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Improved monochromatic loose cycle partitions in hypergraphs^{*}



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ABSTRACT

Improving our earlier result we show that every large enough complete *k*-uniform *r*-colored hypergraph can be partitioned into at most $50rk \log(rk)$ vertex disjoint monochromatic loose cycles. The proof uses a strong hypergraph Regularity Lemma due to Rödl and Schacht and the new, powerful hypergraph Blow-up Lemma of Keevash.

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1. Monochromatic cycle partitions

Suppose first that K_n is a complete graph on n vertices whose edges are colored with r colors ($r \ge 1$). How many monochromatic cycles are needed to partition the vertex set of K_n ? This question received a lot of attention in the last few years. Throughout the paper, single vertices and edges are considered to be cycles. Let p(r, n) denote the minimum number of monochromatic cycles needed to partition the vertex set of any r-colored K_n . In [4] Erdős, Gyárfás and Pyber proved that there exists an absolute constant c such that $p(r, n) \le cr^2 \log r$ (throughout this paper log denotes natural logarithm). Furthermore, in [4] (see also [6]) the authors conjectured the following.

Conjecture 1. p(r, n) = r.

Thus the number of monochromatic cycles needed would be independent of the order of the complete graph. The special case r = 2 of this conjecture was asked earlier by Lehel and for $n \ge n_0$ was first proved by Łuczak, Rödl and Szemerédi [19]. Allen improved on the value of n_0 [1] and recently Bessy and Thomassé [3] proved the original conjecture for r = 2. For general r the current best bound is due to Gyárfás, Ruszinkó, Sárközy and Szemerédi [7] who proved that for $n \ge n_0(r)$ we have $p(r, n) \le 100r \log r$. For r = 3 the conjecture was asymptotically proved in [8], i.e. it was proved that every 3-coloring of K_n admits 3 vertex disjoint monochromatic cycles covering all but o(n) vertices. However, surprisingly Pokrovskiy [20] found a counterexample to the conjecture when $r \ge 3$. In the counterexample all but one vertex can be covered by r vertex disjoint monochromatic cycles. Thus a slightly weaker version of the conjecture still can be true, say that apart from a constant number of vertices the vertex set can be covered by r vertex disjoint monochromatic cycles.

Let us also note that the above problem was generalized in various directions; for complete bipartite graphs (see [4,12]), for graphs which are not necessarily complete (see [2,24]) and for vertex partitions by monochromatic connected k-regular subgraphs (see [25,26]).

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In this paper we study the generalization of this problem for *r*-edge colorings of hypergraphs. This question was initiated in [10]. We will consider two kinds of hypergraph cycles: a *loose* cycle in a *k*-uniform hypergraph is a sequence of edges, e_1, \ldots, e_t such that for $1 \le i \le t$, $e_i \cap e_{i+1} = v_i$ where $e_{t+1} = e_1$ and all v_i -s are distinct (non-consecutive edges do not share any vertices). A *tight* cycle is a sequence of *t* vertices (v_0, \ldots, v_{t-1}) such that $\{v_i, \ldots, v_{i+k-1}\}$ is an edge for each *i*, where *i* is taken modulo *t*.

In [10] we proved that for all integers $r \ge 1$, $k \ge 3$ there exists a constant c = c(r, k) such that in every *r*-coloring of the edges of the complete *k*-uniform hypergraph $K_n^{(k)}$ the vertex set can be partitioned into at most c(r, k) vertex disjoint monochromatic loose cycles. Thus again we have the same phenomenon as for graphs; the partition number does not depend on the order of the hypergraph. Although the bound c(r, k) was not computed explicitly in [10], it is quite weak; it is exponential in *r* and *k*. Here we give a significant improvement of this result.

Theorem 1. For all integers $r, k \ge 2$ there exists a constant $n_0 = n_0(r, k)$ such that if $n \ge n_0$ and the edges of the complete *k*-uniform hypergraph $K_n^{(k)}$ are colored with r colors then the vertex set can be partitioned into at most 50rk log(rk) vertex disjoint monochromatic loose cycles.

We believe that there is still room for improvement but we do not risk an exact conjecture. Furthermore, it would be nice to obtain a similar result for tight cycles.

2. Sketch of the proof of Theorem 1

We may assume throughout that $k \ge 3$, since for graphs we have the result in [7]. A matching in a *k*-uniform hypergraph (or *k*-graph) *H* is called *connected* if its edges are all in the same connected component of *H*, where two vertices are in the same connected component if there is a walk between them. The matching is *self-connected* if the connecting walks can be found within the vertex set of the matching. The study of connected matchings for graphs was initiated in [18] and played an important role in several recent papers (e.g. [7,9]).

To prove Theorem 1 we will follow our proof technique from [7] which is using the Regularity Lemma of Szemerédi [27] and the graph Blow-up Lemma [16,17]. However, here we have to adapt the method to hypergraphs, so we are going to use the technique from [15] (where they find loose Hamiltonian cycles in dense hypergraphs) which in turn is using the strong hypergraph Regularity Lemma due to Rödl and Schacht [22] and the new, powerful hypergraph Blow-up Lemma of Keevash [14].

