bar{G}$ such that $A(\bar{G})$ induces a 2-factor with an Eulerian orientation, and $\bar{G}$ admits a complete $\{P_5, T_5\}$-decomposition.*



(a) A cycle of TT-couples in position 1. (b) A cycle of TT-couples in position 2.

Fig. 3. Examples of cycles of TT-couples.

4. Disentanglement of couples

Given a $\{P_5, T_5\}$-decomposition $\mathcal{D}$ of a mixed 5-regular graph $\bar{G}$, we denote by $\tau(\mathcal{D})$ the number of copies of T_5 in $\mathcal{D}$. To prove our main result, we start with a complete $\{P_5, T_5\}$-decomposition $\mathcal{D}$ of $\bar{G}$ and show that, if $\tau(\mathcal{D}) \neq 0$, then TP-couples and sets of TT-couples can be disentangled, yielding a pure P_5-decomposition. As we will see, it is simpler to disentangle TP-couples, but to deal with TT-couples, we have to introduce further definitions.

Let $k \geq 3$ and let $\bar{B}_1, \ldots, \bar{B}_k$ be copies of T_5 in $\mathcal{D}$. We say that $\bar{B}_1 \cdots \bar{B}_k$ is a *sequence of couples* of $\mathcal{D}$ if $(\bar{B}_i, \bar{B}_{i+1})$ is a TT-couple for $1 \leq i \leq k-1$. If $(\bar{B}_k, \bar{B}_1)$ is also a TT-couple, then we say that $\bar{B}_1 \cdots \bar{B}_k$ is a *cycle of couples* of $\mathcal{D}$. Furthermore, if such a sequence (resp. cycle) is composed only by TT-couples in position i, then we say that it is *in position i*, for $i = 1, 2$ (see Fig. 3). A sequence (resp. cycle) of couples is called *mixed* if it contains couples in positions 1 and 2.

The main results of this section are Lemmas 4.1, 4.2 and 4.6. Most of the subsequent proofs are easier understood by drawing the (sequence or cycle of) couples of the given $\{P_5, T_5\}$-decomposition and redrawing the paths that we claim to define a P_5-decomposition. The reader may convince himself without following the detailed proofs. In these proofs, all additions on the indices are taken modulo k.

We note that, although in this paper we are mainly interested in complete $\{P_5, T_5\}$-decompositions, some of the next lemmas also hold for $\{P_5, T_5\}$-decompositions that are not necessarily complete (as in the case of Lemmas 4.1 and 4.4).

4.1. TP-couples in position 1

We shall prove that, under certain conditions, TP-couples $(\bar{B}, \bar{C})$ in position 1 are solvable.

Lemma 4.1. *Let $\bar{G}$ be a triangle-free 5-regular mixed graph and let $\mathcal{D}$ be a $\{P_5, T_5\}$-decomposition of $\bar{G}$. Let $(\bar{B}, \bar{C})$ be a TP-couple of $\mathcal{D}$ in position 1. If $\bar{C}$ is a roofed path in $\bar{G}$ such that its roof is not in $\bar{B}$, then $(\bar{B}, \bar{C})$ is solvable.*

Proof. Let $\bar{G}$, $\mathcal{D}$, and $(\bar{B}, \bar{C})$ be as stated in the hypothesis of the lemma. Suppose $\bar{B} = b_0b_1b_2b_3b_4b_1$, $\mathrm{cv}(\bar{B}) = b_4 = c_1$, and $\bar{C} = c_0c_1c_2c_3c_4c_5$.

We have to prove that $\bar{B}' = b_0b_1b_2b_3b_4c_0$ and $\bar{C}' = b_1c_1c_2c_3c_4c_5$ are paths of length five. For that, it suffices to prove that $c_0 \notin \{b_0, b_1, b_2, b_3, b_4\}$ and $b_1 \notin \{c_1, c_2, c_3, c_4, c_5\}$.

Firstly, we prove that $c_0 \notin \{b_0, b_1, b_2, b_3, b_4\}$. Since c_0b_4 is an edge of $\bar{C}$, we know that $c_0 \notin \{b_1, b_3, b_4\}$, because $\bar{B}$ and $\bar{C}$ are edge-disjoint. If $c_0 \in \{b_0, b_2\}$, then the set $\{c_0, b_1, b_4\}$ induces a triangle, a contradiction.

Since b_1c_1 is an edge of $\bar{B}$, we know that $b_1 \notin \{c_1, c_2\}$. If $b_1 = c_3$, then the set $\{b_1, c_1, c_2\}$ induces a triangle, a contradiction. If $b_1 = c_4$, then c_1c_4 is an edge of $\bar{B}$, contradicting the fact that the roof of $\bar{C}$ is not in $\bar{B}$. If $b_1 = c_5$, then the set $\{b_1, c_4, c_1\}$ induces a triangle, a contradiction. □

4.2. TT-couples in position 1

The next result shows that if a complete $\{P_5, T_5\}$-decomposition of a triangle-free 5-regular mixed graph contains a cycle of TT-couples in position 1, then such a cycle can be decomposed into special copies of P_5.

Lemma 4.2. *Let $\bar{G}$ be a triangle-free 5-regular mixed graph and let $\mathcal{D}$ be a complete $\{P_5, T_5\}$-decomposition of $\bar{G}$. If $\bar{B}_1 \cdots \bar{B}_k$ is a cycle of TT-couples of $\mathcal{D}$ in position 1, then $\bigcup_{i=1}^{k} \bar{B}_i$ admits a P_5-decomposition $\mathcal{P}$ such that all directed edges of $\bigcup_{i=1}^{k} \bar{B}_i$ are internal to $\mathcal{P}$.*

Proof. Let $\bar{G}$ and $\mathcal{D}$ be as stated in the lemma. Let $\bar{B}_1 \cdots \bar{B}_k$ be a cycle of couples of $\mathcal{D}$ in position 1, where $\bar{B}_i = b_{i,0}b_{i,1}b_{i,2}b_{i,3}b_{i,4}b_{i,1}$ and $\mathrm{cv}(\bar{B}_i) = b_{i,4} = b_{i+1,1}$ for $1 \leq i \leq k$. For every $1 \leq i \leq k$, let $\bar{B}'_i = b_{i,0}b_{i,1}b_{i,2}b_{i,3}b_{i,4}b_{i+1,4}$. We shall prove that $\{\bar{B}'_1, \ldots, \bar{B}'_k\}$ is a P_5-decomposition of $\bigcup_{i=1}^{k} \bar{B}_i$ such that all directed edges of $\bigcup_{i=1}^{k} \bar{B}_i$ are internal to $\{\bar{B}'_1, \ldots, \bar{B}'_k\}$.

To prove that $\bar{B}'_i$ is a path of length five, it suffices to show that $b_{i+1,4} \notin \{b_{i,4}, b_{i,3}, b_{i,2}, b_{i,1}, b_{i,0}\}$. Since B_i and B_{i+1} are edge-disjoint, $b_{i+1,4} \notin \{b_{i,4}, b_{i,3}, b_{i,1}\}$. We also know that $b_{i+1,4} \notin \{b_{i,2}, b_{i,0}\}$, otherwise $\{b_{i+1,4}, b_{i,1}, b_{i,4}\}$ would induce a triangle. It is easy to check that, by the construction of $\bar{B}'_1, \ldots, \bar{B}'_k$, they are pairwise edge-disjoint, and all directed edges of $\bigcup_{i=1}^{k} \bar{B}_i$ are internal to $\{\bar{B}'_1, \ldots, \bar{B}'_k\}$. □

4.3. TT-*couples in position 2*

The aim of this subsection is to prove Lemma 4.6, which is a version of Lemma 4.2 for TT-couples in position 2. Before we state this lemma, we prove some auxiliary results.

Lemma 4.3. *Let $\bar{G}$ be a triangle-free 5-regular mixed graph and let $\mathcal{D}$ be a complete $\{P_5, T_5\}$-decomposition of $\bar{G}$. Then, for every vertex v, there is exactly one element $B \in \mathcal{D}$ such that $d_B(v)$ is odd.*

Proof. Let $\bar{G} = (V, E, A)$ and $\mathcal{D}$ be as stated in the claim. For each vertex v in V, let $\mathcal{D}(v)$ be the number of elements B of $\mathcal{D}$ such that $d_B(v)$ is odd. As G is 5-regular, we have that $\mathcal{D}(v) \geq 1$.

