Discrete Mathematics 338 (2015) 2042–2050

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Hamiltonian claw-free graphs with locally disconnected vertices

Runli Tian [a,b], Liming Xiong [a,*]

[a] School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, PR China

[b] School of Science, Central South University of Forestry and Technology, Changsha 410004, PR China

ARTICLE INFO

Article history:
Received 23 June 2013
Received in revised form 3 April 2015
Accepted 20 April 2015
Available online 6 June 2015

Keywords:
Claw-free graph
Hamiltonian
Closure
Locally disconnected vertex
Singular edge

ABSTRACT

An edge of G is *singular* if it does not lie on any triangle of G; otherwise, it is *non-singular*. A vertex u of a graph G is called *locally connected* if the induced subgraph $G[N(u)]$ by its neighborhood is connected; otherwise, it is called *locally disconnected*.

In this paper, we prove that if a connected claw-free graph G of order at least three satisfies the following two conditions: For each locally disconnected vertex v of G with degree at least 3, there is a nonnegative integer s such that v lies on an induced cycle of length at least 4 with at most s non-singular edges and with at least $s-3$ locally connected vertices; for each locally disconnected vertex v of G with degree 2, there is a nonnegative integer s such that v lies on an induced cycle C with at most s non-singular edges and with at least $s-2$ locally connected vertices and such that the subgraph induced by those vertices of C that have degree two in G is a path or a cycle, then G is Hamiltonian, and it is best possible in some sense.

Our result is a common extension of two known results in Bielak (2000) and in Li (2002) ; hence also of the results in Oberly and Sumner (1979) and in Ryjáček (1990).

1. Introduction

We consider only finite undirected simple graphs, unless otherwise stated. For terminology and notation not defined in this paper we refer to [9].

If H is a graph, then the *line graph* of H, denoted by $L(H)$, is the graph with $E(H)$ as its vertex set, in which two vertices are adjacent if and only if the corresponding edges have a vertex in common. For a family $\mathcal{F}$ of a connected graphs, a graph is called $\mathcal{F}$-free if it contains no induced copies of any member of $\mathcal{F}$. The graph $K_{1,3}$ is called a *claw*. It is a well-known fact that every line graph is claw-free, hence the class of the claw-free graphs can be considered as a natural generalization of the class of line graphs.

The neighborhood of a vertex v in G is denoted by $N_G(v)$. Denote $N_G[v] = N_G(v) \cup \{v\}$. A vertex v of G is *locally connected* if $G[N_G(v)]$ is connected; otherwise, it is *locally disconnected*. Let $LC(G)$ denote the set of all locally connected vertices of G. A graph G is called *locally connected* if every vertex of G is locally connected, *i.e.*, $LC(G) = V(G)$. Oberly and Sumner proved the following well-known result.

Theorem 1 (*Oberly and Sumner [5]*)**.** *Every connected, locally connected claw-free graph on at least three vertices is Hamiltonian.*

* Corresponding author.
E-mail address: lmxiong@bit.edu.cn (L. Xiong).

http://dx.doi.org/10.1016/j.disc.2015.04.020
0012-365X/

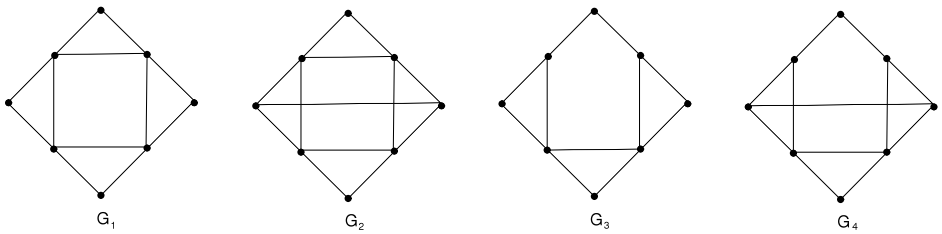

Fig. 1. The graphs G_1, G_2, G_3 and G_4.

We say that a vertex v of a graph G is N_2-*locally connected* if the subgraph of G induced by the edge set $\{e = xy \in E(G) : v \notin \{x, y\}$ and $\{x, y\} \cap N(v) \neq \emptyset\}$ is connected. A graph G is called N_2-*locally connected* if every vertex of G is N_2-locally connected. It follows from the definitions that every locally connected graph is N_2-locally connected, but the converse is not true.

In 1990, Ryjáček [7] considered the graphs with some locally disconnected vertices in claw-free graphs and strengthened Theorem 1 by using this concept of N_2-locally connected. He showed that every connected N_2-locally connected claw-free graph G with $\delta(G) \geq 2$ satisfying that G has no induced subgraph H isomorphic to either G_1 or G_2 (in Fig. 1) such that every vertex of degree 4 in H is locally disconnected in G is Hamiltonian. Bielak later improved this result by weakening the condition. Their result can be restated as the following theorem, where $V_i(G) = \{x : d_G(x) = i\}$ and $V_{\geq i}(G) = \{x : d_G(x) \geq i\}$.

Theorem 2 (*Bielak [1]*). *Let G be a connected, N_2-locally connected claw-free graph with $\delta(G) \geq 2$ such that*

(1) *every induced subgraph H of G isomorphic to either G_1 or G_2 (in Fig. 1) has at least one locally connected vertex of G in $V_3(H) \cup V_4(H)$.*

Then G is Hamiltonian.

In this paper, we shall continue to extend the above result which will need some notation. We say that a vertex v of a graph G is N^2-*locally connected* if the subgraph of G induced by the vertices $\{x \in V(G) : 1 \leq d(x, v) \leq 2\}$ is connected, where $d(x, v)$ denotes the distance between x and v. A graph G is called N^2-*locally connected* if every vertex of G is N^2-locally connected. Obviously, every N_2-locally connected graph is N^2-locally connected, but the converse is not generally true.

Theorem 3. *Let G be a connected, N^2-locally connected claw-free graph with $\delta(G) \geq 2$ satisfying*

(2) *every induced subgraph isomorphic to one of $\{G_1, G_2, G_3, G_4\}$ (in Fig. 1) has at least one locally connected vertex of G in $V_3(H) \cup V_4(H)$.*

Then G is Hamiltonian.

From Theorem 3, one can obtain the following known result immediately.

Corollary 4 (*Li [4]*). *Every connected N^2-locally connected $\{G_1, G_2, G_3, G_4, K_{1,3}\}$-free graph G with $\delta(G) \geq 2$ is Hamiltonian.*

Let G_0 be the graph obtained from some graph G_i in Fig. 1 by joining all vertices of an additional complete graph of arbitrarily larger order to some vertex of degree four or three in G_i and to its neighbors. Then G_0 satisfies the conditions of Theorem 3 but not Corollary 4. This shows that Theorem 3 is stronger than Corollary 4.