Consider an *r*-edge coloring $(H_1, H_2, ..., H_r)$ of $K_n^{(k)}$. Let us take the color in this coloring that has the most edges. For simplicity assume that this is H_1 and call this color red. We apply the strong hypergraph Regularity Lemma to H_1 . Then we introduce the so called *reduced hypergraph* R, the hypergraph whose vertices are associated to the clusters in the partition. Then we find a large self-connected matching in this reduced graph.

Following our method from [7] we establish the bound on the number of monochromatic loose cycles needed to partition the vertex set in the following steps.

- Step 1: We find a sufficiently large self-connected matching *CM* in *R*. It is implicitly proved in [15] using the hypergraph Blow-up Lemma of Keevash that we can remove a small number of exceptional vertices from each cluster of *CM* so that there is a spanning loose red cycle in the remainder of *CM* (assuming that the total number of remaining vertices is divisible by k 1).
- Step 2: We remove the non-exceptional vertices of the clusters of *CM* from $K_n^{(k)}$. We greedily remove a number (depending on *r* and *k*) of vertex disjoint monochromatic loose cycles from the remainder in $K_n^{(k)}$ until the number of leftover vertices is much smaller than the number of vertices associated to *CM*.
- Step 3: Using a lemma about loose cycle covers of *r*-colored unbalanced complete "bipartite" *k*-uniform hypergraphs we combine the leftover vertices with some vertices of the clusters associated with vertices of *CM*.
- Step 4: We remove an additional at most k 2 vertices (degenerate loose cycles) from the vertices of *CM* to make sure that the total number of remaining vertices is divisible by k 1. Finally we find a red cycle spanning the remaining vertices of *CM* (using the above remark).

The organization of the proof follows this outline. After giving the definitions and tools, we discuss each step one by one. We are using techniques from [7,15], but to make the present paper more self-contained we repeat several definitions and arguments here.

3. Tools

3.1. Strong hypergraph regularity and the hypergraph blow-up lemma

In this section we will state the version of the strong hypergraph Regularity Lemma we will use and the hypergraph Blow-up Lemma. We follow the notation and terminology necessary for the statement of these results from [15].

By [r] we denote the set of integers from 1 to r. We write $x = y \pm z$ to mean that $y - z \le x \le y + z$. For a set A, $\binom{A}{k}$

denotes the collection of subsets of *A* of size *k*, and similarly $\begin{pmatrix} A \\ \leq k \end{pmatrix}$ denotes the collection of non-empty subsets of *A* of size at most *k*.

A *k*-graph is a hypergraph in which all the edges are of size *k*. We say that a hypergraph *H* is a *k*-complex if every edge has size at most *k* and *H* forms a simplicial complex, that is, if $e_1 \in H$ and $e_2 \subseteq e_1$ then $e_2 \in H$. We identify a hypergraph *H* with the set of its edges. So |H| is the number of edges in *H*, and if *G* and *H* are hypergraphs then $G \setminus H$ is formed by removing from *G* any edge which also lies in *H*. If *H* is a hypergraph with vertex set *V* then for any $V' \subseteq V$ the restriction H[V'] of *H* to V' is defined to have vertex set *V* and all edges of *H* which are contained in *V* as edges. Also, for any hypergraphs *G* and *H* we define G - H to be the hypergraph $G[V(G) \setminus V(H)]$.

We say that a hypergraph *H* is *r*-partite if its vertex set *X* is divided into *r* pairwise disjoint parts X_1, \ldots, X_r , in such a way that for any edge $e \in H$, $|e \cap X_i| \le 1$ for each *i*. We call X_i the *vertex classes* of *H* and say that the partition X_1, \ldots, X_r of *X* is *equitable* if all X_i have the same size. We say that a set $A \subseteq X$ is *r*-partite if $|A \cap X_i| \le 1$ for each *i*. Similarly we may define *r*-partite *k*-complexes. Given a *k*-graph *H*, we define a *k*-complex $H^{\leq} = \{e_1 : e_1 \subseteq e_2 \text{ and } e_2 \in H\}$ and a (k - 1)-complex $H^{<} = \{e_1 : e_1 \subseteq e_2 \text{ and } e_2 \in H\}$. Conversely, for a *k*-complex *H* we define the *k*-graph $H_{=}$ to be the 'top level' of *H*, i.e. $H_{=} = \{e \in H : |e| = k\}$.

Let $X = X_1 \cup \cdots \cup X_r$. Given $A \in {\binom{[r]}{\leq k}}$, we write $K_A(X)$ for the complete |A|-partite |A|-graph whose vertex classes are all X_i with $i \in A$. The *index* of an *r*-partite subset *S* of *X* is $i(S) = \{i \in [r] : S \cap X_i \neq \emptyset\}$. Given any set $B \subseteq i(S)$, we write $S_B = S \cap \bigcup_{i \in B} X_i$. Similarly, given $A \in {\binom{[r]}{\leq k}}$ and an *r*-partite *k*-graph or *k*-complex *H* on the vertex set *X* we write H_A for the collection of edges in *H* of index *A* and let $H_{\emptyset} = \{\emptyset\}$. In particular, if *H* is a *k*-complex then $H_{\{i\}}$ is the set of all those vertices in X_i which lie in an edge of *H* (and thus form a (singleton) edge of *H*). Also, given a *k*-complex *H* we similarly write $H_{A\leq} = \bigcup_{B\subseteq A} H_B$ and $H_{A<} = \bigcup_{B\subseteq A} H_B$. We write H_A^* for the |A|-graph whose edges are those *r*-partite sets $S \subseteq X$ of index *A* for which all proper subsets of *S* belong to *H*. Then the *relative density of H at index A* is $d_A(H) = |H_A|/|H_A^*|$. The *absolute density of H* a singlet H on $X = X_1 \cup \cdots \cup X_k$ is $d(H) = |H|/|K_{[k]}(X)|$.

