Discrete Mathematics 338 (2015) 1947–1952

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Describing tight descriptions of 3-paths in triangle-free normal plane maps

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ARTICLE INFO

Article history:
Received 26 October 2014
Received in revised form 3 May 2015
Accepted 4 May 2015
Available online 6 June 2015

Keywords:
Normal plane map
Planar graph
Structure properties
3-path
Weight

ABSTRACT

Lebesgue (1940) proved that every normal plane map of girth 5 has a path on three vertices (3-path) of degree 3. A description is tight if no its parameter can be strengthened, and no alternative dropped. Borodin et al. (2013) gave a tight description of 3-paths in arbitrary normal plane maps.

We give seven tight descriptions of 3-paths in triangle-free normal plane maps. Furthermore, we prove that this set of descriptions is complete, which is a result of a bit new type in the structural theory of plane graphs.

1. Introduction

A normal plane map (NPM) is a plane pseudograph in which loops and multiple edges are allowed, but the degree of each vertex and face is at least three. Let δ be the minimum vertex degree, and w_k be the minimum degree-sum of a path on k vertices in an NPM or a graph. The degree of a vertex v or a face f, that is, the number of edges incident with v or f (loops and cut-edges are counted twice), is denoted by $d(v)$ or $d(f)$, respectively. A *k-vertex* is a vertex v with $d(v) = k$. By k^+ or k^- we denote any integer not smaller or not greater than k, respectively. Hence, a k^+-vertex v satisfies $d(v) \geq k$, etc. An edge uv is an *(i, j)-edge* if $d(u) \leq i$ and $d(v) \leq j$. A path uvw is a *path of type* (i, j, k) or *(i, j, k)-path* if $d(u) \leq i$, $d(v) \leq j$, and $d(w) \leq k$.

Already in 1904, Wernicke [14] proved that every NPM M_5 with $\delta(M_5) = 5$ contains a 5-vertex adjacent to a 6^--vertex, and Franklin [7] strengthened this to the existence of at least two 6^--neighbors, which implies that M_5 satisfies $w_3 \leq 17$. Franklin's bound 17 is precise, as shown by putting a vertex inside each face of the dodecahedron and joining it with the five boundary vertices.

It follows from Lebesgue's results in [12] that each NPM has an edge $e = uw$ of weight $w(e) = d(u) + d(w)$ at most 14 (more specifically, a (3, 11)-, or (4, 7)-, or (5, 6)-edge, where bounds 7 and 6 are sharp). For 3-connected plane graphs, Kotzig [11] proved a precise result: $w_2 \leq 13$.

Note that $\delta(K_{2,t}) = 2$ and $w_2(K_{2,t}) = t+2$, so w_2 is unbounded if $\delta \leq 2$. In 1972, Erdős (see [8]) conjectured that Kotzig's bound $w_2 \leq 13$ holds for all planar graphs with $\delta \geq 3$. Barnette (see [8]) announced to have proved this conjecture, but the proof has never appeared in print. The first published proof of Erdős' conjecture is due to Borodin [2]. More generally, Borodin [3,4] proved that every NPM contains a (3, 10)-, or (4, 7)-, or (5, 6)-edge (as easy corollaries of several stronger structural facts with applications to coloring).

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http://dx.doi.org/10.1016/j.disc.2015.05.006

Theorem 1 (*Ando, Iwasaki, Kaneko [1]*). *Every* 3*-polytope satisfies* $w_3 \le 21$, *which is sharp.*

The sharpness of the bound $w_3 \le 21$ in Theorem 1 is witnessed by the Jendrol' construction [9]. Jendrol' [10] proves that each 3-polytope has a 3-path uvw such that $\max\{d(u), d(v), d(w)\} \le 15$ (the bound is precise). Jendrol' [9] further shows that such a 3-path must belong to one of ten types, in which $d(u) + d(v) + d(w)$ varies from 23 to 16:

Theorem 2 (*Jendrol' [9]*). *Every* 3*-polytope has a* 3*-path of one of the following types:* $(10, 3, 10)$, $(7, 4, 7)$, $(6, 5, 6)$, $(3, 4, 15)$, $(3, 6, 11)$, $(3, 8, 5)$, $(3, 10, 3)$, $(4, 4, 11)$, $(4, 5, 7)$, *or* $(4, 7, 5)$.