Since each element D of $\mathcal{D}$ contains exactly two vertices of odd degree in D, we have that $\sum_{v\in V} \mathcal{D}(v) = 2|\mathcal{D}| = n$. Suppose there is a vertex x such that $\mathcal{D}(x) \geq 2$. Then, we have

$$n = \sum_{v\in V} \mathcal{D}(v) = \mathcal{D}(x) + \sum_{v\in V\setminus\{x\}} \mathcal{D}(v) \geq 2 + n - 1 > n,$$

a contradiction. □

Lemma 4.4. *Let $\bar{G}$ be a triangle-free 5-regular mixed graph and let $\mathcal{D}$ be a $\{P_5, T_5\}$-decomposition of $\bar{G}$. If $(\bar{B}, \bar{C})$ is a* TT-*couple of $\mathcal{D}$ in position* 2*, where $\bar{B} = b_0b_1b_2b_3b_4b_1$ and $\bar{C} = c_0c_1c_2c_3c_4c_1$, then $\bar{B}' = b_0b_1b_2b_3b_4c_1$ and $\bar{C}' = c_0c_1c_4c_3c_2b_1$ are roofed paths in $\bar{B} \cup \bar{C}$. Furthermore, all directed edges of $\bar{B}'$ and $\bar{C}'$ are internal.*

Proof. Let $\bar{G}$, $\mathcal{D}$, and $(\bar{B}, \bar{C})$ be as stated in the lemma, with $\mathrm{cv}(\bar{B}) = b_4 = c_2$. Let $\bar{B}' = b_0b_1b_2b_3b_4c_1$ and $\bar{C}' = c_0c_1c_4c_3c_2b_1$. We claim that $\bar{B}'$ and $\bar{C}'$ are roofed paths. Since b_1b_4 and c_1c_2 belong to $E(\bar{G}) \cup A(\bar{G})$, we only have to prove that $\bar{B}'$ and $\bar{C}'$ are paths, i.e., $c_1 \notin \{b_0, b_1, b_2, b_3, b_4\}$ and $b_1 \notin \{c_0, c_1, c_2, c_3, c_4\}$.

Since $c_1b_4 \in E(\bar{C})$, we know that $c_1 \neq b_4$. Furthermore, $c_1 \notin \{b_1, b_3\}$ because $\bar{B}$ and $\bar{C}$ are edge-disjoint. We also know that $c_1 \notin \{b_0, b_2\}$, otherwise $\{c_1, b_1, b_4\}$ would induce a triangle in G, a contradiction. Analogously, since $b_1c_2 \in E(\bar{B})$, we know that $b_1 \neq c_2$. Furthermore, $b_1 \notin \{c_1, c_3\}$ because $\bar{B}$ and $\bar{C}$ are edge-disjoint. We also know that $b_1 \notin \{c_0, c_4\}$, otherwise $\{b_1, c_1, c_2\}$ would induce a triangle in G, a contradiction. It is clear that all directed edges of $\bar{B}'$ and $\bar{C}'$ are internal, as required. □

The next lemma refer to sequences $\bar{B}_1\bar{B}_2\bar{B}_3$ of TT-couples in position 2 of a complete $\{P_5, T_5\}$-decomposition. It shows when such sequences can be decomposed into paths of length five. In this lemma, and thereafter, the following terminology will be useful. If $\bar{T} = v_0v_1v_2v_3v_4$ is a copy of T_5, we say that v_0 is the *pending vertex* of $\bar{T}$.

Lemma 4.5. *Let $\bar{G}$ be a triangle-free 5-regular mixed graph and let $\mathcal{D}$ be a complete $\{P_5, T_5\}$-decomposition of $\bar{G}$. Let $\bar{B}_1\bar{B}_2\bar{B}_3$ be a sequence of* TT-*couples of $\mathcal{D}$ in position* 2*. Then, in each of the following cases, $\bar{B}_1 \cup \bar{B}_2 \cup \bar{B}_3$ admits a P_5-decomposition $\mathcal{P}$ such that all directed edges of $\bar{B}_1 \cup \bar{B}_2 \cup \bar{B}_3$ are internal to $\mathcal{P}$.*

(a) *when the pending vertex of $\bar{B}_1$ is the connection-vertex of $\bar{B}_2$;*
(b) *when the pending vertex of $\bar{B}_3$ is the connection-vertex of $\bar{B}_1$;*
(c) *when the pending vertex of $\bar{B}_2$ is not the connection-vertex of $\bar{B}_3$, and the pending vertex of $\bar{B}_3$ is not the connection-vertex of $\bar{B}_1$.*

Proof. Let $\bar{G}$ and $\mathcal{D}$ be as in the hypothesis of the lemma, and let $\bar{B}_1\bar{B}_2\bar{B}_3$ be a sequence of TT-couples of $\mathcal{D}$ in position 2, where $\bar{B}_i = b_{i,0}b_{i,1}b_{i,2}b_{i,3}b_{i,4}b_{i,1}$, for $i = 1, 2, 3$.

Proof of Case (a). In this case, $b_{1,0} = \mathrm{cv}(\bar{B}_2)$, $\mathrm{cv}(\bar{B}_1) = b_{1,4} = b_{2,2}$ and $\mathrm{cv}(\bar{B}_2) = b_{2,4} = b_{3,2}$. Let $\bar{P}_1 = b_{1,1}b_{1,2}b_{1,3}b_{1,4}b_{2,3}b_{2,4}$, $\bar{P}_2 = b_{2,0}b_{2,1}b_{2,2}b_{1,1}b_{2,4}b_{3,1}$, and $\bar{P}_3 = b_{2,1}b_{2,4}b_{3,3}b_{3,4}b_{3,1}b_{3,0}$. We shall prove that $\mathcal{P} = \{\bar{P}_1, \bar{P}_2, \bar{P}_3\}$ is a P_5-decomposition of $\bar{B}_1 \cup \bar{B}_2 \cup \bar{B}_3$ such that all of its directed edges are internal.

For $\bar{P}_1$, we have to prove that $b_{2,3}, b_{2,4} \notin \{b_{1,1}, b_{1,2}, b_{1,3}, b_{1,4}\}$. Since $b_{1,0} = b_{2,4}$, we know that $b_{2,4} \notin \{b_{1,1}, b_{1,2}, b_{1,3}, b_{1,4}\}$, because $|V(\bar{B}_1)| = 5$. Since $b_{2,3}b_{1,4}$ is an edge of $\bar{G}$ and $\bar{B}_1$ and $\bar{B}_2$ are edge-disjoint, $b_{2,3} \notin \{b_{1,1}, b_{1,3}, b_{1,4}\}$. Finally, $b_{2,3} \neq b_{1,2}$, otherwise $\{b_{2,3}, b_{1,3}, b_{1,4}\}$ would induce a triangle in G. Clearly, $b_{1,2}b_{1,1}$, which is the unique directed edge of $\bar{P}_1$, is internal to $\mathcal{P}$.

For $\bar{P}_2$, we have to prove that $b_{1,1}, b_{3,1} \notin \{b_{2,0}, b_{2,1}, b_{2,2}, b_{2,4}\}$ and that $b_{1,1} \neq b_{3,1}$. Clearly, $b_{1,1} \neq b_{3,1}$, otherwise $b_{3,1}$ would have degree six. Since $b_{1,1}b_{2,2}$ is an edge of $\bar{G}$ and $\bar{B}_1$ and $\bar{B}_2$ are edge-disjoint, $b_{1,1} \notin \{b_{2,1}, b_{2,2}\}$. If $b_{1,1} = b_{2,0}$ or $b_{1,1} = b_{2,4}$, then $\{b_{1,1}, b_{2,1}, b_{2,2}\}$ would induce a triangle in G. Since $b_{3,1}b_{2,4}$ is an edge of $\bar{G}$ and $\bar{B}_2$ and $\bar{B}_3$ are edge-disjoint, then $b_{3,1} \notin \{b_{2,1}, b_{2,4}\}$. If $b_{3,1} = b_{2,0}$ or $b_{3,1} = b_{2,2}$, then $\{b_{3,1}, b_{2,1}, b_{2,4}\}$ would induce a triangle in G. Clearly, $b_{2,1}b_{2,0}$, $b_{2,2}b_{2,1}$, $b_{1,1}b_{1,0}$ and $b_{3,2}b_{3,1}$, which are the possible directed edges of $\bar{P}_2$, are internal to $\mathcal{P}$.

For $\bar{P}_3$, we have to prove that $b_{2,1} \notin \{b_{3,0}, b_{3,1}, b_{3,4}, b_{3,3}, b_{3,2}\}$. Since $b_{2,1}b_{3,2}$ is an edge of $\bar{G}$ and $\bar{B}_2$ and $\bar{B}_3$ are edge-disjoint, then $b_{2,1} \notin \{b_{3,2}, b_{3,3}, b_{3,1}\}$. If $b_{2,1} = b_{3,4}$ or $b_{2,1} = b_{3,0}$, then $\{b_{2,1}, b_{3,2}, b_{3,1}\}$ would induce a triangle. Clearly, $b_{3,1}b_{3,0}$, which is the only possible directed edge of $\bar{P}_3$, is internal to $\bar{P}_3$. This completes the proof of case (a).

Proof of Case (b). In this case, $b_{3,0} = \text{cv}(\bar{B}_1)$, $\text{cv}(\bar{B}_1) = b_{1,4} = b_{2,2}$ and $\text{cv}(\bar{B}_2) = b_{2,4} = b_{3,2}$. Take $\bar{P}_1 = b_{1,0}b_{1,1}b_{1,2}b_{1,3}b_{1,4}b_{2,1}$, $\bar{P}_2 = b_{2,0}b_{2,1}b_{2,4}b_{3,1}b_{2,2}b_{1,1}$, and $\bar{P}_3 = b_{3,1}b_{3,4}b_{3,3}b_{3,2}b_{2,3}b_{2,2}$. We claim that $\mathcal{P} = \{\bar{P}_1, \bar{P}_2, \bar{P}_3\}$ is a P_5-decomposition with the desired properties.