For any vertex v of a hypergraph H, we define the vertex degree d(v) of v to be the number of edges of H which contain v. The maximum vertex degree of H is then the maximum of d(v) taken over all vertices $v \in V(H)$. The vertex neighborhood VN(v) of v is the set of all vertices $u \in V(H)$ for which there is an edge of H containing both u and v. For a k-partite k-complex H on the vertex set $X_1 \cup \cdots \cup X_k$ we also define the neighborhood complex H(v) of a vertex $v \in X_i$ for some i to be the (k - 1)-partite (k - 1)-complex with vertex set $\bigcup_{i \neq i} X_i$ and edge set $\{e \in H : e \cup \{x\} \in H\}$.

Next we will define the concept of regular complexes. For any $A \in \binom{[r]}{\leq k}$, we say that G_A is ϵ -regular if for any $H \subseteq G_{A^<}$ with $|H_A^*| \geq \epsilon |G_A^*|$ we have

$$\frac{|G_A \cap H_A^*|}{|H_A^*|} = d_A(G) \pm \epsilon.$$

We say *G* is ϵ -regular if *G*_{*A*} is ϵ -regular for every $A \in {[r] \\ <k}$.

We will also need the notion of a 'partition complex', which is a certain partition of the edges of a complete *k*-complex. As before, let $X = X_1 \cup \cdots \cup X_r$ be an *r*-partite vertex set. A *partition k-system P* on *X* consists of a partition P_A of the edges of $K_A(X)$ for each $A \in {[r] \\ \leq k}$. We refer to the partition classes of P_A as cells. So every edge of $K_A(X)$ is contained in precisely one cell of P_A . *P* is a *partition k-complex on X* if it also has the property that whenever *S*, $S' \in K_A(X)$ lie in the same cell of P_A , we have that S_B and S'_B lie in the same cell of P_B for any $B \subseteq A$. This property of *S*, S' forms an equivalence relation on the edges of $K_A(X)$, which we refer to as *strong equivalence*.

Let *P* be a partition *k*-complex on $X = X_1 \cup \cdots \cup X_r$. For $i \in [k]$, the cells of $P_{\{i\}}$ are called *clusters*. We say that *P* is *vertex-equitable* if all clusters have the same size. *P* is *a-bounded* if $|P_A| \leq a$ for every *A*. Also, for any *r*-partite set $Q \in \begin{pmatrix} X \\ \leq k \end{pmatrix}$, we write C_Q for the set of all edges lying in the same cell of *P* as *Q*, and write $C_{Q^{\leq}}$ for the *r*-partite *k*-complex whose vertex set is *X* and whose edge set is $\bigcup_{Q' \subseteq Q} C_{Q'}$. The partition *k*-complex *P* is ϵ -regular if $C_{Q^{\leq}}$ is ϵ -regular for every *r*-partite $Q \in \begin{pmatrix} X \\ \leq k \end{pmatrix}$.

Given a partition (k - 1)-complex P on X and $A \in \binom{[r]}{k}$, we can define an equivalence relation on the edges of $K_A(X)$, namely that $S, S' \in K_A(X)$ are equivalent if and only if S_B and S'_B lie in the same cell of P for any strict subset $B \subset A$. We refer to this as *weak equivalence*. Note that if the partition complex P is a-bounded, then $K_A(X)$ is divided into at most a^k classes by weak equivalence. If we let G be an r-partite k-graph on X, then we can use weak equivalence to refine the partition $\{G_A, K_A(X) \setminus G_A\}$ of $K_A(X)$ (i.e. two edges of G_A are in the same cell if they are weakly equivalent and similarly for the edges not in G_A). Together with P, this yields a partition k-complex which we denote by G[P]. If G[P] is ϵ -regular then we say that G is perfectly ϵ -regular with respect to P.

Finally, we say that *r*-partite *k*-graphs *G* and *H* on *X* are ν -close if $|G_A \triangle H_A| < \nu |K_A(X)|$ for every $A \in {\binom{[r]}{k}}$, that is, if there are few edges contained in *G* but not in *H* and vice versa.

Now we are ready to state the version of the strong hypergraph Regularity Lemma due to Rödl and Schacht we will need. It states that for any large enough k-graph H there exists another k-graph G that is close to H and which is regular with respect to some partition complex. There are other versions of the strong hypergraph Regularity Lemma which give information on H itself (see [23,5]) but these do not have the density conditions that are needed for the hypergraph Blow-up Lemma (see the discussion in [14]).