Motivated by the above observation, in this paper, we intend to generality them by avoiding using the concept of N_2-(or N^2-)connected and use certain technical conditions on locally disconnected vertices instead. Here we need divide all edges of the graphs into two kinds of edges: An edge e of G is *singular* if it does not lie on any triangle of G; otherwise, it is *non-singular*. We have the following result that can deduce Theorem 3, as showed in Section 4.

Theorem 5. *Let G be a connected claw-free graph of order at least three such that*

(i) *for each locally disconnected vertex v of degree at least 3 in G, there is a nonnegative integer s such that v lies on an induced cycle of length at least four with at most s non-singular edges and with at least $s - 3$ locally connected vertices;*
(ii) *for each locally disconnected vertex v of degree 2 in G, there is a nonnegative integer s such that v lies on an induced cycle C with at most s non-singular edges and with at least $s - 2$ locally connected vertices and such that $G[V(C) \cap V_2(G)]$ is a path or a cycle.*

Then G is Hamiltonian.

In Section 2, we shall present Ryjáček's closure concept in claw-free graphs and some auxiliary results, which are then applied to the proof of our main result in Section 3. Section 4 is devoted to the proof of Theorem 3. In the last section, we discuss the sharpness of our main results, point out a flaw in the original proof of Corollary 4 and show that Theorem 5 is stronger than Theorem 3, and hence also than Corollary 4.

2. The closure of claw-free graphs

A locally connected vertex v is said to be *eligible* if $G[N_G(v)]$ is not complete. For a vertex x of a graph G, the graph G^*_x with $V(G^*_x) = V(G)$ and $E(G^*_x) = E(G) \cup \{uv : u, v \in N_G(x)\}$ is called the *local completion* of G at x. For a claw-free graph G, let $G_1 = G$. For $i \geq 1$, if G_i is defined and if it has an eligible vertex x_i, then let $G_{i+1} = (G_i)^*_{x_i}$. If $G_s = (G_{s-1})^*_{x_{s-1}}$ has no eligible vertex, then let $\mathrm{cl}(G) = G_s$ and call it the *closure of* G; $G_1, \ldots, G_s$ is called a locally complete sequence of graphs that yields $\mathrm{cl}(G)$. The above operation was introduced in [8] and the following theorem sums up some properties.

Theorem 6 (*Ryjáček [8]*). *If G is a claw-free graph, then there is a closed claw-free graph $\mathrm{cl}(G)$ such that*

(3) *the closure $\mathrm{cl}(G)$ is well-defined;*
(4) *there is a triangle-free graph H such that $\mathrm{cl}(G) = L(H)$;*
(5) *G is Hamiltonian if and only if $\mathrm{cl}(G)$ is Hamiltonian.*

In a claw-free graph G, the locally disconnected vertices can be partitioned into three classes, depending on the structure of the graphs $G[N(v)]$: Let $LD_i(G)$ be the set of locally disconnected vertices v for which there are exactly i components in $G[N(v)]$ of order greater than one. (Note that the notations here are something different from [6].) Note that for a locally disconnected vertex v, $G[N(v)]$ consists of exactly two complete subgraphs of G. Pfender proved the following.

Lemma 7 (*Pfender, [6]*). *$(LD_0(\mathrm{cl}(G)) \cup LD_1(\mathrm{cl}(G))) \subseteq (LD_0(G) \cup LD_1(G))$ and $LD_2(\mathrm{cl}(G)) \subseteq LD_2(G)$ for every claw-freegraph G.*

We need the following lemma, which follows from Lemma 7.

Lemma 8. *For $i \in \{0, 1, 2\}$, $LD_i(\mathrm{cl}(G)) \subseteq LD_i(G)$ for every claw-free graph G.*

Proof of Lemma 8. Suppose that $x \in LD_0(\mathrm{cl}(G))$, i.e, x is locally disconnected in $\mathrm{cl}(G)$ and $d_{\mathrm{cl}(G)}(x) = 2$. By Lemma 7, x is locally disconnected in G, hence $d_G(x) = 2$. Thus $x \in LD_0(G)$.

Suppose that $x \in LD_1(\mathrm{cl}(G))$. We claim that $x \notin LD_0(G)$: otherwise, let $N_G(x) = \{y_1, y_2\}$. Note that every edge in $E(\mathrm{cl}(G)) \setminus E(G)$ is non-singular, so either xy_1 or xy_2 is singular in $\mathrm{cl}(G)$. Thus $y_1y_2 \notin E(\mathrm{cl}(G))$. Let $G = G_1, \ldots, G_s = \mathrm{cl}(G)$ be a locally complete sequence of graphs that yields $\mathrm{cl}(G)$. Then $x \in LD_0(G_i)$ for some i. We can deduce that $x \in LD_0(G_{i+1})$ by the fact that both y_1 and y_2 are not eligible vertices in G_i and $y_1y_2 \notin E(G_{i+1})$. Hence $x \in LD_0(\mathrm{cl}(G))$, a contradiction. By Lemma 7, $x \in LD_1(G)$. $\square$

The following result is useful for proving our main result.

Lemma 9. *Let G be a graph satisfying all conditions of Theorem 5. Then $\mathrm{cl}(G)$ is a connected claw-free graph such that*

(6) *every locally disconnected vertex of degree at least 3 in $\mathrm{cl}(G)$ lies on an induced cycle of length at least 4 with at most 3 non-singular edges;*
(7) *every locally disconnected vertex of degree 2 in $\mathrm{cl}(G)$ lies on an induced cycle C' with at most 2 non-singular edges such that $\mathrm{cl}(G)[V(C') \cap V_2(\mathrm{cl}(G))]$ is a path or a cycle.*

In order to prove Lemma 9, we need the following lemmas. A *branch* in G is a nontrivial path with end vertices that do not lie in $V_2(G)$ and with internal vertices of degree 2 (if existing). If a branch has length 1, then it has no internal vertices of degree 2. We use $\mathcal{B}(G)$ to denote the set of branches in G.