For $\bar{P}_1$, we have to prove that $b_{2,1} \notin \{b_{1,0}, b_{1,1}, b_{1,2}, b_{1,3}, b_{1,4}\}$. Since $b_{2,1}b_{1,4}$ is an edge of G, and $\bar{B}_2$ and $\bar{B}_3$ are edge-disjoint, $b_{2,1} \notin \{b_{1,1}, b_{1,3}, b_{1,4}\}$. If $b_{2,1} = b_{1,0}$ or $b_{2,1} = b_{1,2}$, then $\{b_{2,1}, b_{1,1}, b_{1,4}\}$ would induce a triangle. Clearly, $b_{1,4}b_{2,1}$, $b_{1,1}b_{1,0}$ and $b_{1,2}b_{1,1}$, the only possible directed edges of $\bar{P}_1$, are internal to $\mathcal{P}$.

For $\bar{P}_2$, we have to show that $b_{3,1}, b_{1,1} \notin \{b_{2,0}, b_{2,1}, b_{2,2}, b_{2,4}\}$ and that $b_{1,1} \neq b_{3,1}$. Since $b_{1,1}b_{2,2}$ is an edge of $\bar{G}$, $\bar{B}_1$ and $\bar{B}_2$ are edge-disjoint, and $\bar{B}_1$ and $\bar{B}_3$ are edge-disjoint, $b_{1,1} \notin \{b_{2,1}, b_{2,2}, b_{3,1}\}$. If $b_{1,1} = b_{2,0}$ or $b_{1,1} = b_{2,4}$, then $\{b_{1,1}, b_{2,1}, b_{2,2}\}$ would induce a triangle in G. Since $b_{3,1}b_{2,4}$ is an edge of $\bar{G}$, and $\bar{B}_2$ and $\bar{B}_3$ are edge-disjoint, $b_{3,1} \notin \{b_{2,1}, b_{2,4}\}$. If $b_{3,1} = b_{2,0}$ or $b_{3,1} = b_{2,2}$, then $\{b_{3,1}, b_{2,1}, b_{2,4}\}$ would induce a triangle in G. Clearly, $b_{2,1}b_{2,0}$, $b_{3,1}b_{3,0}$ and $b_{3,2}b_{3,1}$, which are the only possible directed edges of $\bar{P}_2$, are internal to $\mathcal{P}$.

For $\bar{P}_3$, we have to prove that $b_{2,2}, b_{2,3} \notin \{b_{3,1}, b_{3,2}, b_{3,3}, b_{3,4}\}$. Since $b_{3,0} = b_{2,2}$, we know that $b_{2,2} \notin \{b_{3,1}, b_{3,2}, b_{3,3}, b_{3,4}\}$, because $|V(\bar{B}_3)| = 5$. Since $b_{2,3}b_{2,4}$ is an edge of $\bar{G}$ and $\bar{B}_2$ and $\bar{B}_3$ are edge-disjoint, we have that $b_{2,3} \notin \{b_{3,1}, b_{3,2}, b_{3,3}\}$. Finally, $b_{2,3} \neq b_{3,4}$, otherwise $\{b_{2,3}, b_{2,4}, b_{3,3}\}$ would induce a triangle in G. Clearly, no edge of $\bar{P}_3$ is directed.

Proof of Case (c). In this case, $\text{cv}(\bar{B}_1) = b_{1,4} = b_{2,2}$, $\text{cv}(\bar{B}_2) = b_{2,4} = b_{3,2}$, $b_{2,0} \neq b_{3,4}$ and $b_{3,0} \neq b_{1,4}$. Let $\bar{P}_1 = b_{1,0}b_{1,1}b_{1,2}b_{1,3}b_{1,4}b_{2,1}$, $\bar{P}_2 = b_{1,1}b_{1,4}b_{2,3}b_{2,4}b_{3,1}b_{3,0}$ and $\bar{P}_3 = b_{2,0}b_{2,1}b_{2,4}b_{3,3}b_{3,4}b_{3,1}$. We shall prove that $\mathcal{P} = \{\bar{P}_1, \bar{P}_2, \bar{P}_3\}$ is a P_5-decomposition with the desired properties. Since it is clear that all directed edges of $\bar{B}_1 \cup \bar{B}_2 \cup \bar{B}_3$ are internal to $\mathcal{P}$, we only have to prove that $\bar{P}_1$, $\bar{P}_2$ and $\bar{P}_3$ are paths of length five.

For $\bar{P}_1$, we have to prove that $b_{2,1} \notin \{b_{1,0}, b_{1,1}, b_{1,2}, b_{1,3}, b_{1,4}\}$. Since $b_{2,2} = b_{1,4}$ and $\bar{B}_1$ and $\bar{B}_2$ are edge-disjoint, we know that $b_{2,1} \notin \{b_{1,1}, b_{1,3}, b_{1,4}\}$. Note that $b_{2,1} \notin \{b_{1,0}, b_{1,2}\}$, otherwise $\{b_{1,1}, b_{1,4}, b_{2,1}\}$ would induce a triangle in G.

For $\bar{P}_2$, we have to prove that $b_{1,1} \notin \{b_{1,4}, b_{2,3}, b_{2,4}, b_{3,1}, b_{3,0}\}$ and $b_{3,0}, b_{3,1} \notin \{b_{1,4}, b_{2,3}, b_{2,4}\}$. Let us start by analyzing $b_{1,1}$. Since $b_{2,2} = b_{1,4}$ and $\bar{B}_1$ and $\bar{B}_2$ are edge-disjoint, we know that $b_{1,1} \notin \{b_{1,4}, b_{2,3}\}$. Note that $b_{1,1} \neq b_{2,4}$, otherwise $\{b_{1,4}, b_{2,1}, b_{1,1}\}$ would induce a triangle in G. By Lemma 4.3, we have that $b_{1,1} \notin \{b_{3,1}, b_{3,0}\}$. We have to analyze $b_{3,1}$. Since $b_{3,2} = b_{2,4}$, we know that $b_{3,1} \notin \{b_{2,3}, b_{2,4}\}$. We also know that $b_{3,1} \neq b_{1,4}$, otherwise $\{b_{2,4}, b_{2,1}, b_{3,1}\}$ would induce a triangle in G. To conclude, we have to analyze $b_{3,0}$. By assumption, we know that $b_{3,0} \neq b_{1,4}$. Since $|V(\bar{B}_3)| = 5$, we know that $b_{3,0} \neq b_{2,4}$, and we know that $b_{3,0} \neq b_{2,3}$, otherwise $\{b_{3,1}, b_{2,4}, b_{3,0}\}$ would induce a triangle in G.

For $\bar{P}_3$, we have to prove that $b_{2,0}, b_{2,1} \notin \{b_{2,4}, b_{3,3}, b_{3,4}, b_{3,1}\}$. Let us start by analyzing $b_{2,0}$. Since $|V(\bar{B}_2)| = 5$, we have $b_{2,0} \neq b_{2,4}$. Note that $b_{2,0} \notin \{b_{3,1}, b_{3,3}\}$, otherwise $\{b_{2,1}, b_{2,4}, b_{2,0}\}$ would induce a triangle in G. Finally, by assumption, $b_{2,0} \neq b_{3,4}$. Now let us analyze $b_{2,1}$. Since $b_{2,4} = b_{3,2}$, we know that $b_{2,1} \notin \{b_{2,4}, b_{3,1}, b_{3,3}\}$. Note that $b_{2,1} \neq b_{3,4}$, otherwise $\{b_{2,4}, b_{3,1}, b_{2,1}\}$ would induce a triangle in G.

Clearly, the paths $\bar{P}_1$, $\bar{P}_2$ and $\bar{P}_3$ are edge-disjoint. This concludes the proof of case (c), and therefore, of the lemma. □

The next result, which is the main result of Section 4.3, states that a cycle of couples in position 2 of a complete $\{P_5, T_5\}$-decomposition of a mixed graph admits a P_5-decomposition.

Lemma 4.6. *Let $\bar{G}$ be a triangle-free 5-regular mixed graph and let $\mathcal{D}$ be a complete $\{P_5, T_5\}$-decomposition of $\bar{G}$. If $\bar{B}_1 \cdots \bar{B}_k$ is a cycle of couples of $\mathcal{D}$ in position 2, then $\bigcup_{i=1}^{k} \bar{B}_i$ admits a P_5-decomposition $\mathcal{P}$ such that all directed edges of $\bar{B}_1 \cup \bar{B}_2 \cup \bar{B}_3$ are internal to $\mathcal{P}$.*

Proof. Let $\bar{G}$, $\mathcal{D}$, and $\bar{B}_1 \cdots \bar{B}_k$ be as stated in the lemma; and let $\bar{B}_i = b_{i,0}b_{i,1}b_{i,2}b_{i,3}b_{i,4}b_{i,1}$ and $\text{cv}(\bar{B}_i) = b_{i,4} = b_{i+1,2}$ for $i = 1, \ldots, k$. We divide the proof in two cases.