Note that the graphs of 3-polytopes are precisely the 3-connected planar graphs due to Steinitz's famous theorem [13]. The requirement of 3-connectedness is essential for the finiteness of w_3, as shown by a graph formed by t copies of the graph $K_4 - e$ by identifying the 2-vertices of these copies. Indeed, Borodin [5] has proved the following refinement of Theorem 1:

Theorem 3 (*Borodin [5]*). *Every* $(K_4 - e)$*-less NPM has*
(i) *either* $w_3 \le 18$ *or a vertex of degree* ≤ 15 *adjacent to two* 3*-vertices, and*
(ii) *either* $w_3 \le 17$ *or* $w_2 \le 7$.

As mentioned above, the bounds $w_3 \le 21$ and $w_3 \le 17$ are tight. It was open whether the bound $w_3 \le 18$ in Theorem 3 is sharp or not; its sharpness was recently confirmed by Borodin et al. [6].

In particular, precise bound $w_3 \le 21$ by Ando, Iwasaki, and Kaneko's [1] is valid for all NPMs in which no two 3-vertices are adjacent:

Corollary 4 (*[5]*). *Every NPM with* $w_2 > 6$ *has* $w_3 \le 21$.

Theorem 3 immediately implies that Franklin's precise bound $w_3 \le 17$ is valid for all NPMs with $\delta \ge 4$:

Corollary 5 (*[5]*). *Every NPM without* 3*-vertices satisfies* $w_3 \le 17$.

The upper bound in the following statement is also immediate:

Corollary 6 (*[5]*). *Every* 3*-polytope with* $\delta \ge 4$ *has a path* uvw *such that* $\max\{d(u), d(v), d(w)\} \le 9$, *which bound is tight.*

A description of 3-paths is *tight* if no its parameter can be strengthened and no term dropped.

Lebesgue (1940) proved that every NPM of girth 5 has a 3-path consisting of 3-vertices. Borodin et al. [6] gave a tight description of 3-paths in arbitrary normal plane maps:

Theorem 7 (*Borodin, Ivanova, Jensen, Kostochka, Yancey [6]*). *Every* $(K_4 - e)$*-less NPM has a* 3*-path of one of the following types:* $(3, 4, 11)$, $(3, 7, 5)$, $(3, 10, 4)$, $(3, 15, 3)$, $(4, 4, 9)$, $(6, 4, 8)$, $(7, 4, 7)$, $(6, 5, 6)$, *which description is tight.*

One of the purposes of this paper is to give seven tight descriptions of 3-paths in triangle-free NPMs (see Theorems and Corollaries 8–14 below).

Theorem 8. *Every triangle-free NPM has a* $(5, 3, 6)$*-path or* $(4, 3, 7)$*-path, which description is tight.*

Corollary 9. *Every triangle-free NPM has a* $(5, 3, 7)$*-path, which is tight.*

Theorem 10. *Every triangle-free NPM has a* $(3, 5, 3)$*-path or* $(3, 4, 4)$*-path, which is tight.*

Corollary 11. *Every triangle-free NPM has a* $(3, 5, 4)$*-path, which description is tight.*

Theorem 12. *Every triangle-free NPM has a* $(5, 3, 6)$*-path or* $(3, 4, 3)$*-path, which is tight.*

Corollary 13. *Every triangle-free NPM has a* $(5, 4, 6)$*-path, which is tight.*

Theorem 14. *Every triangle-free NPM has a* $(3, 5, 3)$*-path or* $(4, 3, 4)$*-path, which description is tight.*

The other purpose of this paper is to show that there are no tight descriptions other than in Theorems and Corollaries 8–14.

Theorem 15. *There exist precisely seven tight descriptions of* 3*-paths in triangle-free NPMs:*
(i) $(5, 3, 6) \vee (4, 3, 7)$,
(ii) $(3, 5, 3) \vee (3, 4, 4)$,
(iii) $(5, 3, 6) \vee (3, 4, 3)$,
(iv) $(3, 5, 3) \vee (4, 3, 4)$,
(v) $(5, 3, 7)$,
(vi) $(3, 5, 4)$,
(vii) $(5, 4, 6)$.