Theorem 2 (Theorem 14 in [22], see also Theorem 3.1 in [15]). Suppose integers n, t, a, k and reals ϵ , ν satisfy $0 < 1/n \ll \epsilon \ll 1/a \ll \nu$, 1/t, 1/k and alt divides n. Suppose also that H is a t-partite k-graph whose vertex classes X_1, \ldots, X_t form an equitable partition of its vertex set X, where |X| = n. Then there is an a-bounded ϵ -regular vertex-equitable partition (k - 1)-complex P on X and a t-partite k-graph G on X that is ν -close to H and perfectly ϵ -regular with respect to P.

Here $a \ll b$ means that *a* is sufficiently small compared to *b*.

The other main tool we will need in this section is the recent hypergraph Blow-up Lemma of Keevash. In this result there is not only a *k*-complex *G*, but also a *k*-graph *M* of "marked" or forbidden edges on the same vertex set. We will find an embedding of *any* spanning bounded vertex-degree *k*-complex in $G \setminus M$, and thus not using any of the marked edges. We will apply this with $M = G \setminus H$ where *G* is the *k*-graph given by Theorem 2, and thus we will find the embedding within the original hypergraph *H*. The use of the Blow-up Lemma will be hidden through the following definition.

Definition 1 (*Robustly universal complexes*). Suppose that J' is a *k*-partite *k*-complex on $V' = V'_1 \cup \cdots V'_k$ and $J'_{\{i\}} = V'_i$ for each $i \in [k]$. We say that J' is (c, c_0) -robustly *D*-universal if whenever

- (i) $V_j \subseteq V'_j$ are sets with $|V_j| \ge c|V'_j|$ for all $j \in [k]$, such that writing $V = \bigcup_{j \in [k]} V_j$, J = J'[V], we have $|J(v)_{=}| \ge c|J'(v)_{=}|$ for any $j \in [k]$, $v \in V_j$,
- (ii) *L* is a *k*-partite *k*-complex of maximum vertex degree at most *D* on some vertex set $U = U_1 \cup \cdots \cup U_k$ with $|U_j| = |V_j|$ for all $j \in [k]$,
- (iii) $U_* \subseteq U$ satisfies $|U_* \cap U_j| \le c_0 |U_j|$ for every $j \in [k]$, and sets $Z_u \subseteq V_{i(u)}$ satisfy $|Z_u| \ge c |V_{i(u)}|$ for each $u \in U_*$, where for each u we let i(u) be such that $u \in U_{i(u)}$,

then *J* contains a copy of *L*, in which for each $j \in [k]$ the vertices of U_j correspond to the vertices of V_j , and *u* corresponds to a vertex of Z_u for every $u \in U_*$.

Then the next lemma by Keevash claims that given a regular *k*-partite *k*-complex *G* with sufficient density (the densities must be much greater than the measure of regularity), and a *k*-partite *k*-graph *M* on the same vertex set which is small relative to *G*, we can delete a small number of vertices so that $G \setminus M$ is robustly universal.

Theorem 3 (Theorem 6.32 in [14], see also Theorem 3.3 in [15]). Suppose that $0 \le 1/n \ll \epsilon \ll c_0 \ll d^* \ll d_a \ll \theta \ll d, c, 1/k, 1/D, 1/C, G is a k-partite k-complex on <math>V = V_1 \cup \cdots V_k$ with $n \le |G_{\{j\}}| = |V_j| \le Cn$ for every $j \in [k]$, G is ϵ -regular with $d_{[k]}(G) \ge d$ and $d(G_{[k]}) \ge d_a$, and $M \subseteq G_=$ with $|M| \le \theta |G_=|$. Then we can delete at most $2\theta^{1/3}|V_j|$ vertices from each V_j to obtain $V' = V'_1 \cup \cdots \cup V'_k$, G' = G[V'] and M' = M[V'] such that

- (i) $d(G') > d^*$ and $|G'(v)| > d^*|G'_{=}|/|V'_{i}|$ for every $v \in V'_{i}$, and
- (ii) $G' \setminus M'$ is (c, c_0) -robustly D-universal.

3.2. Other tools

The following lemma will be used repeatedly in Step 2 of the proof of Theorem 1.

Lemma 1. If the edges of the complete k-uniform hypergraph $K_n^{(k)}$ are colored with r colors then there is a monochromatic loose cycle covering at least $\frac{n}{2rk}$ vertices.

For Ramsey results of loose cycles for two colors see [11,13]. The proof of this lemma also uses Theorems 2 and 3 and will be given in Section 4. In fact the proof of Theorem 1 will proceed identically to the proof of this lemma up to a point in Step 1.

We will use the following three lemmas explicitly from [15]. The first lemma claims that regular complexes remain regular when restricted to a large subset of their vertex set.

Lemma 2 (Restriction of regular complexes, Lemma 3.2 in [15]). Suppose $\epsilon \ll \epsilon' \ll d \ll c \ll 1/k$, and that *G* is an ϵ -regular *k*-partite *k*-complex on the vertex set $X = X_1 \cup \cdots \cup X_k$ such that $G_{\{i\}} = X_i$ for each *i* and d(G) > d. Let *W* be a subset of *X* such that $|W \cap X_i| \ge c|X_i|$ for each *i*. Then the restriction *G*[*W*] of *G* to *W* is ϵ' -regular, with d(G[W]) > d(G)/2 and $d_{[k]}(G[W]) > d_{[k]}(G)/2$.

The next lemma gives a lower bound on the density of a subgraph of a k-partite k-graph chosen uniformly at random.