Lemma 10. *Let G be a claw-free graph. If the length of $L \in \mathcal{B}(G)$ is at least 3 in G, then $L \in \mathcal{B}(\mathrm{cl}(G))$.*

Proof of Lemma 10. Let $G = G_1, \ldots, G_s = \mathrm{cl}(G)$ be a locally complete sequence of graphs that yields $\mathrm{cl}(G)$. Then L is a branch in G_i for some i. Since every vertex of $V(L)$ is not eligible in G_i, $x_i \in V(G) \setminus V(L)$, where x_i is the eligible vertex such that $G_{i+1} = (G_i)^*_{x_i}$. Noticing that $(V(L) \cap V_2(G)) \cap N(x_i) = \emptyset$ and $|V(L) \cap V_2(G)| \geq 2$, L is a branch in G_{i+1}. By recursively performing this operation, we can obtain that L is a branch of $\mathrm{cl}(G)$. $\square$

Lemma 11. *Let G be a claw-free graph and C be an induced cycle with at most s non-singular edges in G and with at least $s - l$ locally connected vertices in G. If $x \in V(C)$ is locally disconnected in $\mathrm{cl}(G)$, then there is an induced cycle C' of length at least 4 in $\mathrm{cl}(G)$ with $x \in V(C') \subseteq V(C)$ and with at most l non-singular edges in $\mathrm{cl}(G)$, where s and l are nonnegative integers.*

Proof of Lemma 11. Since $x \in V(C)$ is locally disconnected in $\mathrm{cl}(G)$, there is an induced cycle C' in $\mathrm{cl}(G)$ such that $x \in V(C') \subseteq V(C)$ and $|V(C')| \geq 4$. It remains to prove that C' has at most l non-singular edges in $\mathrm{cl}(G)$.

Note that every vertex of C' is locally disconnected in $\mathrm{cl}(G)$. By Lemma 8, $V(C') \cap LD_i(\mathrm{cl}(G)) \subseteq V(C) \cap LD_i(G)$ for $i \in \{0, 1, 2\}$. Hence the number of non-singular edges in C' is no more than s, the number of non-singular edges in C. If C has no locally connected vertex in G, then $s = l$, hence we are done. Now consider $s \neq l$.

Suppose that $\{u_1, \ldots, u_{s-l}\} \subseteq V(C) \cap LC(G)$. By Condition (3) in Theorem 6, $\mathrm{cl}(G)$ is uniquely determined by the graph G, i.e., $\mathrm{cl}(G)$ is independent of the order of eligible vertices during the construction. Note that each u_i is an eligible vertex in G by the hypothesis that C is an induced cycle. Let $G_1 = G^*_{u_1}$ and $N_G(u_1) \cap V(C) = \{v_1, v_2\}$. Then there exists an induced cycle

C_1 in G_1 with $V(C_1) = V(C) \setminus \{u_1\}$ and $E(C_1) = (E(C) \setminus \{u_1v_1, u_1v_2\}) \cup \{v_1v_2\}$. Since u_1v_1, u_1v_2, v_1v_2 are non-singular in G, C_1 has at most $s-1$ non-singular edges in G_1. Since C_1 is an induced cycle, u_i is an eligible vertex in G_1 for $i \in \{2, \ldots, s-l\}$. By recursively performing the local completion on u_i for $i \in \{1, \ldots, s-l\}$, we can obtain an induced cycle C_{s-l} in G_{s-l} such that C_{s-l} has at most $s-(s-l) = l$ non-singular edges in G_{s-l} and $V(C_{s-l}) = V(C) \setminus \{u_1, \ldots, u_{s-l}\}$. By Lemma 8, $(V(C') \cap LD_i(\mathrm{cl}(G))) \subseteq (V(C_{s-l}) \cap LD_i(G_{s-l}))$ for $i \in \{0, 1, 2\}$. Hence the number of non-singular edges of C' in $\mathrm{cl}(G)$ is no more than the number l of non-singular edges of C_{s-l} in $\mathrm{cl}(G)$. □

Now we provide the proof of Lemma 9.

Proof of Lemma 9. First suppose that x is a locally disconnected vertex of degree at least 3 in $\mathrm{cl}(G)$. Then either $x \in LD_1(\mathrm{cl}(G))$ or $x \in LD_2(\mathrm{cl}(G))$. By Lemma 8, either $x \in LD_1(G)$ or $x \in LD_2(G)$. This implies that x is locally disconnected in G and $d_G(x) \geq 3$. By Condition (i) of Theorem 5, x lies on an induced cycle of length at least 4 with at most s non-singular edges and with at least $s-3$ locally connected vertices. By Lemma 11, x satisfies Condition (6) of Lemma 9.

Next suppose that x is a locally disconnected vertex of degree 2 in $\mathrm{cl}(G)$. Then x is a locally disconnected vertex of degree 2 in G. By Condition (ii) of Theorem 5, x lies on an induced cycle C of length at least 4 with at most s non-singular edges and with at least $s-2$ locally connected vertices such that $G[V(C) \cap V_2(G)]$ is a path or a cycle. By Lemma 11, x lies on an induced cycle C' with $V(C') \subseteq V(C)$ and with at most 2 non-singular edges.

If $G[V(C) \cap V_2(G)]$ is a cycle, then since G is connected, G is a cycle. Hence $\mathrm{cl}(G)$ is a cycle and we are done. If $V(C) \cap V_2(G) = \{x\}$, then since $x \in V(C') \subseteq V(C)$ and $x \in V_2(\mathrm{cl}(G)) \subseteq V_2(G)$, $V(C') \cap V_2(\mathrm{cl}(G)) = \{x\}$ and we are also done. Suppose that $|V(C) \cap V_2(G)| \geq 2$ and L is the branch such that $(V(C) \cap V_2(G)) \subseteq V(L)$. By Condition (ii) of Theorem 5, $L \in \mathcal{B}(G)$ is the unique branch in C. By Lemma 10, $L \in \mathcal{B}(\mathrm{cl}(G))$ is the unique branch in C'. This implies that $\mathrm{cl}(G)[V(C') \cap V_2(\mathrm{cl}(G))]$ is a path (of L). Now we complete the proof of Lemma 9. □

3. Proof of Theorem 5

In this section, we present the proof of the main result of this paper. A graph is *Eulerian* if it is connected and has no vertex of odd degree. For a graph G with an Eulerian subgraph H, we call H *a spanning Eulerian subgraph* of G if $V(G) = V(H)$; and *a dominating Eulerian subgraph* of G if $G - V(H)$ is edgeless.

Theorem 12 (*Lai [3]*). *Let G be a 2-connected graph with $\delta(G) \geq 3$. If every edge of G lies on a cycle of length at most 4, then G has a spanning Eulerian subgraph.*

Note that the graphs in consideration in Theorem 12 may have multiple edges. A well-known relationship between dominating Eulerian subgraphs in G and Hamiltonian cycles in $L(G)$ was given by Harary and Nash-Williams.