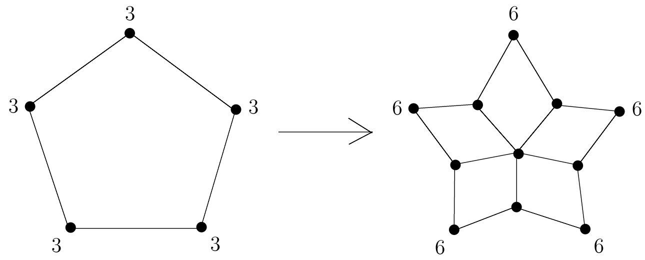

Fig. 1. The graph G_1 without paths smaller than those of types (3, 5, 3) and (5, 3, 6).

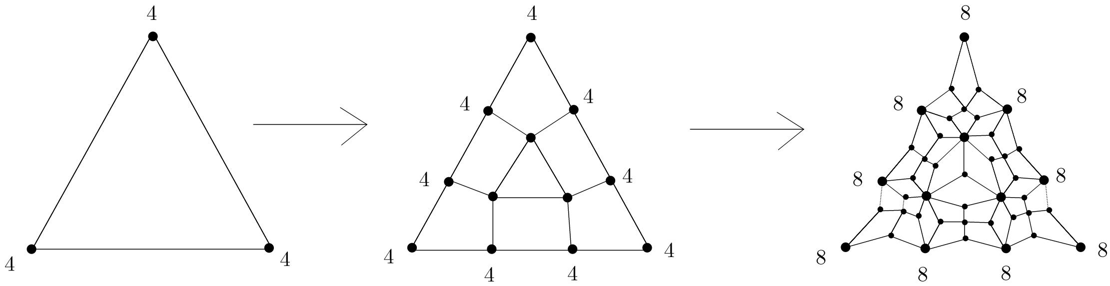

Fig. 2. The graph G_2 without paths smaller than those of types (3, 4, 3) and (4, 3, 7).

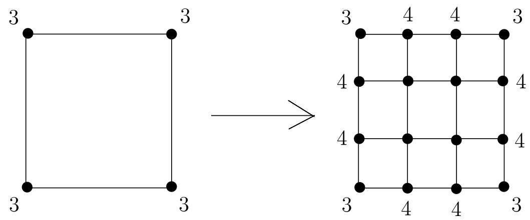

Fig. 3. The graph G_3 without paths smaller than those of types (4, 3, 4) and (3, 4, 4).

2. The tightness of Theorems 8, 10, 12 and 14 and Corollaries 9, 11 and 13

In Fig. 1, we see a graph G_1 formed by replacing each face of the dodecahedron by a copy of the graph from the right side of Fig. 1. As a result, all 3-paths of G_1 majorize those of types (3, 5, 3) and (5, 3, 6). In Fig. 2, a graph G_2 is similarly constructed from the octahedron and contains no 3-paths strictly majorized by (4, 3, 7)- and (3, 4, 3)-paths. Finally, a graph G_3 (constructed from the cube) contains no 3-paths of types smaller than (4, 3, 4) and (3, 4, 4) (see Fig. 3).

For example, we now explain the tightness of the descriptions in Theorem 8 and its Corollary 9.

For the first description, we have to prove that each of the following strengthenings of Theorem 8 is wrong: "Every triangle-free NPM has a 3-path of one of the types: (a) $(4, 3, 6) \vee (4, 3, 7)$, (b) $(5, 3, 5) \vee (4, 3, 7)$, (c) $(5, 3, 6) \vee (3, 3, 7)$, and (d) $(5, 3, 6) \vee (4, 3, 6)$". In fact, (a) is reduced to saying that there is a (4, 3, 7)-path, and (d) is equivalent to the claim on the existence of (5, 3, 6)-paths.