Lemma 3 (Density of random subhypergraphs, Lemma 4.4 in [15]). Suppose $1/n \ll c, \beta, 1/k, 1/b < 1$, and that H is a k-partite k-graph on the vertex set $X = X_1 \cup \cdots \cup X_k$, where $n \le |X_i| \le bn$ for each $i \in [k]$. Suppose also that H has density $d(H) \ge c$ and that for each i we have $\beta |X_i| \le t_i \le |X_i|$. If we choose a subset $W_i \subseteq X_i$ with $|W_i| = t_i$ uniformly at random and independently for each i, and let $W = W_1 \cup \cdots \cup W_k$, then the probability that H[W] has density d(H[W]) > c/2 is at least $1 - 1/n^2$. Moreover, the same holds if we choose W_i by including each vertex of X_i independently with probability $t_i/|X_i|$.

Finally we will use a lemma about loose paths in complete k-partite k-graphs. Note that the maximum vertex degree of a loose path is two, and so this lemma will tell us when we can find a loose path in a robustly universal k-complex.

Lemma 4 (Loose paths in complete k-graphs, Lemma 4.2 in [15]). Let G be a complete k-partite k-graph on the vertex set $V_1 \cup \cdots \cup V_k$. Let b_1, \ldots, b_k be integers with $0 \le b_i \le |V_i|$ for each i. Suppose that

- $n := \frac{1}{k-1}((\sum_{i=1}^{k} b_i) 1)$ is an integer, and $\frac{n}{2} + 1 \le b_i \le n$ for all *i*.

Then for any s, $t \in [k]$, there exists a loose path in G with an initial vertex in V_s , a final vertex in V_t , and containing b_i vertices from V_i for each $i \in [k]$.

4. Proof of Lemma 1

We will use the following main parameters

$$0 < \frac{1}{n} \ll \epsilon \ll d^* \ll d_a \ll \frac{1}{a} \ll \nu \ll \theta \ll d \ll c \ll \frac{1}{k}, \frac{1}{r},$$
(1)

where again $a \ll b$ means that a is sufficiently small compared to b. Furthermore, for any of these constants α , we might also use $\alpha \ll \alpha' \ll \alpha'' \ll \ldots$ and assume that the above hierarchy extends for these constants as well, say $d'' \ll c \ll c' \ll c'' \ll \frac{1}{k}$, etc.

Consider an *r*-edge colored complete *k*-graph $K_n^{(k)}$. We remove at most *a*!*k* vertices so that the number of remaining vertices is divisible by a!k. Let $T = T_1 \cup \cdots \cup T_k$ be an equitable k-partition of the remaining vertices and let $|T_i| = n'$, $1 \le i \le k$. Consider the r-edge coloring (H_1, \ldots, H_r) of the k-partite (crossing) edges only. Let us take the color in this coloring that has the most edges. For simplicity assume that this is H_1 and call this color red. We have

$$|H_1| \ge \frac{1}{r} (n')^k.$$
 (2)

Following the technique in [15] (Section 5.1.1) we apply the strong hypergraph Regularity Lemma (Theorem 2) to H_1 with t = k (this is possible as the number of vertices is divisible by a!k). We get an *a*-bounded ϵ -regular vertex-equitable partition (k-1)-complex P on T and a k-partite k-graph G on T that is v-close to H_1 and perfectly ϵ -regular with respect to P.

Let $M = G \setminus H_1$. Thus any edge of $G \setminus M$ is indeed also an edge of H_1 . Let V_1, \ldots, V_m be the clusters of P. So $T = V_1 \cup \cdots \cup V_m$ and G is m-partite with vertex classes $V_1 \cup \cdots \cup V_m$. Let $a_1 = m/k$ and $n_1 = |V_i| = n'/a_1$ (P is equitable). We have $a_1 \leq a_1$ as P is a-bounded.