Theorem 13 (*Harary and Nash-Williams [2]*). *Let G be a graph with at least 3 edges. Then the line graph $L(G)$ is Hamiltonian if and only if G has a dominating Eulerian subgraph.*

The following result is immediately from Condition (7) in Lemma 9, which is also necessary for our proof.

Lemma 14. *Let G be a graph satisfying the conditions of Theorem 5. Then every branch $L \in \mathcal{B}(\mathrm{cl}(G))$ of length at least 2 lies on an induced cycle C such that C has at most 2 non-singular edges and L is an unique branch of length at least 2 in C.*

Before presenting the proofs of main results, we give some additional notation. Let M and M' be the two sets of edges of a graph G. We use $M \triangle M'$ to denote the symmetric difference of M and M', i.e, $M \triangle M' = (M \cup M') \setminus (M \cap M')$. An edge e is called a *pendant* edge if the degree of an end vertex of e is 1; otherwise, it is non-pendant. The graph H for which $L(H) = G$ will be called the *preimage* of G and denote $H = L^{-1}(G)$. Note that for subgraph $G_1 \subseteq G$, $L^{-1}(G_1)$ is possible not unique. However $L^{-1}(G_1)$ would be unique if $G_1 \subseteq G$ is an induced subgraph of order at least three. Therefore, for any induced subgraph C of a line graph G, we let $L^{-1}(C)$ denote the preimage of C.

Given 2-connected block B of a simple graph H that is not a cycle, let $U(B) = \{u : d_B(u) = 2 \text{ and } d_H(u) \geq 3\}$ and $U_1(B) = \{u : u \in U(B) \text{ and } N_B(u) \cap V_2(H) = \emptyset\}$ and $U_2(B) = U(B) \setminus U_1(B)$.

Now we present the proof of our main result.

Proof of Theorem 5. Suppose firstly that G is itself a cycle, then it is clearly Hamiltonian and we are done. Now suppose that G is not a cycle. By (5) in Theorem 6, it suffices to prove that its closure $\mathrm{cl}(G)$ is Hamiltonian. By (4) in Theorem 6, we may assume that $\mathrm{cl}(G) = L(H)$ is the line graph of a triangle-free graph H.

In order to use Theorem 13, it suffices to find a dominating Eulerian subgraph in H. For this, in the following, we use Theorem 12 to get a dominating Eulerian subgraph in each block with some properties and then prove that the union of these Eulerian subgraphs is the desired dominating Eulerian subgraph.

Taking any block B of H that is not a cycle or a pendant edge, we may show the following claims.

Claim 1. *Every edge $e = uv \in E(B)$ lies on some cycle C such that either*

(8) *C has exactly 2 vertices of degree greater than 2 in H and C has exactly one branch of length at least 3 in H; or*
(9) *C has at most 3 vertices of degree greater than 2 in H and C has no branch of length at least 3 in H.*

Proof of Claim 1. By Lemma 9, every locally disconnected vertex in $\mathrm{cl}(G)$ satisfies Condition (6) or (7) of Lemma 9. Note that if $e \in E(\mathrm{cl}(G))$ is singular, then $d_H(e) = 2$ because G is claw-free. We first prove the following fact.

Claim 1.0. *Every branch $L \in \mathcal{B}(H)$ of length at least 3 lies on a cycle C such that C has exactly 2 vertices of degree greater than 2 in H and L is the unique branch of length 3 in C.*

Proof of Claim 1.0. Because $G = L(H)$, the $V(G)$ is identified with $E(H)$. Thus, $L' \in \mathcal{B}(\mathrm{cl}(G))$ is the line graph of L. Note that $|L'| = |L| - 1 \geq 2$. By Lemma 14, there exists an induced cycle C' such that C' has at most 2 non-singular edges and L' is the unique branch of length at least 2 in C'.

By the fact that $\mathrm{cl}(G) = L(H)$, $L^{-1}(C')$ is a cycle in H such that $L^{-1}(C')$ has at most 2 vertices of degree greater than 2 in H and L is the unique branch of length at least 3 in H. Moreover, since B is not a cycle, $L^{-1}(C')$ has at least 2 vertices of degree greater than 2 in H. Thus $L^{-1}(C')$ has exactly 2 vertices of degree greater than 2 in H. □

Now we start to prove Claim 1. If e lies on a branch $L \in \mathcal{B}(H)$ of length at least 3, then e lies on a cycle satisfying (8) by Claim 1.0. Now suppose that e lies on a branch $L \in \mathcal{B}(H)$ of length 1 or 2. Let $v_e \in V(\mathrm{cl}(G))$ be the vertex corresponding to the edge e in H. Then $d_{\mathrm{cl}(G)}(v_e) \geq 3$. Since H is triangle free, we have $N_H(u) \cap N_H(v) = \emptyset$. By the fact that B is 2-connected, $N_H(u) \neq \emptyset$ and $N_H(v) \neq \emptyset$. Furthermore, since $\mathrm{cl}(G)$ is claw-free, $\mathrm{cl}(G)[N_{\mathrm{cl}(G)}(v_e)]$ is composed of two vertex-disjoint cliques, i.e., v_e is locally disconnected in $\mathrm{cl}(G)$.

By the hypotheses of Lemma 9, v_e lies on an induced cycle C_e of length at least four with at most 3 non-singular edges. Hence, by the fact that $\mathrm{cl}(G) = L(H)$, $L^{-1}(C_e)$ is a cycle in H such that $e \in E(L^{-1}(C_e))$ and $|V(L^{-1}(C_e)) \cap V_{\geq 3}(H)| \leq 3$. Since B is not a cycle, $|V(L^{-1}(C_e)) \cap V_{\geq 3}(H)| \geq 2$. Note that $L^{-1}(C_e)$ has a branch of length 1 or 2. Therefore, if $|V(L^{-1}(C_e)) \cap V_{\geq 3}(H)| = 2$, then $L^{-1}(C_e)$ has at most one branch of length at least 3 which implies that $L^{-1}(C_e)$ satisfies (8); if $|V(L^{-1}(C_e)) \cap V_{\geq 3}(H)| = 3$ and $L^{-1}(C_e)$ has t branches of length at least 3, then $t \leq 2$. Suppose that L_1, L_2 are the two possible branches of length at least 3 in $L^{-1}(C_e)$ ($L_1 = L_2$ if $t = 1$). By Claim 1.0, L_i lies on a cycle C_i satisfying (8) for $i = 1, 2$. Thus, either $C = H[E(L^{-1}(C_e)) \triangle E(C_1)) \triangle E(C_2)]$ (if $t = 2$), or $C = H[E(L^{-1}(C_e)) \triangle E(C_1)]$ (if $t = 1$), or $C = C_e$(if $t = 0$) is a cycle such that $e \in E(C)$ and such that C satisfies (9), which completes the proof of Claim 1. □