Indeed, (a) fails at graph G_1 because G_1 has no 3-vertex adjacent to a 4^--vertex. As for (b), we see that each 3-vertex in G_1 is adjacent to a 5-vertex and two 6-vertices, a contradiction. In G_2, no two 3-vertices are adjacent, and each 3-vertex has at least two 7^+-neighbors, which contradicts (c) and (d).

To confirm the tightness of Corollary 9, we must check that the type (5, 3, 7) cannot be replaced by (4, 3, 7) or (5, 3, 6). Indeed, the first claim fails at G_1, while the second fails at G_2.

Checking the tightness of the other five theorems and corollaries is provided by similar arguments based on graphs G_1, G_2, and G_3, and is left to the reader.

3. Proving the main statements of Theorems 8, 10, 12 and 14 and Corollaries 9, 11 and 13

For example, by the main statement of Theorem 8 we mean that every triangle-free NPM has a (5, 3, 6)-path or (4, 3, 7)-path. Throughout this section we discuss only the main statements of these seven theorems and corollaries.

To deduce Corollaries 9, 11 and 13 is easy. For instance, (the main statement of) Corollary 9 claims that there exists a (5, 3, 7)-path. However, we know by Theorem 8 that there exists either a (5, 3, 6)-path or (4, 3, 7)-path, each of which is a (5, 3, 7)-path, and we are done.

Now suppose that M is a counterexample to any of Theorems 8, 10, 12 and 14.

Euler's formula $|V| - |E| + |F| = 2$ for M may be written as

$$\sum_{x \in V \cup F} (d(x) - 4) = -8, \tag{1}$$

where V, E, and F are the sets of vertices, edges and faces of M, respectively.

Every $x \in V \cup F$ contributes the *charge* $\mu(x) = d(x) - 4$ to (1), so only the charge of 3-vertices is negative. Using the properties of M as a counterexample, we define a local redistribution of μ's, preserving their sum, such that the *new charge* $\mu'(x)$ is non-negative for all $x \in V \cup F$. This will contradict the fact that the sum of the new charges is, by (1), equal to -8, which will complete the proof of any of the four theorems.

Throughout the paper, we denote the vertices adjacent to a vertex v in a cyclic order by $v_1, \dots, v_{d(v)}$.

3.1. Proving Theorems 8 and 12

We apply the following rules of discharging.

R1. *Every* 4^+*-vertex* v *gives* $\frac{d(v)-4}{d(v)}$ *to each incident face if* $d(v) \le 7$ *or* $\frac{1}{2}$ *otherwise.*

Let $\rho(f)$ be the total donation to a face f by R1, and $n_3(f)$ the number of 3-vertices incident with f.

R2. *Every* 3*-vertex* v *receives* $\min\{\frac{1}{2}, \frac{\mu(f)+\rho(f)}{n_3(f)}\}$ *from each incident face* f.

Lemma 16. *Every* 3*-vertex* v *receives* $\frac{1}{2}$ *from each incident* 5^+*-face* f.

Proof. Suppose that $d(f) = 5$. Since M has no $(3, 3, 3)$-paths, it follows that $n_3(f) \le 3$. If $n_3(f) \le 2$, then we are readily done because $\mu(f) = 1$. So suppose $n_3(f) = 3$, which means that f is incident with two 7^+-vertices due to the absence of $(3, 3, 6)$-paths in M for each of Theorems 8 and 12. Thus $\mu(f) + \rho(f) > n_3(f) \times \frac{1}{2}$, as desired.

If $d(f) \ge 6$, then $\mu(f)$ is enough for f to give $\frac{1}{2}$ to each incident 3-vertex. Indeed, since $n_3(f) \le \frac{2d(f)}{3}$, we have $\mu(f) = d(f) - 4 \ge \frac{2d(f)}{3} \times \frac{1}{2}$. □

Now we check that the new charge μ' is non-negative for all faces and vertices of M.

It follows from Lemma 16 that $\mu'(f) \ge \mu(f) - n_3(f) \times \frac{1}{2} \ge 0$ whenever $d(f) \ge 5$. Suppose f is a 4-face. If $n_3(f) = 0$, then $\mu'(f) \ge \mu(f) = 0$. Otherwise, $\mu'(f) = 0$ by R2.