We define the reduced k-graph R: the vertices of R correspond to the clusters and a k-tuple S of vertices of R corresponds to the *k*-partite union $S' = \bigcup_{i \in S} V_i$ of clusters. The edges of *R* are those $S \in {\binom{[m]}{k}}$ for which *G* has high density and *M* has low density, more precisely G[S'] has density at least $c''(\text{i.e. } |G[S']| \ge c''|K_S(S')|)$ and M[S'] has density at most $v^{1/2}$ (i.e. $|M[S']| \le c''|K_S(S')|$) and M[S'] has density at most $v^{1/2}$ (i.e. $|M[S']| \le c''|K_S(S')|$) and M[S'] has density at most $v^{1/2}$ (i.e. $|M[S']| \le c''|K_S(S')|$) and M[S'] has density at most $v^{1/2}$ (i.e. $|M[S']| \le c''|K_S(S')|$) and M[S'] has density at most $v^{1/2}$ (i.e. $|M[S']| \le c''|K_S(S')|$) and M[S'] has density at most $v^{1/2}$ (i.e. $|M[S']| \le c''|K_S(S')|$) and M[S'] has density at most $v^{1/2}$ (i.e. $|M[S']| \le c''|K_S(S')|$) and M[S'] has density at most $v^{1/2}$ (i.e. $|M[S']| \le c''|K_S(S')|$) and M[S'] has density at most $v^{1/2}$ (i.e. $|M[S']| \le c''|K_S(S')|$) and M[S'] has density at most $v^{1/2}$ (i.e. $|M[S']| \le c''|K_S(S')|$) and M[S'] has density at most $v^{1/2}$ (i.e. $|M[S']| \le c''|K_S(S')|$) and M[S'] has density at most $v^{1/2}$ (i.e. $|M[S']| \le c''|K_S(S')|$) and M[S'] has density at most $v^{1/2}$ (i.e. $|M[S']| \le c''|K_S(S')|$) and M[S'] has density at most $v^{1/2}$ (i.e. $|M[S']| \le c''|K_S(S')|$) and M[S'] has density at most $v^{1/2}$ (i.e. $|M[S']| \le c''|K_S(S')|$) and M[S'] has density at most $v^{1/2}$ (i.e. $|M[S']| \le c''|K_S(S')|$) and M[S'] has density at most $v^{1/2}$ (i.e. $|M[S']| \le c''|K_S(S')|$) and M[S'] has density at most $v^{1/2}$ (i.e. $|M[S']| \le c''|K_S(S')|$) and M[S'] has density at most $v^{1/2}$ (i.e. $|M[S']| \le c''|K_S(S')|$) and M[S'] has density at most $v^{1/2}$ (i.e. $|M[S']| \le c''|K_S(S')|$) and M[S'] has density at most $v^{1/2}$ (i.e. $|M[S']| \le c''|K_S(S')|$) and M[S'] has density at most $v^{1/2}$ (i.e. $|M[S']| \le c''|K_S(S')|$) and M[S'] has density at most $v^{1/2}$ (i.e. $|M[S']| \le c''|K_S(S')|$) and M[S'] has density at most $v^{1/2}$ (i.e. $|M[S']| \le c''|K_S(S')|$) and M[S'] has density at $\nu^{1/2}|K_{S}(S')|).$

Consider an edge $S \in R$ and $S' = \bigcup_{i \in S} V_i$. The cells of P induce a partition $C^{S,1}, \ldots, C^{S,m_S}$ of the edges of $K_S(S')$, where $m_S \leq a^k$. We would like to select a cell $C^{S,i}$ with nice properties. We can have at most $c''|K_S(S')|/3$ edges of $K_S(S')$ within cells $C^{S,i}$ with $|C^{S,i}| \le c'' |K_S(S')|/(3a^k)$. Furthermore, we can have at most $v^{1/4}|K_S(S')|$ edges of $K_S(S')$ within cells $C^{S,i}$ with $|M \cap C^{S,i}| \ge v^{1/4}|C^{S,i}|$ since $|M[S']| \le v^{1/2}|K_S(S')|$. This and the fact that $|G[S']| \ge c''|K_S(S')|$ implies that at least $c''|K_S(S')|/2$ edges of G[S'] lie in cells $C^{S,i}$ with $|C^{S,i}| > c''|K_S(S')|/(3a^k)$ and $|M \cap C^{S,i}| < v^{1/4}|C^{S,i}|$. Thus there must exist such a set $C^{S,i}$ that also satisfies $|G \cap C^{S,i}| > c''|C^{S,i}|/2$. Fix one such a choice for $C^{S,i}$ and denote it by C^S . Let G^S be the k-partite k-complex. on the vertex set S' consisting of $G \cap C^S$ and the cells of P that underlie C^S . We also define the k-partite k-graph $M^S = G^S \cap M$ on the vertex set S'. Then the k-partite k-complex G^{S} has the following properties:

(A1) G^{S} is ϵ -regular,

- (A2) G^{S} has kth level relative density $d_{[k]}(G^{S}) > c''/2(\gg d)$, (A3) G^{S} has absolute density $d(G^{S}) \ge (c'')^{2}/6a^{k}(\gg d_{a})$, (A4) M^{S} satisfies $|M^{S}| < 2\nu^{1/4}|G^{S}|/c''(\ll \theta|G^{S}|)$,

(A5)
$$(G^{S})_{\{i\}} = V_i$$
 for any $i \in S$.

Indeed, (A1) follows from the fact that G is perfectly ϵ -regular with respect to P. To see (A2), note that $(G_{[k]}^S)^* = C^S$ and so

$$d_{[k]}(G^{S}) = \frac{|G_{[k]}^{S}|}{|(G_{[k]}^{S})^{*}|} = \frac{|G^{S} \cap C^{S}|}{|C^{S}|} > c''/2$$

by our choice of C^S . Similarly, (A3) follows from our choice of C^S since

$$d(G^{S}) = \frac{|G_{[k]}^{S}|}{|K_{S}(S')|} = \frac{|G^{S} \cap C^{S}|}{|C^{S}|} \cdot \frac{|C^{S}|}{|K_{S}(S')|} > \frac{(c'')^{2}}{6a^{k}}.$$

Finally, (A4) holds since $|G^{S}| \ge |G \cap C^{S}| > c'' |C^{S}|/2$ and $|M^{S}| \le |M \cap C^{S}| < \nu^{1/4} |C^{S}|$ and (A5) follows from the construction.