Claim 2. *Every vertex u of $U(B)$ lies on a cycle C such that C satisfies (9) of Claim 1 and $|V(C) \cap V_{\geq 3}(B)| = 2$ and $V(C) \cap U(B) = \{u\}$.*

Proof of Claim 2. Let e be an edge incident with u in B. By Claim 1, e lies on a cycle C satisfying Condition (8) or (9) of Claim 1. This implies that $|V(C) \cap V_{\geq 3}(H)| \leq 3$. Since B is not a cycle, $|V(C) \cap V_{\geq 3}(B)| \geq 2$. Note that $u \in V(C) \cap (V_{\geq 3}(H) \setminus V_{\geq 3}(B))$, $|V(C) \cap V_{\geq 3}(H)| \geq 3$. Thus $|V(C) \cap V_{\geq 3}(H)| = 3$ and $|V(C) \cap V_{\geq 3}(B)| = 2$. This implies that C satisfies Condition (9) of Claim 1 and $V(C) \cap U(B) = \{u\}$. □

Claim 3. *Every vertex u of $U_1(B)$ lies on a cycle C of length 4 such that $|V(C) \cap V_2(B)| = 2$, $|V(C) \cap V_{\geq 3}(H)| = 3$ and $|V(C) \cap V_2(H)| = 1$.*

Proof of Claim 3. By Claim 2, we may take a cycle C such that C satisfies Condition (9) of Claim 1. Note that $N_B[u] \subseteq V(C)$ and $N_B[u] \subseteq V_{\geq 3}(H)$, $|V(C) \cap V_{\geq 3}(H)| = 3$. Since H is triangle-free and there is no branch of length at least 3 in C, $|V(C)| = 4$ and $|V(C) \cap V_2(H)| = 1$. Since $u \in V_2(B) \setminus V_2(H)$, $|V(C) \cap V_2(B)| = 2$. □

Using the above claims, we obtain three kinds of blocks B_1, B_2, B_3 from B by the following three steps. Note that possibly $B_i = B_{i-1}$ for some ≤ 2, where $B_0 = B$.

(I) Let B_1 be obtained from B by replacing each branch of length 2 with an edge. Then we have the following fact.

Fact 1. *If B_1 has a spanning Eulerian subgraph, then B has a dominating Eulerian subgraph F such that $(V_3(B) \cup (V_{\geq 3}(H) \cap V(B))) \subseteq V(F)$, i.e., $(V_3(B) \cup U(B)) \subseteq V(F)$.*

Clearly $V_1(H) \cap V(F) = \emptyset$. Furthermore, there may be some vertices in $V_2(B) \cap V_2(H)$ that are not in $V(F)$, i.e., possibly $(V_2(B) \cap V_2(H)) \setminus V(F) \neq \emptyset$. However, Fact 1 shows that $(V_{\geq 3}(H) \cap V_2(B)) \setminus V(F) = \emptyset$.

Proof of Fact 1. Suppose B_1 has a spanning Eulerian subgraph F_1. By the definition of B_1 in Step (I), B has a dominating Eulerian subgraph F such that $V_{\geq 3}(B) \subseteq V(F_1) \subseteq V(F)$. Note that $(V_{\geq 3}(H) \cap V(B)) \setminus V_{\geq 3}(B) = U(B)$. Since any $u \in U_2(B)$ lies on a branch of length at least 3 in B, $U_2(B) \subseteq V(F)$ by the definition of B_1 in Step (I). Now suppose $u \in U_1(B)$, then u lies on a branch of length 2 in B. Furthermore, by Claim 3, u lies on a 4-cycle C with two branches of length 2 in B with $|V(C) \cap V_2(B)| = 2$, $|V(C) \cap V_{\geq 3}(H)| = 3$ and $|V(C) \cap V_2(H)| = 1$ and there would exist two parallel edges in B_1 by Step (I). Note that one of the parallel edges is obtained by replacing branch containing u and the other is

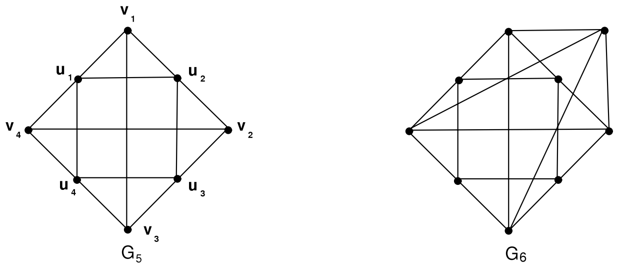

Fig. 2. The graphs G_5 and G_6.

obtained by replacing the branch of length 2 containing the vertex of $V(C) \cap V_2(H)$. Note also that F_1 may be chosen such that it contains at least one of the parallel edges. Hence we may assume that F_1 is a spanning Eulerian subgraph of B_1 such that the edge obtained by replacing the branch containing u belong to F_1. This implies that $U_1(B) \subseteq V(F)$. Thus, $(V_{\geq 3}(H) \cap V(B)) \subseteq V(F)$. By the definition of B_1, $V_3(B) \subseteq V(F)$. This completes the proof of Fact 1. □

(II) Let B_2 be obtained from B_1 by replacing each branch L of length at least 3 with a branch of length 3. Then

Fact 2. *If B_2 has a spanning Eulerian subgraph, then B_1 has a spanning Eulerian subgraph.*

(III) Let $v, v' \in V(L) \cap V_2(B_2)$ and let B_3 be a graph obtained from B_2 and two additional vertices $w_v, w_{v'}$ by adding all the edges $\{w_v v, w_v v', w_{v'} v, w_{v'} v', w_v w_{v'}\}$. Then $vv' \in E(B_2)$ and

Fact 3. *If B_3 has a spanning Eulerian subgraph, then B_2 has a spanning Eulerian subgraph.*

By the definition of B_3, and by Claims 1 and 2, we may obtain the following fact immediately.

Fact 4. *Every edge of B_3 lies in a cycle of length at most 4 and $\delta(B_3) \geq 3$.*

Note that the vertex of degree greater than 2 is the dominating Eulerian subgraph of a block that is a pendant edge and H has no block of order 2 that is not a pedant edge by Lemma 9. Hence, since B_3 is 2-connected, B_3 has an spanning Eulerian subgraph by Fact 4 and by Theorem 12. Combining Facts 1–3, one may obtain the following claim.