Suppose $v \in V$. If $d(v) \ge 8$, then $\mu'(v) \ge d(v) - 4 - d(v) \times \frac{1}{2} \ge 0$ by R1. If $4 \le d(v) \le 7$, then $\mu'(v) \ge d(v) - 4 - d(v) \times \frac{d(v)-4}{d(v)} = 0$.

To complete the proof, it suffices to check that every 3-vertex v in M satisfies $\mu'(v) \ge 0$.

Let v be incident with faces $f_1 = v_1vv_2x\dots$, $f_2 = v_3vv_2y\dots$, and $f_3 = v_1vv_3\dots$.

CASE 1. $d(v_2) = 3$. Here, $d(x) \ge 7$, $d(y) \ge 7$, $d(v_1) \ge 7$, and $d(v_3) \ge 7$ due to the absence of $(3, 3, 6)$-paths in M for each of Theorems 8 and 12. By R1, each of the incident faces gives v at least $\frac{3}{7}$, which results in $\mu'(v) \ge -1 + 3 \times \frac{3}{7} > 0$.

CASE 2. $d(v_2) = 4$. In this case, our proof of Theorems 8 and 12 splits.

Under the assumptions of Theorem 8, we have $d(v_1) \ge 8$ and $d(v_3) \ge 8$ due to the absence of $(4, 3, 7)$-paths in M. Now v receives at least $\frac{1}{4}$ from f_2 (which bound is attained only when $d(f_2) = 4$ and $d(y) = 3$). The same is true for f_1. From f_3, our v receives $\frac{1}{2}$. So, we have $\mu'(v) = 0$.

In Theorem 12, we have $d(v_1) \ge 7$ and $d(v_3) \ge 7$, while $d(x) \ge 4$ and $d(y) \ge 4$ due to the absence of $(5, 3, 6)$-paths and $(3, 4, 3)$-paths. So v receives at least $\frac{3}{7}$ from each incident face, and we have $\mu'(v) \ge -1 + 3 \times \frac{3}{7} > 0$.

CASE 3. $d(v_2) = 5$. Now we have $d(v_1) \ge 7$ and $d(v_3) \ge 7$ due to the absence of $(5, 3, 6)$-paths. Each of f_1 and f_2 gives v at least $(\frac{1}{5} + \frac{3}{7}) \times \frac{1}{2} = \frac{11}{35}$, while f_3 gives at least $\frac{3}{7}$, which implies $\mu'(v) \ge -1 + 2 \times \frac{11}{35} + \frac{3}{7} > 0$.

CASE 4. $d(v_i) \ge 6$, where $1 \le i \le 3$. Now each incident face gives v at least $\frac{1}{3}$. ■

3.2. Proving Theorems 10 and 14

This time we apply the following rules of discharging.

R1. *Every* 3*-vertex receives* $\frac{1}{3}$ *from each incident face.*

R2. *Every* 6^+*-vertex gives* $\frac{1}{3}$ *to each incident face.*

R3. *Every* 5*-vertex gives zero to each incident* 5^+*-face.*

R4. *Every* 5*-vertex gives to each incident* 4*-face* f:

(a) $\frac{1}{3}$ *if* f *is incident with either two* 3*-vertices or one* 3*-vertex and two* 4*-vertices,*

(b) $\frac{1}{6}$ *if* f *is incident with a* 3*-vertex and either three* 5*-vertices or a* 4*-vertex and two* 5*-vertices,*

(c) *nothing otherwise.*

Suppose $f \in F$. If $d(f) \ge 6$, then $\mu'(f) \ge d(f) - 4 - d(f) \times \frac{1}{3} \ge 0$ by R1.

If $d(f) = 5$, then f is incident with at most three 3-vertices due to the absence of (3, 3, 3)-paths in M, which implies that $\mu'(f) \geq 1 - 3 \times \frac{1}{3} = 0$.

Finally suppose that $f = v_1vv_2x$. Note that f is incident with at most two 3-vertices.