As is usual in this type of proof we have to show that the reduced *k*-graph satisfies similar density conditions as the original *k*-graph H_1 (see (2)):

$$|R| \ge \left(\frac{1}{r} - 2c''\right)a_1^k. \tag{3}$$

For this purpose first we estimate how many edges of H_1 do not belong to G[S'] for some edge $S \in R$. There are three possible reasons why an edge $e \in H_1$ does not belong to such a restriction:

- (i) *e* is not an edge of *G*. There are at most $\nu(n')^k$ edges of this type since H_1 and *G* are ν -close.
- (ii) $e \in G$ contains vertices from V_{i_1}, \ldots, V_{i_k} such that the restriction of M to $S' = \bigcup_{i \in S} V_i$ satisfies $|M[S']| > \nu^{1/2} |K_S(S')| = \nu^{1/2} n_1^k$, where $S = \{i_1, \ldots, i_k\}$ (note that since G and thus M is m-partite, i_1, \ldots, i_k are all distinct). There are at most $\nu^{1/2}(n')^k$ edges of this type since H_1 and G are ν -close.
- (iii) $e \in G$ contains vertices from V_{i_1}, \ldots, V_{i_k} such that the restriction of G to $S' = \bigcup_{i \in S} V_i$ has density less than c''. There are at most $c''(n')^k$ edges of this type.

Therefore using $\nu \ll c$ there are fewer than $2c''(n')^k$ edges of H_1 that do not belong to the restriction of G to S' for some $S \in R$. Then using (2) and $n' = a_1n_1$ we get

$$\frac{1}{r}(n')^k \le |H_1| \le 2c''(n')^k + |R|n_1^k,$$

from which we get (3).

Next we will use the following simple lemma from [21].

Lemma 5 (*Claim 4.1 in [21]*). Given c > 0 and $k \ge 2$, every k-partite k-graph with at most m vertices in each partition set and with at least cm^k edges contains a tight path on at least cm vertices.

Applying this lemma for the *k*-partite *k*-graph *R* using (3) we can find a tight path on at least $(\frac{1}{r} - 2c'')a_1$ vertices in *R*. By taking consecutive disjoint edges along this path until we can we get a self-connected matching *CM* with *t* edges in *R*, such that the number of vertices covered is

$$kt \ge \left(\frac{1}{r} - 3c''\right)a_1 = \left(\frac{1}{r} - 3c''\right)\frac{m}{k}.$$
(4)

Denote the *i*th edge of *CM* by *S*(*i*), the corresponding clusters by $\{X'_{i,1}, \ldots, X'_{i,k}\}, X'_i = \bigcup_{j=1}^k X'_{i,j}, G'_i = G^{S(i)}$ and $M'_i = M^{S(i)}$ (where the *k*-partite *k*-complex $G^{S(i)}$ and the *k*-partite *k*-graph $M^{S(i)}$ were defined above). We have

$$d(H_1[X_i']) \ge c''/2 \quad \text{for all } i \in [t].$$
(5)

Indeed, since $S(i) \in R$, $G[X'_i]$ has absolute density at least c'' and $M[X'_i]$ has density at most $v^{1/2}$. Then $G \setminus M \subseteq H_1$ and $v \ll c''$ imply (5). Furthermore, (A1)–(A5) imply that we have the following situation: G'_i is an ϵ -regular k-partite k-complex on the vertex set X'_i , with absolute density $d(G'_i) \ge (c'')^2/6a^k \gg d_a$, relative density $d_{[k]}(G'_i) > c''/2 \gg d$, $(G'_i)_{[j]} = X'_{i,j}$ for any $j \in S(i)$ and $|M'_i| < 2v^{1/4}|G'_i|/c'' \ll \theta|G'_i|$. Thus by applying Theorem 3, we can delete at most $\theta'|X'_{i,j}|$ vertices from each $X'_{i,j}$ to obtain a (c, ϵ') -robustly 2^k -universal complex. Let $X_{i,j} \subseteq X'_{i,j}$ and $X_i = \bigcup_{j=1}^k X_{i,j}$ denote the remaining vertices and $G_i = G'_i[X_i]$, $M_i = M'_i[X_i]$. Thus $G_i \setminus M_i$ is (c, ϵ') -robustly 2^k -universal with $d(G_i) > d^*$ and $|G_i(v)_{=}| > d^*|(G_i)_{=}|/|X_{i,j}|$ for every $v \in X_{i,j}$. Then we have the following properties:

- (B1) For each *i*, G_i is a *k*-partite sub-*k*-complex of *G* on the vertex set X_i . M_i is the *k*-partite *k*-graph $M \cap G_i$, and $G_i \setminus M_i \subseteq H_1$. Clearly these statements remain true after the deletion of up to ϵn_1 vertices of X_i .
- (B2) Even after the deletion of up to ϵn_1 vertices of X_i , the following statement holds. Let L be a k-partite k-complex on the vertex set $U = U_1 \cup \cdots \cup U_k$, where $|U_j| = |X_{i,j}|$ for each j, and let L have maximum vertex degree at most 2^k . Let $\ell \leq 2t$ and suppose we have $u_1, \ldots, u_\ell \in U$ and sets $Z_s \subseteq X_{i,j(u_s)}$ with $|Z_s| \geq c |X_{i,j(u_s)}|$ for each $s \in [\ell]$ (where $j(u_s)$ is such that $u_s \in U_{j(u_s)}$). Then $G_i \setminus M_i$ contains a copy of L, in which for each j the vertices of U_j correspond to the vertices of $X_{i,j}$, and each u_s corresponds to a vertex in Z_s .
- (B3) For each *i*, $H_1[X_i]$ has density at least c', even after the deletion of up to ϵn_1 vertices of X_i .
- (B4) If we delete up to ϵn_1 vertices from any X_i , and let $t_j = |X_{i,j}|$ for each $j \in [k]$ after these deletions, and let $n'_i = \frac{(\sum t_j) 1}{k-1}$, then $n'_i/2 + 1 \le t_j \le n'_i$ for all j.