Case 1: *k is even.*

Applying Lemma 4.4 to every TT-couple $(\bar{B}_i, \bar{B}_{i+1})$, for $i = 1, 3, 5, \ldots, k-1$, we obtain that $\bar{B}'_i = b_{i,0}b_{i,1}b_{i,2}b_{i,3}b_{i,4}b_{i+1,1}$ and $\bar{B}'_{i+1} = b_{i+1,0}b_{i+1,1}b_{i+1,4}b_{i+1,3}b_{i+1,2}b_{i,1}$ are roofed paths and all directed edges of $\bar{B}'_i$ and $\bar{B}'_{i+1}$ are internal. Thus, clearly $\bigcup_{i=1}^{k} \bar{B}_i$ admits a P_5-decomposition $\mathcal{P}$ such that all directed edges of $\bigcup_{i=1}^{k} \bar{B}_i$ are internal to $\mathcal{P}$.

Case 2: *k is odd.*

Suppose there exists $i \in \{1, \ldots, k\}$ such that $b_{i,0} = b_{i+1,4}$. We may suppose w.l.o.g. that $i = 1$. Applying Lemma 4.5(a) to the sequence $\bar{B}_1\bar{B}_2\bar{B}_3$, we obtain a P_5-decomposition $\mathcal{P}' = \{\bar{B}'_1, \bar{B}'_2, \bar{B}'_3\}$ of $\bar{B}_1 \cup \bar{B}_2 \cup \bar{B}_3$ such that all directed edges of $\bar{B}_1 \cup \bar{B}_2 \cup \bar{B}_3$ are internal to $\mathcal{P}'$. If $k = 3$, then the lemma is proved. Thus, we may assume that $k \geq 5$. In this case, $\bar{B}_4\bar{B}_5 \cdots \bar{B}_k$ is a sequence in position 2 of even size (size $k-3$). Then, applying Lemma 4.4 to every TT-couple $(\bar{B}_i, \bar{B}_{i+1})$, for $i = 4, 6, 8, \ldots, k-1$, as in Case 1, we obtain a P_5-decomposition $\mathcal{R}'$ of $\bar{B}_4 \cup \bar{B}_5 \cup \cdots \cup \bar{B}_k$ such that all directed edges of $\bar{B}_4 \cup \bar{B}_5 \cup \cdots \cup \bar{B}_k$ are internal to $\mathcal{R}'$. Then $\mathcal{P} = \mathcal{P}' \cup \mathcal{R}'$ is a P_5-decomposition of the cycle $\bar{B}_1 \cdots \bar{B}_k$ such that all directed edges of $\bigcup_{i=1}^{k} \bar{B}_i$ are internal to $\mathcal{P}$.

Suppose there exists $i \in \{1, \ldots, k\}$ such that $b_{i,0} = b_{i-2,4}$. We may suppose w.l.o.g. that $i = 3$. Then, applying Lemma 4.5(b) to the sequence $\bar{B}_1\bar{B}_2\bar{B}_3$, we obtain a P_5-decomposition $\mathcal{P}' = \{\bar{B}'_1, \bar{B}'_2, \bar{B}'_3\}$ of $\bar{B}_1 \cup \bar{B}_2 \cup \bar{B}_3$ such that all directed edges of $\bar{B}_1 \cup \bar{B}_2 \cup \bar{B}_3$ are internal to $\mathcal{P}'$. If $k = 3$, then the lemma is proved. Thus, we may assume that $k \geq 5$. In this case, the sequence $\bar{B}_4\bar{B}_5\cdots\bar{B}_k$ is a sequence in position 2 of even size (size $k-3$). Then, applying Lemma 4.4 to every TT-couple $(\bar{B}_i, \bar{B}_{i+1})$, for $i = 4, 6, 8, \dots, k-1$, as in Case 1, we obtain a P_5-decomposition $\mathcal{R}'$ of $\bar{B}_4 \cup \bar{B}_5 \cup \cdots \cup \bar{B}_k$ such that all directed edges of $\bar{B}_4 \cup \bar{B}_5 \cup \cdots \cup \bar{B}_k$ are internal to $\mathcal{R}'$. Then $\mathcal{P} = \mathcal{P}' \cup \mathcal{R}'$ is a P_5-decomposition of the cycle $\bar{B}_1\cdots\bar{B}_k$ such that all directed edges of $\bigcup_{i=1}^{k}\bar{B}_i$ are internal to $\mathcal{P}$.

We assume now that for every $i \in \{1, \dots, k\}$ we have that $b_{i,0} \neq b_{i+1,4}$ and $b_{i,0} \neq b_{i-2,4}$. Thus, we have that $b_{2,0} \neq b_{3,4}$ and $b_{3,0} \neq b_{1,4}$. Then, applying Lemma 4.5(c) to the sequence $\bar{B}_1\bar{B}_2\bar{B}_3$, we obtain a P_5-decomposition $\mathcal{P}'$ of $\bar{B}_1 \cup \bar{B}_2 \cup \bar{B}_3$ such that all directed edges of $\bar{B}_1 \cup \bar{B}_2 \cup \bar{B}_3$ are internal to $\mathcal{P}'$. If $k = 3$, then the lemma is proved. Therefore, we may assume that $k \geq 5$. In this case, $\bar{B}_4\bar{B}_5\cdots\bar{B}_k$ is a sequence in position 2 of even size. Then, applying Lemma 4.4 to every TT-couple $(\bar{B}_i, \bar{B}_{i+1})$, for $i = 4, 6, 8, \dots, k-1$, as in Case 1, we obtain a P_5-decomposition $\mathcal{R}'$ of $\bar{B}_4 \cup \bar{B}_5 \cup \cdots \cup \bar{B}_k$ such that all directed edges of $\bar{B}_4 \cup \bar{B}_5 \cup \cdots \cup \bar{B}_k$ are internal to $\mathcal{R}'$. Then $\mathcal{P} = \mathcal{P}' \cup \mathcal{R}'$ is a P_5-decomposition of the cycle $\bar{B}_1\cdots\bar{B}_k$ such that all directed edges of $\bigcup_{i=1}^{k}\bar{B}_i$ are internal to $\mathcal{P}$. □

5. Main result

We start proving a lemma which plays an important role in the proof of the main result. For that, we have to introduce some concepts. Given a mixed graph $\bar{G}$ and a complete $\{P_5, T_5\}$-decomposition $\mathcal{D}$ of $\bar{G}$, we say that a copy of T_5 in $\mathcal{D}$ is an *initial element* of $\mathcal{D}$ if it is not the base of any couple in $\mathcal{D}$. If $\mathcal{D}$ has the least number of copies of T_5 among all complete $\{P_5, T_5\}$-decompositions of $\bar{G}$, then $\mathcal{D}$ is called a *minimal complete* $\{P_5, T_5\}$*-decomposition* of $\bar{G}$. Furthermore, if there is at least one copy of T_5 in $\mathcal{D}$, that is, $\tau(\mathcal{D}) \neq 0$, then we say that $\mathcal{D}$ is *nontrivial*. Such decompositions have some properties that are summarized in the next lemma.

Lemma 5.1. *Let G be a triangle-free 5-regular graph. Suppose that G is the underlying graph of a mixed graph $\bar{G}$ such that $A(\bar{G})$ induces a 2-factor with an Eulerian orientation. If there is a nontrivial minimal complete $\{P_5, T_5\}$-decomposition $\mathcal{D}$ of $\bar{G}$, then the following properties hold.*

(a) *Every copy of T_5 in $\mathcal{D}$ is the top of exactly one couple of $\mathcal{D}$;*
(b) *$\mathcal{D}$ contains no initial element;*
(c) *Every copy of T_5 in $\mathcal{D}$ is the base of exactly one couple of $\mathcal{D}$. Furthermore, every copy of P_5 in $\mathcal{D}$ is not the base of any couple of $\mathcal{D}$.*

Proof. We divide the proof in three parts, one part for each item. For all cases, let G and $\mathcal{D}$ be as stated in the lemma and note that $\mathrm{cv}(\bar{B}) \neq \mathrm{cv}(\bar{B}')$ for all pairs of elements $B, B' \in \mathcal{D}$ with $B \neq B'$. In fact, suppose by contradiction that $\mathrm{cv}(\bar{B}) = \mathrm{cv}(\bar{B}')$ for some $\bar{B}, \bar{B}' \in \mathcal{D}$. Let $\bar{X}$ be an element of $\mathcal{D}$ such that $(\bar{B}, \bar{X})$ and $(\bar{B}', \bar{X})$ are couples of $\mathcal{D}$. Note that each of $\bar{B}$, $\bar{B}'$ and $\bar{X}$ contains two edges that are incident to $\mathrm{cv}(\bar{B})$. Since $\bar{B}$, $\bar{B}'$ and $\bar{X}$ are edge-disjoint, $\mathrm{cv}(\bar{B})$ has degree at least 6, a contradiction.

• *Proof of item* (a).