CASE 1. f is incident with two 3-vertices. If $d(v_1) = d(x) = 3$, then $d(v_2) \geq 5$ and $d(v) \geq 5$ due to the absence of (3, 3, 4)-paths. Each of v and v_2 gives $\frac{1}{3}$ to f by R2 or R4a, so that $\mu'(f) \geq 0 - 2 \times \frac{1}{3} + 2 \times \frac{1}{3} = 0$ by R1.

If $d(v) = d(x) = 3$, then $d(v_1) \geq 6$ and $d(v_2) \geq 6$ due to the absence of (3, 5, 3)-paths, and we are done by R1 and R2.

CASE 2. f is incident with precisely one 3-vertex x. If there is a 6^+-vertex incident with f, then $\mu'(f) \geq -\frac{1}{3} + \frac{1}{3} = 0$ by R1 and R2. So assume that f is not incident with 6^+-vertices.

If f is incident at most one 4-vertex, then $\mu'(f) \geq -\frac{1}{3} + 2 \times \frac{1}{6} = 0$ by R1 and R4b. Note that f cannot be incident with three 4-vertices due to the absence of (3, 4, 4)-paths in M if we are in Theorem 10, and (4, 3, 4)-paths in M if we deal with Theorem 14.

So suppose that f is incident with two 4-vertices and one 5-vertex. Note that these two 4-vertices are adjacent if we are in Theorem 14, and not adjacent in Theorem 10. So, we are done by R1 and R4a.

CASE 3. f is not incident with 3-vertices. Since f does not participate in R1, we have $\mu'(f) \geq 0$.

If $v \in V$ is a 6^+-vertex, then $\mu'(v) \geq d(v) - 4 - d(v) \times \frac{1}{3} \geq 0$ by R2. If $d(v) = 3$, then $\mu'(v) \geq -1 + 3 \times \frac{1}{3} = 0$ by R1. If $d(v) = 4$, then v does not participate in discharging, so $\mu'(v) = \mu(v) = 0$.

To complete the proof, it suffices to check that every 5-vertex v satisfies $\mu'(v) \geq 0$. If v gives at most $\frac{1}{6}$ to each incident face by R3, R4b, R4c, then $\mu'(v) \geq 1 - 5 \times \frac{1}{6} > 0$.

So suppose v gives $\frac{1}{3}$ to a face $f_1 = vv_1xv_2$ by R4a. Let there be faces $f_2 = vv_2y\ldots$, $f_3 = vv_3u_3\ldots$, $f_4 = vv_4u_4\ldots$, and $f_5 = v_5vv_1z\ldots$.

CASE 1. f_1 is incident with two 3-vertices. Since M does not contain (3, 5, 3)-paths, we can assume that $d(v_1) = d(x) = 3$, and $d(v_2) \geq 5$ due to the absence of (3, 3, 4)-paths in M for both Theorems 10 and 14. Note that $d(v_i) \geq 4$ with $3 \leq i \leq 5$, and $d(u_1) \geq 5$. We see that v gives at most $\frac{1}{6}$ to each of the faces f_5, f_2. This means that we are done unless v gives $\frac{1}{3}$ to one of the faces f_3 and f_4 and at least $\frac{1}{6}$ to the other by R4. In particular, $f_3 = vv_3u_3v_4$ and $f_4 = vv_4u_4v_5$. By symmetry, we can assume that f_3 receives $\frac{1}{3}$ from v. This implies that $d(u_3) = d(u_4) = 3$ and $d(v_3) = d(v_4) = 4$. We have a (4, 3, 4)-path $v_3u_3v_4$ and a (3, 4, 3)-path $u_3v_4u_4$, which is impossible for M.

CASE 2. f_1 is incident with precisely one 3-vertex and two 4-vertices.

Subcase 2.1. $d(v_1) = d(v_2) = 4$, $d(x) = 3$. This is possible only if we deal with Theorem 10. Note that $d(y) \geq 5$ and $d(z) \geq 5$ since M has no (3, 4, 4,)-paths. If both v_3 and v_5 are 4^+-vertices, then v gives nothing to f_2 and f_5, which implies that $\mu'(v) \geq 1 - 3 \times \frac{1}{3} = 0$.