Claim 4. *Every block B of H that is not a cycle has a dominating Eulerian subgraph containing all vertices of $V_3(B) \cup (V_{\geq 3}(H) \cap V(B))$.*

Note that every block of H that is a cycle has a spanning Eulerian subgraph which is itself the cycle. This together with Claim 4 implies that the union of those dominating Eulerian subgraphs of these blocks is a dominating Eulerian subgraphs of H. (Note that every cut vertex of H has degree at least three in H and then, by Claim 4, it lies on a dominating Eulerian subgraph of the nontrivial blocks containing it. Thus, the union of these dominating Eulerian subgraphs of these blocks is connected by its definition.) By Theorem 13, $\text{cl}(G) = L(H)$ is Hamiltonian. The proof of Theorem 5 is complete. □

4. Proof of Theorem 3 — Theorem 5 implies Theorem 3

Before proving Theorem 3, we start with the following lemmas.

Lemma 15. *Let G be a connected claw-free graph satisfying Condition (1) of Theorem 2. If G has an induced subgraph $H \cong G_5$ such that every edge of H is non-singular in G and such that H has a locally disconnected vertex x of G with $d_H(x) = 4$ that does not satisfy Condition (i) of Theorem 5, then $G \cong G_6$ (see Fig. 2).*

Proof of Lemma 15. Let H be an induced subgraph of G isomorphic to the graph G_5 in Fig. 2. If H has a locally connected vertex in G, then every locally disconnected vertex x of H with $d_H(x) = 4$ satisfies Condition (i) of Theorem 5, contradicting the hypothesis. Now suppose that

(∗) every vertex of H is locally disconnected in G.

All the subscripts are taken module by 4 in the whole proof.

Claim 5. *For any $x \in V(G - H)$, if $\{xv_i, xv_{i+2}\} \subset E(G)$ for some $i \in \{1, 2\}$, then $xu_j \notin E(G)$ for any $j \in \{1, 2, 3, 4\}$.*

Proof of Claim 5. By contradiction. Without loss of generality, we may assume that there exists a vertex $x_0 \in V(G-H)$ such that $\{x_0v_1, x_0v_3\} \subset E(G)$ and $x_0u_1 \in E(G)$. Hence $\{v_1, v_3, u_1\} \subseteq N(x_0)$ and $G[\{v_1, v_3, u_1\}]$ is connected but not complete by the hypothesis that H is an induced subgraph of G. This implies that x_0 is an eligible vertex of G.

Noticing that u_1 is locally disconnected in G and $\{u_2, v_1, x_0\}$ belongs to a connected component of $G[N(u_1)]$, we have that $x_0u_2 \in E(G)$ and $x_0u_4 \notin E(G)$. Since u_2 is locally disconnected, $x_0u_3 \notin E(G)$. Hence we can obtain two induced cycles $C_1 = u_1u_4v_3x_0u_1$ and $C_2 = u_2u_3v_3x_0u_2$ such that they have exactly four non-singular edges and have a locally connected vertex x_0 of G. This implies that every locally disconnected vertex of degree 4 in H satisfies (i) of Theorem 5, contradicting the hypothesis. □

Claim 6. *Let C_1 and C_2 be any two triangles containing v_1v_3 and v_2v_4, respectively. Then $|V(C_1) \cap V(C_2)| = 1$.*

Proof of Claim 6. Suppose, otherwise, that $C_1 = w_1v_1v_3w_1$ and $C_2 = w_2v_2v_4w_2$ and $w_1 \neq w_2$. Since $H \cong G_5$ is an induced subgraph, $\{w_1, w_2\} \subseteq V(G-H)$. Consider $S_j = (V(H) \setminus \{v_{j+1}\}) \cup \{w_j\}$ for $j \in \{1, 2\}$. By Claim 5, $w_ju_i \notin E(G)$ for any $i \in \{1, 2, 3, 4\}$. Suppose $w_jv_{j+3} \notin E(G)$, then $G[S_j] \cong G_2$. By the hypothesis of Lemma 15, there is at least a locally connected vertex in $V(H) \setminus \{v_{j+1}\}$. This contradicts $(*)$. Thus $w_jv_{j+3} \in E(G)$. By symmetry, we can obtain $w_jv_{j+1} \in E(G)$.

Since $\{u_1, u_2, w_1, w_2\} \subset N(v_1)$, $w_1w_2 \in E(G)$: otherwise $G[\{v_1, w_1, w_2, u_1\}]$ would be a claw by Claim 5, a contradiction. Hence $\{w_2, v_1, v_2, v_3, v_4\} \subseteq N(w_1)$ and $G[\{w_2, v_1, v_2, v_3, v_4\}]$ is connected but not complete. This implies that w_1 is an eligible vertex. Hence we can obtain that $C_i = u_iv_iw_1v_{i-1}u_i$ is an induced cycle with a locally connected vertex w_1 that implies u_i satisfying Condition (i) of Theorem 5 for any $i \in \{1, 2, 3, 4\}$, contradicting the hypothesis. □

By Claim 6, we may assume that w is the common neighbor of $\{v_1, v_2, v_3, v_4\}$. Noticing that H is an induced subgraph in G and $wu_i \notin E(G)$ by Claim 5, we can obtain that $G[V(H) \cup \{w\}] \cong G_6$. Now suppose that $x \in V(G) \setminus (V(H) \cup \{w\})$. If $xw \in E(G)$, then since $\{v_1, v_2, v_3, v_4, x\} \subseteq N(w)$ and by the fact that G is claw-free, then $xv_i, xv_{i+2} \in E(G)$ for some $i \in \{1, 2\}$, say $i = 1$. Hence we can obtain two triangles xv_1v_3x and wv_2v_4w. By Claim 6, $x = w$. However, this contradicts that $x \in V(G) \setminus (V(H) \cup \{w\})$. Hence $xw \notin E(G)$.