Let $\bar{B}$ be a copy of T_5 in $\mathcal{D}$. Note that there exists at least one element $\bar{C} \in \mathcal{D}$ such that $(\bar{B}, \bar{C})$ is a couple, that is, C is the element that contains the outgoing edge of $\mathrm{cv}(\bar{B})$, which we denote by e. If $\bar{C}$ is a copy of T_5, then, since $\bar{C}$ is well-oriented (recall that $\mathcal{D}$ is complete), $(\bar{B}, \bar{C})$ is a TT-couple. On the other hand, if $\bar{C}$ is a copy of P_5, then, since e is an internal directed edge, $(\bar{B}, \bar{C})$ is a TP-couple.

Suppose that there are at least two elements of $\mathcal{D}$, say $\bar{C}$ and $\bar{D}$, such that $(\bar{B}, \bar{C})$ and $(\bar{B}, \bar{D})$ are couples. By the definition of couple, we know that $d_{\bar{B}}(\mathrm{cv}(\bar{B})), d_{\bar{C}}(\mathrm{cv}(\bar{B})), d_{\bar{D}}(\mathrm{cv}(\bar{B})) \geq 2$ and, since $\mathcal{D}$ is a decomposition, $\bar{B}$, $\bar{C}$ and $\bar{D}$ are edge-disjoint. Then, we have $d_G(\mathrm{cv}(\bar{B})) \geq 6$, a contradiction.

• *Proof of item* (b).

Suppose that there is an initial element $\bar{B}$ in $\mathcal{D}$. From item (a), we know that $\bar{B}$ is the top of exactly one couple $(\bar{B}, \bar{C})$, for $\bar{C} \in \mathcal{D}$. We analyze three cases depending on whether $(\bar{B}, \bar{C})$ is a TT-couple in position 1 or 2, or a TP-couple in position 1 (since $\mathcal{D}$ is complete, every TP-couple is solvable, and therefore, is in position 1).

Case 1: $(\bar{B}, \bar{C})$ *is a* TT*-couple in position* 1.

Suppose that $(\bar{B}, \bar{C})$ is a TT-couple of $\mathcal{D}$ in position 1. Let $\bar{B} = b_0b_1b_2b_3b_4b_1$, $\bar{C} = c_0c_1c_2c_3c_4c_1$ and $\mathrm{cv}(\bar{B}) = b_4 = c_1$. Let $\bar{B}' = b_0b_1b_2b_3b_4c_0$ and $\bar{C}' = b_1c_1c_2c_3c_4c_1$. We claim that $\bar{B}'$ is a roofed path and $\bar{C}'$ is a copy of T_5. Clearly, b_1b_4 is an edge of G. It is immediate that $c_0 \notin \{b_1, b_3, b_4\}$, because $\bar{B}$ and $\bar{C}$ are edge-disjoint. We also know that $c_0 \notin \{b_0, b_2\}$, otherwise $\{c_0, b_1, b_4\}$ would induce a triangle in G. Thus, $\bar{B}'$ is a roofed path. Note that $b_1 \notin \{c_1, c_2, c_4\}$, because $\bar{B}$ and $\bar{C}$ are edge-disjoint; furthermore, $b_1 \neq c_3$, otherwise $\{b_1, c_4, c_1\}$ would induce a triangle in G. Thus, $\mathcal{D}' = \mathcal{D} \setminus \{\bar{B}, \bar{C}\} \cup \{\bar{B}', \bar{C}'\}$ is a $\{P_5, T_5\}$-decomposition of $\bar{G}$ such that $\tau(\mathcal{D}') < \tau(\mathcal{D})$.

To obtain a contradiction, we have to show that $\mathcal{D}'$ is complete. Since all elements of $\mathcal{D}$ are well-oriented, we know that $c_2c_1 \in A(\bar{G})$. Thus, $\bar{C}'$ is also well-oriented, from where we conclude that item (i) of Definition 3.4 holds. Since $\bar{B}$ and $\bar{C}$ are well-oriented, we have that besides b_2b_1, the possible directed edges of $\bar{B}'$ are b_1b_0 and c_1c_0, and the only possible directed edge of $\bar{C}'$ is c_2c_1. It is easy to check that all these directed edges are internal to $\mathcal{D}'$. Therefore, item (ii) of Definition 3.4 holds. It remains to check (item (iii)) that every TP-couple of $\mathcal{D}'$ is solvable. We only need to prove this for TP-couples of $\mathcal{D}'$ such that either $\bar{B}'$ is the base of the couple or $\bar{C}'$ is the top of the couple.

First, we prove that there is no TP-couple $(\bar{X}, \bar{B}')$ in $\mathcal{D}'$. Suppose by contradiction that there is a TP-couple $(\bar{X}, \bar{B}')$ in $\mathcal{D}'$. Since the possible directed edges of $\bar{B}'$ are b_1b_0, b_2b_1, b_4c_0, we have that $\mathrm{cv}(\bar{X}) \in \{b_1, b_2, b_4\}$. Since $\mathrm{cv}(\bar{C}') = \mathrm{cv}(\bar{C}) =$

$c_4 \notin \{b_1, b_2, b_4\}$, we conclude that $\bar{X} \neq C'$. Therefore, $\bar{X} \in \mathcal{D}$. Since $\mathrm{cv}(\bar{X}) \neq \mathrm{cv}(\bar{B})$, we have that $\mathrm{cv}(\bar{X}) \neq b_4$. Also, if $\mathrm{cv}(\bar{X}) \in \{b_1, b_2\}$, then $(\bar{X}, \bar{B})$ is a TP-couple in $\mathcal{D}$, a contradiction to the hypothesis that $\bar{B}$ is an initial element.

Now, suppose there is a TP-couple $(\bar{C}', \bar{X})$ in $\mathcal{D}'$, where $\bar{X} = x_0x_1x_2x_3x_4x_5$. Since $\bar{B}'$ is an initial element, $\bar{X} \neq \bar{B}'$, and hence $\bar{X}$ is an element of $\mathcal{D}$. By the definition of couple, $\bar{X}$ is the element of $\mathcal{D}'$ that contains the directed edge that leaves $\mathrm{cv}(\bar{C}')$. Since $\mathrm{cv}(\bar{C}') = c_4 = \mathrm{cv}(\bar{C})$, we have that $(\bar{C}, \bar{X})$ is a TP-couple of $\mathcal{D}$. Moreover, the couple $(\bar{C}', \bar{X})$ is in position 1, because $(\bar{C}, \bar{X})$ is a TP-couple of $\mathcal{D}$ in position 1 (since $\mathcal{D}$ is complete, every TP-couple of $\mathcal{D}$ is solvable and, hence, in position 1). To prove that $(\bar{C}', \bar{X})$ is solvable, we must prove that $\bar{C}^+ = b_1c_1c_2c_3c_4x_0$ and $\bar{X}^+ = c_1x_1x_2x_3x_4x_5$ are edge-disjoint paths. Since $\bar{C}' = b_1c_1c_2c_3c_4c_1$ is a copy of T_5, we have that $b_1c_1c_2c_3c_4$ is a path; moreover $x_0 \notin \{c_1, c_2, c_3, c_4\}$ (because $(\bar{C}, \bar{X})$ is a TP-couple of $\mathcal{D}$ in position 1), and $b_1 \neq x_0$, otherwise $\{b_1, c_4, c_1\}$ would induce a triangle in G. Thus, $\bar{C}^+$ is a path of length five. Moreover, $\bar{X}^+$ is a path of length five, because $(\bar{C}, \bar{X})$ is solvable.

Since we proved that $\mathcal{D}'$ is a complete $\{P_5, T_5\}$-decomposition of $\bar{G}$ such that $\tau(\mathcal{D}') < \tau(\mathcal{D})$, we have a contradiction.

Case 2: *$(\bar{B}, \bar{C})$ is a* TT-*couple in position* 2.

Suppose that $(\bar{B}, \bar{C})$ is a TT-couple of $\mathcal{D}$ in position 2. Let $\bar{B} = b_0b_1b_2b_3b_4b_1$, $\bar{C} = c_0c_1c_2c_3c_4c_1$ and $\mathrm{cv}(\bar{B}) = b_4 = c_2$. Let $\bar{B}' = b_0b_1b_2b_3b_4c_1$ and $\bar{C}' = c_0c_1c_4c_3c_2b_1$. By Lemma 4.4, the elements $\bar{B}'$ and $\bar{C}'$ are roofed paths and all their directed edges are internal to $\mathcal{D}'$, where $\mathcal{D}' = \mathcal{D} \setminus \{\bar{B}, \bar{C}\} \cup \{\bar{B}', \bar{C}'\}$. Note that $\mathcal{D}'$ is a $\{P_5, T_5\}$-decomposition of $\bar{G}$ such that $\tau(\mathcal{D}') < \tau(\mathcal{D})$.