By symmetry, suppose that $d(v_5) = 3$. So $d(v_3) \geq 4$ and $d(v_4) \geq 4$ due to the absence of (3, 5, 3)-paths. Note that v gives at most $\frac{1}{6}$ to f_5 and 0 to f_2 by R4b and R4c. We are done unless v gives $\frac{1}{3}$ to each of f_3, f_4, which means that M contains the (3, 4, 4)-path $u_3v_4u_4$, a contradiction.

Subcase 2.2. $d(v_1) = 3$, $d(v_2) = d(x) = 4$. This is possible only if we are in Theorem 14. Here $d(z) \geq 5$ and $d(v_i) \geq 4$ with $3 \leq i \leq 5$. We see that v gives at most $\frac{1}{6}$ to each of f_i, where $2 \leq i \leq 4$, since M has no (4, 3, 4,)-paths. The same is true for f_5 by R4b, R4c. This implies that $\mu'(v) \geq 1 - 4 \times \frac{1}{6} - \frac{1}{3} = 0$. ∎

4. Proving Theorem 15

Suppose $D = x_1y_1z_1 \vee \cdots \vee x_ky_kz_k$ is a tight description of 3-paths in triangle-free NPMs. This means that

(1) every NPM has a (x_i, y_i, z_i)-path for at least one i with $1 \leq i \leq k$, and

(2) if we delete any term $x_iy_iz_i$ from D or decrease any parameter in D by one without changing the other $3k-1$ parameters, then the new description is not satisfied by at least one NPM.

Note that, due to its tightness, the description D cannot have triplets XYZ and $X'Y'Z'$ such that $X \leq X'$, $Y \leq Y'$, and $Z \leq Z'$, for $D' = D \setminus \{XYZ\}$ is equivalent to D but shorter.

CASE 1. D has a term $XYZ = 3^+5^+3^+$.

Subcase 1.1. $X \geq 4$ or $Z \geq 4$. By Corollary 11, D is true and not stronger than the tight description 354, which implies that $D = 354$.

Subcase 1.2. $X = Z = 3$. We note that the graph G_3 (see Fig. 3) has no $(3, 5^+, 3)$-paths. However, G_3 has both (3, 4, 4)-paths and (4, 3, 4)-paths. This implies that to be satisfied by G_3, our D should contain one of the terms $3^+4^+4^+$ and $4^+3^+4^+$.

In the first case, we deduce from Theorem 10 that $D = 353 \vee 344$, because $D = 3^+5^+3^+ \vee 3^+4^+4^+ \vee \cdots$, which is not stronger than $353 \vee 344$, but D is tight by assumption. In the second case, we see from Theorem 14 that $D = 353 \vee 434$.

CASE 2. D has no term $XYZ = 3^+5^+3^+$. We now look at the graph G_1 (see Fig. 1). Since D has no $3^+5^+3^+$, it follows that D must contain a term $5^+3^+6^+$ to be satisfied by G_1.

Subcase 2.1. D has a term 5^+46^+. By Corollary 13, we have $D = 546$.

Subcase 2.2. D has a term $XYZ = 5^+36^+$.

Subcase 2.2.1. $X \geq 7$ or $Z \geq 7$. By Corollary 9, we have $D = 537$.

Subcase 2.2.2. $5 \leq X \leq 6$ and $5 \leq Z \leq 6$, i.e. $D = 536 \vee \cdots$ or $D = 636 \vee \cdots$. However, G_2 (see Fig. 2) has neither term 536 nor 636. This implies that to be satisfied by G_2, our D should contain one of the terms $3^+4^+3^+$ and $4^+3^+7^+$.

In the first case, we deduce from Theorem 12 that $D = 536 \vee 343$. In the second case, Theorem 8 implies that $D = 536 \vee 437$.

Thus we have proved that there are precisely seven tight descriptions of 3-paths in normal triangle-free plane maps. ■

Acknowledgments

The first author was supported by the Russian Foundation for Basic Research (grants 12-01-00631, 15-01-05867) and by President Grants for Government Support of the Leading Scientific Schools of the Russian Federation (grant NSh-1939.2014.1) The second author's work was performed as a part of government work "Organizing research".

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