If $xu_i \in E(G)$ for some $i \in \{1, 2, 3, 4\}$, then since $\{u_{i+1}, u_{i-1}, v_{i-1}, v_i, x\} \subset N(u_i)$ and by the fact that G is claw-free, $\{xv_i, xu_{i+1}\} \subset E(G)$ or $\{xv_{i-1}, xu_{i-1}\} \subset E(G)$, say $\{xv_i, xu_{i+1}\} \subset E(G)$. Consider $S' = (V(H) \setminus \{v_i\}) \cup \{x\}$. By Claim 5, $xv_{i+2} \notin E(G)$. Since u_i, u_{i+1} are locally disconnected, $\{xu_{i+2}, xu_{i+3}, xv_{i-1}, xv_{i+1}\} \cap E(G) = \emptyset$. Thus $G[S'] \cong G_2$. By the hypothesis of Lemma 15, there is a locally connected vertex in $V(H) \setminus \{v_i\}$, contradicting $(*)$. Thus $xu_i \notin E(G)$. Similarly, we can obtain $xv_i \notin E(G)$ for any $i \in \{1, 2, 3, 4\}$. Since G is connected, $V(G) = V(H) \cup \{w\}$. Thus $G \cong G_6$. We have now completed the proof of Lemma 15. □

Lemma 16. *Let G be a claw-free graph satisfying Condition* (1) *of Theorem 2. If a locally disconnected u_1 lies on an induced cycle of length* 4, *then either u_1 satisfies Condition* (i) *or* (ii) *of Theorem 5; or $G \cong G_6$.*

Proof of Lemma 16. Suppose that C is an induced cycle of length 4 with $u_1 \in V(C)$. If either $V(C) \cap LC(G) \neq \emptyset$ or C has a singular edge, then u_1 satisfies Condition (i) or (ii) of Theorem 5 (Note that if $d(u_1) = 2$, then the two edges incident with u_1 are singular edges and clearly u_1 satisfies Condition (ii) of Theorem 5.) and we are done.

Now consider the case when $V(C) \cap LC(G) = \emptyset$ and C has no singular edge. Let $C = u_1u_2u_3u_4u_1$ and $T_i = v_iu_iu_{i+1}v_i$ be a triangle containing u_iu_{i+1}, and let $S = \bigcup_{i=1}^{i=4} V(T_i)$, where $i \in \{1, 2, 3, 4\}$ and all the subscripts are taken module by 4 in the proof. Note that $v_iv_{i+1} \notin E(G)$ and $v_i \neq v_{i+1}$ since u_{i+1} is locally disconnected. Similarly, v_1, v_2, v_3, v_4 are four distinct vertices, since, otherwise, say, $v_1 = v_3$, then u_1 would be locally connected, a contraction. Since C is an induced cycle in G and every vertex of C is locally disconnected in G, $\{u_iv_{i+1}, u_iv_{i+2}, u_iu_{i+2}, v_iv_{i+1} : i \in \{1, 2, 3, 4\}\} \cap E(G) = \emptyset$.

Suppose first that $\{v_1v_3, v_2v_4\} \cap E(G) = \emptyset$, then $G[S] \cong G_1$. By the hypothesis of Theorem 2, some $u_i \neq u_1$ is locally connected in G. Hence u_1 satisfies Condition (i) of Theorem 5.

Suppose next that exactly one of $\{v_1v_3, v_2v_4\}$, say v_1v_3, belongs to $E(G)$, then $G[S] \cong G_2$. By the hypothesis of Theorem 2, at least one of $\{u_1, u_2, u_3, u_4, v_1, v_3\}$ is locally connected in G. Hence u_1 satisfies Condition (i) of Theorem 5.

Suppose finally that $\{v_1v_3, v_2v_4\} \subset E(G)$. Then $G[S] \cong G_5$. If exactly one of $\{v_1v_3, v_2v_4\}$, say v_2v_4, is non-singular in G, then we can obtain an induced cycle $C' = u_1v_1v_3u_4u_1$ such that u_1 satisfies Condition (i) of Theorem 5. Now suppose that both v_1v_3 and $v_2v_4 \in E(G)$ are non-singular in G. If u_1 satisfies Condition (i) of Theorem 5, then we are done. Otherwise by Lemma 15, $G \cong G_6$ which is Hamiltonian. This completes the proof of Lemma 16. □

Now we prove Theorem 3.

Proof of Theorem 3. We shall prove that the locally disconnected vertices in G satisfy Condition (i) or (ii) of Theorem 5 with only one exceptional case when $G \cong G_6$. Suppose that u is a locally disconnected vertex of G. Since G is N^2-locally connected, u lies on an induced cycle C of length 4 or 5. If $|V(C)| = 4$, then by Lemma 16, either u satisfies Condition (i) or (ii) of Theorem 5 or $G \cong G_6$ and we are done.

Now suppose $|V(C)| = 5$. If either $|V(C) \cap LC(G)| \geq 2$ or C has at most 2 non-singular edges, then u satisfies Condition (i) or (ii) of Theorem 5 (Note that if C has a vertex of degree two, then C has at most three non-singular edges.) and we are done. It remains to consider the case when $|V(C) \cap LC(G)| \leq 1$ and C has at least 3 non-singular edges. Let $C = u_1u_2u_3u_4u_5u_1$, where $u = u_i$ for some i. All the subscripts are taken by 5 in the whole proof.

Case 1. C has a path P of length 3 such that every vertex of P is locally disconnected and every edge of P is non-singular.

Without loss of generality, we may assume that $P = u_2u_3u_4u_5$. Let $T_i = v_iu_{i+1}u_{i+2}v_i$ be a triangle for $i = 1, 2, 3$, and let $S = V(C) \bigcup \bigcup_{i=1}^{i=3} V(T_i)$. Since each vertex of P is locally disconnected, v_1, v_2, v_3 are distinct vertices. Consider the induced subgraph $G[S]$. Since each $u_i \neq u_1$ is locally disconnected in G, $\{v_1v_2, v_2v_3, v_iu_i, v_iu_{i+3} : i \in \{1, 2, 3\}\} \cap E(G) = \emptyset$. Thus $v_iu_{i-1} \notin E(G)$: Otherwise, $G[\{u_{i-1}, v_i, u_i, u_{i-2}\}]$ is a claw since C is an induced cycle, a contradiction. If $v_1v_3 \notin E(G)$, then this implies that $G[S] \cong G_3$; this is impossible by Condition (2) of Theorem 3. Thus $v_1v_3 \in E(G)$, implying that $G[S] \cong G_4$ and hence at least one of $\{v_1, v_3\}$ is locally connected by Condition (2) of Theorem 3.

First suppose that either u_1 is locally connected or at least one of $\{u_1u_5, u_1u_2\}$ is singular, then u lies on either an induced cycle $C' = u_1u_2v_1v_3u_5u_1$ or an induced cycle $C'' = v_1u_3u_4v_3v_1$, hence u satisfies Condition (i) or (ii) of Theorem 5 and we are done.

Next suppose that u_1 is locally disconnected and both u_1u_5 and u_1u_2 are non-singular, then let $T_i = v_iu_{i+1}u_{i+2}v_i$ be a triangle for $i \in \{4, 5\}$, and let $S' = V(C) \cup \bigcup_{i=3}^{i=5} V(T_i)$. Then $v_4 \neq v_5$. Now consider the induced subgraph $G[S']$. Similar to the discussion on $G[S]$ of the fact that $v_1v_3 \in E(G)$, we can obtain that $v_3v_5 \in E(G)$. But this implies that $G[\{v_3, u_4, v_1, v_5\}]$ is a claw, a contradiction.