Next, we show that $\mathcal{D}'$ is complete. Since every copy $\bar{X}$ of T_5 in $\mathcal{D}'$ is an element of $\mathcal{D}$, and $\mathcal{D}$ is complete, we conclude that $\bar{X}$ is well-oriented. Thus, item (i) of Definition 3.4 holds. Since all directed edges of $\bar{B}'$ and $\bar{C}'$ are internal, we conclude that all directed edges of $\bar{G}$ are internal to $\mathcal{D}'$. Then, item (ii) of Definition 3.4 holds. It remains to verify (item (iii)) that every TP-couple of $\mathcal{D}'$ is solvable. We only need to prove this for TP-couples of $\mathcal{D}'$ in which either $\bar{B}'$ or $\bar{C}'$ is the base of the couple.

First, we prove that there is no TP-couple $(\bar{X}, \bar{B}')$ in $\mathcal{D}'$. Suppose by contradiction that there is a TP-couple $(\bar{X}, \bar{B}')$ in $\mathcal{D}'$. Since the possible directed edges of $\bar{B}'$ are b_1b_0, b_2b_1, b_4c_1, we conclude that $\mathrm{cv}(\bar{X}) \in \{b_1, b_2, b_4\}$. Since $\mathrm{cv}(\bar{C}') = \mathrm{cv}(\bar{C}) = c_4 \notin \{b_1, b_2, b_4\}$, we have that $\bar{X} \neq C'$. Therefore, $\bar{X} \in \mathcal{D}$. Since $\mathrm{cv}(\bar{X}) \neq \mathrm{cv}(\bar{B})$, it follows that $\mathrm{cv}(\bar{X}) \neq b_4$. Also, if $\mathrm{cv}(\bar{X}) \in \{b_1, b_2\}$, then $(\bar{X}, \bar{B})$ is a TP-couple in $\mathcal{D}$, a contradiction to the hypothesis that $\bar{B}$ is an initial element.

Now, suppose that there is a TP-couple $(\bar{X}, \bar{C}')$ in $\mathcal{D}'$, where $\bar{X} = x_0x_1x_2x_3x_4x_5$. Since the only possible directed edge of $\bar{C}'$ is c_1c_0, we have that $\mathrm{cv}(\bar{X}) = c_1$, and hence $(\bar{X}, \bar{C}')$ is a TP-couple in position 1. Thus, $\bar{X}$ is a copy of T_5, and hence $\bar{X} \neq \bar{B}'$ and $\bar{X} \neq \bar{C}'$. Therefore, $\bar{X}$ is an element of $\mathcal{D}$, and $\bar{X}$ is well-oriented (because $\mathcal{D}$ is complete). Since $(\bar{X}, \bar{C}')$ is a TP-couple of $\mathcal{D}'$ in position 1, $\bar{X}$ is well-oriented and $\bar{C}'$ is a roofed path such that its roof is not in $\bar{X}$, we conclude, by Lemma 4.1, that $(\bar{X}, \bar{C}')$ is solvable.

Since we proved that $\mathcal{D}'$ is a complete $\{P_5, T_5\}$-decomposition of $\bar{G}$ such that $\tau(\mathcal{D}') < \tau(\mathcal{D})$, we have a contradiction.

Case 3: *$(\bar{B}, \bar{C})$ is a* TP-*couple in position* 1.

Suppose that $(\bar{B}, \bar{C})$ is a TP-couple of $\mathcal{D}$ in position 1. Let $\bar{B} = b_0b_1b_2b_3b_4b_1$, $\bar{C} = c_0c_1c_2c_3c_4c_5$ and $\mathrm{cv}(\bar{B}) = b_4 = c_1$. Let $\bar{B}' = b_0b_1b_2b_3b_4c_0$ and $\bar{C}' = b_1c_1c_2c_3c_4c_5$. By the definition of a solvable couple, we know that $\bar{B}'$ is a roofed path and $\bar{C}'$ is a path of length five. Thus, $\mathcal{D}' = \mathcal{D} \setminus \{\bar{B}, \bar{C}\} \cup \{\bar{B}', \bar{C}'\}$ is a $\{P_5, T_5\}$-decomposition of $\bar{G}$ such that $\tau(\mathcal{D}') < \tau(\mathcal{D})$.

Now let us prove that $\mathcal{D}'$ is complete. Since every copy $\bar{X}$ of T_5 in $\mathcal{D}'$ is an element of $\mathcal{D}$, and $\mathcal{D}$ is complete, $\bar{X}$ is well-oriented. Since $\bar{B}$ is well-oriented and every directed edge is internal to $\mathcal{D}$, the possible directed edges of $\bar{B}'$ are b_1b_0, b_2b_1 and c_1c_0, which are clearly internal to $\mathcal{D}'$. Since $\bar{B}$ is well-oriented, we know that b_1b_4 is not directed. Since each directed edge of $\bar{C}$ is internal, it follows that each directed edge of $\bar{C}'$ is internal.

It remains to show that every TP-couple $(\bar{X}, \bar{Y})$ of $\mathcal{D}'$ is solvable. We only need to prove this for TP-couples of $\mathcal{D}'$ in which either $\bar{B}'$ or $\bar{C}'$ is the base.

First, we prove that there is no TP-couple $(\bar{X}, \bar{B}')$ in $\mathcal{D}'$. Suppose by contradiction that there is a TP-couple $(\bar{X}, \bar{B}')$ in $\mathcal{D}'$. Since the possible directed edges of $\bar{B}'$ are b_1b_0, b_2b_1, b_4c_0, we have that $\mathrm{cv}(\bar{X}) \in \{b_1, b_2, b_4\}$. Since C' is a copy of P_5, it follows that $\bar{X} \neq C'$. Therefore $\bar{X} \in \mathcal{D}$. Since $\mathrm{cv}(\bar{X}) \neq \mathrm{cv}(\bar{B})$, we conclude that $\mathrm{cv}(\bar{X}) \neq b_4$. Also, if $\mathrm{cv}(\bar{X}) \in \{b_1, b_2\}$, then $(\bar{X}, \bar{B})$ is a TP-couple in $\mathcal{D}$, a contradiction to the hypothesis that B is an initial element.

Now, suppose that there is a TP-couple $(\bar{X}, \bar{C}')$ in $\mathcal{D}'$, where $\bar{X} = x_0x_1x_2x_3x_4x_5$. Since the only possible directed edges of $\bar{C}'$ are c_1c_0 and c_4c_5, we conclude that $\mathrm{cv}(\bar{X}) \in \{c_1, c_4\}$. Therefore, $(\bar{X}, \bar{C}')$ is a couple in position 1. Since B' and C' are copies of P_5, and $\bar{X}$ is a copy of T_5, we have that $\bar{X} \neq B'$ and $\bar{X} \neq C'$, and hence $\bar{X} \in \mathcal{D}$. Since $\bar{X}$ is also an element of $\mathcal{D}'$, we have that $\bar{X} \neq \bar{B}$, and hence $\mathrm{cv}(\bar{X}) \neq \mathrm{cv}(\bar{B})$. Therefore, $\mathrm{cv}(\bar{X}) \neq c_1$, from where we conclude that $\mathrm{cv}(\bar{X}) = c_4$. Let us prove that $(\bar{X}, \bar{C}')$ is solvable. We must show that $\bar{X}^+ = x_0x_1x_2x_3x_4c_5$ and $\bar{C}^+ = x_1c_4c_3c_2c_1b_1$ are edge-disjoint paths. Since $(\bar{X}, \bar{C})$ is solvable, we know that $\bar{X}^+$ and $x_1c_4c_3c_2c_1c_0$ are edge-disjoint paths. Thus, $x_1 \notin \{c_1, c_2, c_3, c_4\}$. Note that $x_1 \neq b_1$. Indeed, if $x_1 = b_1$, since $d_{\bar{X}}(x_1) = d_{\bar{B}}(b_1) = 3$ and $\bar{B}$ and $\bar{X}$ are edge-disjoint in G, the degree of x_1 would be at least six, a contradiction. Therefore, we conclude that $\bar{C}^+$ is a path of length five.

Since we proved that $\mathcal{D}'$ is a complete $\{P_5, T_5\}$-decomposition of $\bar{G}$ such that $\tau(\mathcal{D}') < \tau(\mathcal{D})$, we have a contradiction.

• *Proof of item* (c).

Given an element $\bar{X}$ of $\mathcal{D}$, denote by $t(\bar{X})$ (resp. $b(\bar{X})$), the number of couples of $\mathcal{D}$ in which $\bar{X}$ is the top (resp. $\bar{X}$ is the base). Define $Q_{i,j} = \{\bar{X} \in \mathcal{D} : b(\bar{X}) = i \text{ and } t(\bar{X}) = j\}$, for $0 \leq i, j \leq 2$. Furthermore, let $q_{i,j} = |Q_{i,j}|$.

Note that $\sum_{\bar{X} \in \mathcal{D}} b(\bar{X}) = \sum_{\bar{X} \in \mathcal{D}} t(\bar{X})$. Thus, we have the following equalities.

$$\sum_{i=0}^{2}\sum_{j=0}^{2} i q_{i,j} = \sum_{\bar{X} \in \mathcal{D}} b(\bar{X}) = \sum_{\bar{X} \in \mathcal{D}} t(\bar{X}) = \sum_{i=0}^{2}\sum_{j=0}^{2} j q_{i,j}.$$

Since there is no initial element in $\mathscr{D}$ (by item (b) of this lemma), we have that $q_{0,j} = 0$ for $j = 1, 2$. Since every copy of T_5 in $\mathscr{D}$ is the top of exactly one couple (by item (a) of this lemma), and by the definition of couples, no copy of P_5 is the top of any couple, we have that $q_{i,2} = 0$ for $i = 0, 1, 2$. Thus,

$$q_{1,0} + q_{1,1} + 2q_{2,0} + 2q_{2,1} = q_{1,1} + q_{2,1}.$$

Therefore

$$q_{1,0} + 2q_{2,0} + q_{2,1} = 0,$$

from where we conclude that $q_{1,0} = q_{2,0} = q_{2,1} = 0$. Therefore, if $q_{i,j} > 0$, then either $i = j = 0$ or $i = j = 1$. Let $\bar{X}$ be a copy of T_5 in $\mathscr{D}$. Since every copy of T_5 in $\mathscr{D}$ is the top of exactly one couple, we have that $\bar{X} \in Q_{1,1}$, i.e., $\bar{X}$ is the base of exactly one couple of $\mathscr{D}$. Furthermore, let Y be a copy of T_5 in $\mathscr{D}$. Since every copy of P_5 in $\mathscr{D}$ is not the top of any couple of $\mathscr{D}$, we have that $Y \in Q_{0,0}$, i.e., Y is not the base of any couple of $\mathscr{D}$. □

5.1. Proof of the main theorem

We are now ready to prove the main result of this paper.

Theorem 5.2. *Every triangle-free 5-regular graph containing a perfect matching admits a P_5-decomposition.*

Proof. Let G be a triangle-free 5-regular graph containing a perfect matching. Applying Corollary 3.6 to G, we obtain that G is the underlying graph of a mixed graph $\bar{G}$ such that $A(\bar{G})$ induces a 2-factor with an Eulerian orientation, and $\bar{G}$ admits a complete $\{P_5, T_5\}$-decomposition. Let $\mathscr{D}$ be a minimal complete $\{P_5, T_5\}$-decomposition of $\bar{G}$. If $\tau(\mathscr{D}) = 0$, then $\mathscr{D}$ is a P_5-decomposition of G and the theorem is proved. Thus, we assume that $\tau(\mathscr{D}) > 0$ ($\mathscr{D}$ is nontrivial) and we aim for a contradiction.

The idea of the proof is to find a complete $\{P_5, T_5\}$-decomposition $\mathscr{D}'$ of $\bar{G}$ such that $\tau(\mathscr{D}') < \tau(\mathscr{D})$, a contradiction to the fact that $\mathscr{D}$ is a minimal complete $\{P_5, T_5\}$-decomposition of $\bar{G}$.

By item (c) of Lemma 5.1, we know that there is no TP-couple in $\mathscr{D}$. Thus, $\mathscr{D}$ contains only TT-couples. By Lemma 5.1, every copy of T_5 is the top of exactly one couple of $\mathscr{D}$ (by item (a)) and the base of exactly one couple of $\mathscr{D}$ (by item (c)). We will prove that every copy of T_5 is an element of a cycle of TT-couples.

Let $\bar{B}_1$ be a copy of T_5 in $\mathscr{D}$. Clearly, there is a sequence $\bar{B}_1 \cdots \bar{B}_k$ of couples in $\mathscr{D}$ such that $(\bar{B}_i, \bar{B}_{i+1})$ is the only couple of $\mathscr{D}$ such that $\bar{B}_i$ is the top and $\bar{B}_{i+1}$ is the base, for $1 \leq i \leq k-1$. Let $\bar{B}_1 \cdots \bar{B}_K$ be such a sequence with maximum number of distinct elements. Since, by item (a) of Lemma 5.1, $\bar{B}_K$ is the top of one couple, $(\bar{B}_K, \bar{B}_j)$ is a TT-couple for some $j \in \{1, \ldots, K-2\}$ (note that $(\bar{B}_K, \bar{B}_{K-1})$ cannot be a TT-couple). Note that $\bar{B}_j$ is the base of $(\bar{B}_{j-1}, \bar{B}_j)$ and is also the base of $(\bar{B}_K, \bar{B}_j)$; thus, by item (c) of Lemma 5.1, we conclude that $j = 1$, and therefore $\bar{B}_1 \cdots \bar{B}_K$ is a cycle of TT-couples.

We divide the proof in cases, depending on whether the cycle $\bar{B}_1 \cdots \bar{B}_K$ is mixed, in position 1 or in position 2.

Case 1: *$\bar{B}_1 \cdots \bar{B}_K$ is mixed.*

In what follows all additions are taken modulo K. Suppose that $\bar{B}_1 \cdots \bar{B}_K$ is a mixed cycle. Then, there is an index $h \in \{1, \ldots, K\}$ such that $(\bar{B}_{h-1}, \bar{B}_h)$ is a TT-couple in position 1 and $(\bar{B}_h, \bar{B}_{h+1})$ is a TT-couple in position 2. Let $\bar{B}_h = x_0x_1x_2x_3x_4x_1$ and $\bar{B}_{h+1} = y_0y_1y_2y_3y_4y_1$, where $\mathrm{cv}(\bar{B}_h) = x_4 = y_2$. Let $\bar{B}'_h = x_0x_1x_2x_3x_4y_1$ and $\bar{B}'_{h+1} = y_0y_1y_4y_3y_2x_1$. By Lemma 4.4, the elements $\bar{B}'_h$ and $\bar{B}'_{h+1}$ are roofed paths and all their directed edges are internal to $\mathscr{D}'$, where $\mathscr{D}' = \mathscr{D} \setminus \{\bar{B}_h, \bar{B}_{h+1}\} \cup \{\bar{B}'_h, \bar{B}'_{h+1}\}$. Note that $\mathscr{D}'$ is a $\{P_5, T_5\}$-decomposition of $\bar{G}$ such that $\tau(\mathscr{D}') < \tau(\mathscr{D})$.

We have to show that $\mathscr{D}'$ is complete. Since every copy $\bar{X}$ of T_5 in $\mathscr{D}'$ is an element of $\mathscr{D}$ and $\mathscr{D}$ is complete, we have that $\bar{X}$ is well-oriented. Therefore, item (i) of Definition 3.4 holds. Since all directed edges of $\bar{B}'_h$ and $\bar{B}'_{h+1}$ are internal, we conclude that all directed edges of $\bar{G}$ are internal to $\mathscr{D}'$. Then, item (ii) of Definition 3.4 holds.

To verify item (iii) we have to prove that every TP-couple $(\bar{X}, \bar{Y})$ of $\mathscr{D}'$ is solvable. We only need to prove it for TP-couples of $\mathscr{D}'$ in which $\bar{B}'_h$ or $\bar{B}'_{h+1}$ is the base. If there is a TP-couple $(\bar{X}, \bar{B}'_h)$ of $\mathscr{D}'$, then clearly $\bar{X} = \bar{B}_{h-1}$ and, since $(\bar{B}_{h-1}, \bar{B}_h)$ is in position 1, we have that $\mathrm{cv}(\bar{B}_{h-1}) = x_1$, from where we conclude that $(\bar{B}_{h-1}, \bar{B}'_h)$ is in position 1. Since $(\bar{B}_{h-1}, \bar{B}'_h)$ is a TP-couple of $\mathscr{D}'$ in position 1, $\bar{B}_{h-1}$ is well-oriented and $\bar{B}'_h$ is a roofed path such that its roof is not in $\bar{B}_{h-1}$, we know, by Lemma 4.1, that $(\bar{B}_{h-1}, \bar{B}'_h)$ is solvable. If there is a TP-couple $(\bar{X}, \bar{B}'_{h+1})$ of $\mathscr{D}'$, then $(\bar{X}, \bar{B}_{h+1})$ is a TT-couple of $\mathscr{D}$. Since $\bar{X} \in \mathscr{D}'$, we have that $\bar{X} \neq \bar{B}_h$. Then, $(\bar{X}, \bar{B}_{h+1})$ is in position 1, from where we conclude that $\bar{B}_{h+1}$ is the base of two couples of $\mathscr{D}$, a contradiction to item (c) of Lemma 5.1.

Case 2: *$\bar{B}_1 \cdots \bar{B}_K$ is in position 1 (resp. in position 2).*

By Lemma 4.2 (resp. Lemma 4.6) applied on the cycle $\bar{B}_1 \cdots \bar{B}_K$, we obtain a P_5-decomposition $\mathscr{P}$ of $\bigcup_{i=1}^{K} \bar{B}_i$ such that all directed edges of $\bigcup_{i=1}^{K} \bar{B}_i$ are internal to $\mathscr{P}$. Then, it is clear that the decomposition $\mathscr{D}' = \mathscr{D} \setminus \{\bar{B}_1, \ldots, \bar{B}_K\} \cup \mathscr{P}$ is a $\{P_5, T_5\}$-decomposition of $\bar{G}$ such that $\tau(\mathscr{D}') < \tau(\mathscr{D})$.