Case 2. C has no a path P of length 3 such that every vertex of P is locally disconnected and every edge of P is non-singular.

If C has no locally connected vertex, then it suffices to consider the case when C has exactly 3 non-singular edges of G which are not consecutive. This implies that the degree of every vertex of C is at least 3 in G. Hence u satisfies Condition (i) of Theorem 5.

If C has a locally connected vertex, then it suffices to consider the case when C has at least one singular edge in G. If $d(u) \geq 3$, then C has at most 4 non-singular edges and a locally connected vertex. If $d(u) = 2$, then C has at most 3 non-singular edges and a locally connected vertex. This implies that u satisfies Condition (i) or (ii) of Theorem 5, in either case above. This completes the proof of Theorem 3. □

5. Concluded remarks

5.1. Sharpness

In this subsection, we discuss the sharpness and show that the conditions of Theorems 5 and 3 are all best possible in some sense.

- Let $k \geq 2$ be an integer. Let G^k be the graph obtained from $K_{2,3}$ by attaching k pendant edges to each vertex of $K_{2,3}$, and then subdividing each original edge once. It is straightforward to check that the graph $H^k = L(G^k)$ is 2-connected claw-free graph with $\delta(H^k) \geq k+1$ such that every locally disconnected vertex of G lies on an induced cycle of length 8 with 4 non-singular edges and without a locally connected vertex. However, it is not Hamiltonian: Otherwise, by Theorem 13, G^k has a dominating Eulerian subgraph, a contradiction. This shows that "$s - 3$ locally connected vertices" in Condition (i) of Theorem 5 cannot be replaced by "$s-4$ locally-connected vertices" even under an additional condition that a graph has any given large minimum degree.
- We demonstrate that the condition "$G[V(C) \cap V_2(G)]$ is a path or a cycle" in Theorem 5(ii) is necessary. The graph G' in Fig. 3 is a graph satisfying Condition (i) but not (ii) since the two locally disconnected vertices of degree two do not satisfy Condition (ii) of Theorem 5 (although they lie on an induced cycle with only two nonsingular edges). It is straightforward to check that G' is not Hamiltonian. This shows that the condition "$G[V(C) \cap V_2(G)]$ is a path or a cycle" in Condition (ii) of Theorem 5 is necessary. One can obtain many such graphs with any large order by joining a clique of any large order to any nontrivial maximal clique of G'.
- Theorem 3 is best possible in the sense that Condition (2) cannot be weakened. To see this, consider G' depicted in Fig. 3. The graph G' satisfies all conditions of Theorem 3 except that every induced subgraph H of G' isomorphic to G_3 does not have a locally connected vertex in $V_3(H) \cup V_4(H)$. Note that G' is not Hamiltonian. This shows that "at least one locally connected vertex" in Condition (2) of Theorem 3 is necessary.

5.2. A flaw in the original proof of Corollary 4

Looking at the original proof of Corollary 4, we find that the author used the assumption that if a claw-free graph satisfies the condition that it is connected N^2-locally connected $\{G_1, G_2, G_3, G_4\}$-free (where G_i is the graph in Fig. 1), then so does its closure. However, this is not generally true: For example, the claw-free graph G'' in Fig. 3 is connected N^2-locally connected $\{G_1, G_2, G_3, G_4\}$-free, however, $\mathrm{cl}(G'')$ has an induced subgraph isomorphic to G_3. This implies that $\mathrm{cl}(G'')$ does not satisfy the conditions of Corollary 4 any more. One can obtain many such graphs of any large order by joining a clique of any large order to any nontrivial maximal clique of G''. Our proof has conquered the flaw.

5.3. Theorem 5 is stronger than Theorem 3 and Corollary 4

We show that some graphs satisfy Theorem 5 but not Theorem 3. As we showed in Section 5.2, the closure of the graph G'' depicted in Fig. 3 does not satisfy the conditions of Corollary 4. In fact, it does not satisfy Theorem 3. However, it satisfies the conditions of Theorem 5. This shows that Theorem 5 is stronger than Theorem 3.

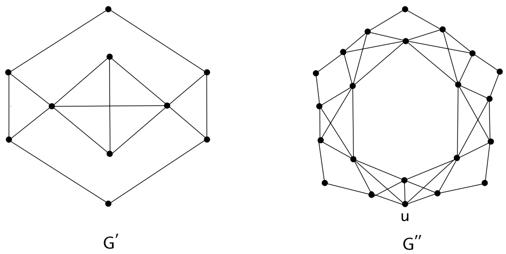

Fig. 3. The graph G', G'': u is a locally connected vertex of G''.

Acknowledgments

The authors thank the referees for their careful reading and valuable comments and also thank the editor for providing the website: http://www.math.uiuc.edu/~west/grammar.html to correct the grammar mistakes. The work is supported by the Natural Science Funds of China and by Specialized Research Fund for the Doctoral Program of Higher Education (No. 20131101110048). The second author is supported by the mathematical foundation in Tianyuan (11426222).

References

[1] H. Bielak, Sufficient condtion for Hamiltonicity of N_2-locally connected claw-free graph, Discrete Math. 213 (2000) 21–24.
[2] F. Harary, C.St.J.A. Nash-Williams, On eulerian and hamiltonian graphs and line graphs, Canad. Math. Bull. 8 (1965) 701–709.
[3] H.-J. Lai, Graph whose edges are in small cycles, Discrete Math. 94 (1991) 11–22.
[4] M.C. Li, Hamiltonian cycles in N^2-locally connected claw-free graphs, Ars Combin. 62 (2002) 281–288.
[5] D.J. Oberly, D.P. Sumner, Every connected, locally connected nontrivial graph with no induced claw is hamiltonian, J. Graph Theory 3 (1979) 351–356.
[6] F. Pfender, Hamiltonicity and forbidden subgraphs in 4-connected graphs, J. Graph Theory 49 (2005) 262–272.
[7] Z. Ryjáček, Hamiltonian circuits in N_2-locally connected $K_{1,3}$-free graphs, J. Graph Theory 14 (1990) 321–331.
[8] Z. Ryjáček, On a closure concept in claw-free graphs, J. Combin. Theory Ser. B 70 (1997) 217–224.
[9] D.B. West, Introduction to Graph Theory, second ed., Prentice Hall, Upper Saddle River, NJ, 2001.