Now we shall prove that $\mathscr{D}'$ is complete. Since every copy $\bar{X}$ of T_5 in $\mathscr{D}'$ is an element of $\mathscr{D}$, and $\mathscr{D}$ is complete, we conclude that $\bar{X}$ is well-oriented. Thus, item (i) of Definition 3.4 holds. Furthermore, since every directed edge of $\bigcup_{i=1}^{K} \bar{B}_i$ is internal to $\mathscr{P}$, we conclude that every directed edge of $\bar{G}$ is internal to $\mathscr{D}'$. Thus, item (ii) of Definition 3.4 holds.

To verify item (iii) we have to prove that every TP-couple $(\bar{X}, \bar{Y})$ of $\mathcal{D}'$ is solvable. We only need to prove it for TP-couples of $\mathcal{D}'$ whose base is an element of $\mathcal{P}$. Suppose that there is a TP-couple $(\bar{X}, \bar{Y})$ of $\mathcal{D}'$ such that $\bar{Y} \in \mathcal{P}$. Since $\bar{X}$ is a copy of T_5 in $\mathcal{D}'$, we have $\bar{X} \neq \bar{B}_i$ for every i in $\{1, \ldots, K\}$. Let e be the directed edge leaving cv$(\bar{X})$. Since every directed edge is internal, the element $\bar{Y}$ must contain e, otherwise, the degree of cv$(\bar{X})$ would be at least six. Thus, we conclude that e is an edge of $\bigcup_{i=1}^{K} \bar{B}_i$. Suppose without loss of generality that $\bar{B}_2$ is the element of $\mathcal{D}$ that contains e. Thus, $(\bar{X}, \bar{B}_2)$ and $(\bar{B}_1, \bar{B}_2)$ are two couples of $\mathcal{D}$ in which $\bar{B}_2$ is the base, contradicting item (c) of Lemma 5.1.

Since we proved that in both possible cases, the decomposition $\mathcal{D}'$ is a complete $\{P_5, T_5\}$-decomposition of $\bar{G}$ such that $\tau(\mathcal{D}') < \tau(\mathcal{D})$, we have a contradiction. □

6. Concluding remarks

We proved that every triangle-free 5-regular graph containing a perfect matching admits a P_5-decomposition. To prove this result, we start deleting a perfect matching and orienting a 2-factor of the remaining graph. This idea was used by Kotzig [25] to show that a 3-regular graph containing a perfect matching admits a P_3-decomposition. For a triangle-free 5-regular graph G this idea does not give straightforwardly a P_5-decomposition of G; it gives a $\{P_5, T_5\}$-decomposition of G. The next step, the elimination of the undesired trails T_5s (if existent), is the core of this work. For that, we use the technique of considering couples consisting of a P_5 and a T_5 or two T_5s, and also sequences (or cycles) of T_5s, satisfying certain properties, and disentangling the undesired trails of such a decomposition. The concept of completeness of a $\{P_5, T_5\}$-decomposition that we have introduced captures the property we need to be able to repeat the process of decreasing the number of undesired trails.

To our knowledge, this technique has not been used in the literature. It is likely that it can be useful to obtain more general results on path decompositions. Indeed, we have used a generalized version of this technique to show results on path decompositions of graphs satisfying different properties [5,6].

Acknowledgments

We thank the referees for their valuable comments and suggestions. F. Botler is supported by FAPESP (Proc. 2011/08033-0 and 2014/01460-8). G.O. Mota is supported by FAPESP (Proc. 2013/11431-2 and 2013/20733-2). Y. Wakabayashi is partially supported by CNPq Grant (Proc. 303987/2010-3).

References

[1] J.C. Bermond, D. Sotteau, Graph decompositions and G-designs, in: Proceedings of the Fifth British Combinatorial Conference (Univ. Aberdeen, Aberdeen, 1975), Utilitas Math., Winnipeg, Man., 1976, pp. 53–72. Congressus Numerantium, No. XV.
[2] B. Bollobás, Modern Graph Theory, in: Graduate Texts in Mathematics, vol. 184, Springer-Verlag, New York, 1998.
[3] J.A. Bondy, X. Buchwalder, F. Mercier, Lexicographic products and a conjecture of Hahn and Jackson, SIAM J. Discrete Math. 23 (2009) 882–887.
[4] J. Bosák, Decompositions of Graphs, in: Mathematics and its Applications (East European Series), vol. 47, Kluwer Academic Publishers Group, Dordrecht, 1990, Translated from the Slovak, With a preface by Štefan Znám.
[5] F. Botler, G.O. Mota, M.T.I. Oshiro, Y. Wakabayashi, Decompositions of highly edge-connected graphs into paths of length five, 2015. Manuscript (an extended abstract will appear in ENDM, volume dedicated to VIII Latin-American Algorithms, Graphs and Optimization Symposium (LAGOS)).
[6] F. Botler, G.O. Mota, M.T.I. Oshiro, Y. Wakabayashi, Path decompositions of regular graphs with prescribed girth, 2015. Manuscript.
[7] A. Bouchet, J.L. Fouquet, Trois types de décompositions d'un graphe en chaînes, in: Combinatorial Mathematics (Marseille-Luminy, 1981), in: North-Holland Math. Stud., vol. 75, North-Holland, Amsterdam, 1983, pp. 131–141.
[8] F.R.K. Chung, R.L. Graham, Recent results in graph decompositions, in: Combinatorics (Swansea, 1981), in: London Math. Soc. Lecture Note Ser., vol. 52, Cambridge Univ. Press, Cambridge, 1981, pp. 103–123.
[9] D. Conlon, J. Fox, B. Sudakov, Cycle packing, 2013. ArXiv:1310.0632.
[10] N. Dean, M. Kouider, Gallai's conjecture for disconnected graphs, Discrete Math. 213 (2000) 43–54. Selected topics in discrete mathematics (Warsaw, 1996).
[11] R. Diestel, Graph Theory, fourth ed., in: Graduate Texts in Mathematics, vol. 173, Springer, Heidelberg, 2010.
[12] J. Doyen, A. Rosa, An extended bibliography and survey of Steiner systems, in: Proceedings of the Seventh Manitoba Conference on Numerical Mathematics and Computing (Univ. Manitoba, Winnipeg, Man., 1977) Congress. Numer., XX, Utilitas Math., Winnipeg, Man., 1978, pp. 297–361.
[13] P. Erdős, On some of my conjectures in number theory and combinatorics, Congr. Numer., in: Proceedings of the Fourteenth Southeastern Conference on Combinatorics, Graph Theory and Computing, vol. 39 (Boca Raton, Fla., 1983) pp. 3–19.
[14] P. Erdős, A.W. Goodman, L. Pósa, The representation of a graph by set intersections, Canad. J. Math. 18 (1966) 106–112.
[15] G. Fan, Path decompositions and Gallai's conjecture, J. Combin. Theory Ser. B 93 (2005) 117–125.
[16] O. Favaron, F. Genest, M. Kouider, Regular path decompositions of odd regular graphs, J. Graph Theory 63 (2010) 114–128.
[17] S. Fiorini, R.J. Wilson, Edge-colourings of Graphs, Pitman, London, 1977, Research Notes in Mathematics, No. 16.
[18] R.L. Graham, B.L. Rothschild, J.H. Spencer, Ramsey Theory, second ed., in: Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons Inc., New York, 1990, A Wiley-Interscience Publication.
[19] F. Harary, R.W. Robinson, Isomorphic factorizations. X. Unsolved problems, J. Graph Theory 9 (1985) 67–86.
[20] K. Heinrich, Path-decompositions, Matematiche (Catania) 47 (1992) 241–258. (1993) Combinatorics 92 (Catania, 1992).
[21] K. Heinrich, J. Liu, M. Yu, P_4-decompositions of regular graphs, J. Graph Theory 31 (1999) 135–143.
[22] M.S. Jacobson, M. Truszczyński, Z. Tuza, Decompositions of regular bipartite graphs, Discrete Math. 89 (1991) 17–27.
[23] F. Jaeger, C. Payan, M. Kouider, Partition of odd regular graphs into bistars, Discrete Math. 46 (1983) 93–94.
[24] K.F. Jao, A.V. Kostochka, D.B. West, Decomposition of Cartesian products of regular graphs into isomorphic trees, J. Comb. 4 (2013) 469–490.
[25] A. Kotzig, Aus der Theorie der endlichen regulären Graphen dritten und vierten Grades, Časopis Pěst. Mat. 82 (1957) 76–92.
[26] M. Kouider, Z. Lonc, Path decompositions and perfect path double covers, Australas. J. Combin. 19 (1999) 261–274.
[27] L. Lovász, On covering of graphs, in: Theory of Graphs (Proc. Colloq., Tihany, 1966), Academic Press, New York, 1968, pp. 231–236.
[28] J. Petersen, Die Theorie der regulären graphs, Acta Math. 15 (1891) 193–220.
[29] H.S.C. Snevily, Combinatorics of finite sets, (Ph.D. thesis) University of Illinois, Urbana-Champaign, ProQuest LLC, Ann Arbor, MI, 1991.