Indeed, (B1) is clear as whenever we deleted vertices we simply restricted *G* and *M* to the remaining vertices. (B2) follows from the fact that $G_i \setminus M_i$ was (c, ϵ') -robustly 2^k -universal (where $X_{i,j}$ plays the role of V_j). (B3) follows from (5) and the fact that X_i was formed by deleting at most $\theta'' n_1 \ll c' |X_i'|$ vertices from X_i' . Finally, for (B4) note that (even after up to ϵn_1 more

deletions) we have deleted at most $2\theta' n_1$ vertices from each $X'_{i,j}$ to get $X_{i,j}$. Hence

- $(1 2\theta')n_1 \le t_j = |X_{i,j}| \le n_1,$ $\frac{k}{k-1}(1 2\theta')n_1 \le n'_i \le \frac{k}{k-1}n_1,$

and thus (B4) follows (using $\theta' \ll 1/k$).

We delete an additional at most k - 2 vertices to make sure that the total number of remaining vertices in $\bigcup_{i=1}^{t} X_i$ is divisible by k - 1. For simplicity let $X_{i,j}$ and $X_i = \bigcup_{j=1}^k X_{i,j}$ still denote the set of remaining vertices. The following lemma can be proved from properties (B1)–(B4)

Lemma 6. There is a loose cycle in H_1 (i.e. a red loose cycle) spanning all the vertices in $\bigcup_{i=1}^{t} X_i$.

Then indeed this red loose cycle has length at least $(\frac{1}{r} - 4c'')\frac{n}{k} \ge \frac{n}{2rk}$ (using (4)), finishing the proof of Lemma 1.

Proof of Lemma 6. We follow the argument from [15], but for the sake of completeness we provide the details.

The idea is that in each $G_i \setminus M_i$ (and thus in H_1) we can find a spanning loose path using the fact that $G_i \setminus M_i$ is robustly universal (assuming that $X_i \equiv 1 \mod (k-1)$). Then we have to join up all these loose paths we find. However, we will construct these short connecting loose paths first in such a way that the divisibility problems are dealt with. Recall that the edges of CM in R are denoted by $S(1), \ldots, S(t)$. First we find a connecting edge e'_i in R between S(i) and S(i + 1) for $1 \le i \le t - 1$: e' contains the last |k/2| clusters from S(i) (on the underlying tight path) and the first [k/2] clusters from S(i + 1) (note that this must be an edge of R as the underlying path is a tight path). We define the supplementary hypergraph R^* on [t] as in [15]: the vertex set is [t] and a subset $e \subseteq [t]$ of size at least 2 is an edge of R^* if there exists an edge $S_e \in R$ such that for all $j \in S_e$ there are $i_j \in e$ and $l_j \in [k]$ with $X_{i_j,l_j} \subseteq V_j$ and $e = \{i_j : j \in S_e\}$. Then the above e'_i edges translate into a graph path $W = e_1, \ldots, e_{t-1}$ in R^* , where $e_i = (i, i+1)$ for all $1 \le i \le t - 1$. (This corresponds to the connecting walk W in [15] but here the situation is much easier because of the underlying tight path, here all clusters appear exactly once as initial, link or final vertices.) The key lemma in [15] is the following lemma which finds a reasonably short connecting loose path in H_1 connecting S(i) and S(i + 1) where we may choose (modulo (k - 1)) how many vertices this path uses from X_i and X_{i+1} . Furthermore the path avoids a number of forbidden vertices to make sure that these connecting loose paths are disjoint.

Lemma 7 (Lemma 5.2 in [15]). Suppose that $e_i = (i, i+1) \in R^*$ is given as above and t_1, t_2 are integers with $0 \le t_1, t_2 \le k-1$ and $t_1 + t_2 \equiv 1 \mod (k-1)$. Moreover, suppose that Z is a set of at most $100t^2k^3$ forbidden vertices of H₁. Then in the sub-k-graph of H_1 induced by $X_i \cup X_{i+1}$ we can find a loose path L with the following properties.

- L contains at most $4k^3$ vertices.
- *L* has an initial vertex $u \in X_i$ and a final vertex $v \in X_{i+1}$.
- $|V(L) \cap X_i| \equiv t_i \mod (k-1)$ for i = 1, 2.
- L contains no forbidden vertices, i.e. $V(L) \cap Z = \emptyset$.
- u lies in at least $|H_1[X_i]|/(2|X_i|)$ edges of $H_1[X_i]$ and v lies in at least $|H_1[X_{i+1}]|/(2|X_{i+1}|)$ edges of $H_1[X_{i+1}]$.

We will also need the notion of a *prepath* from [15]: given a loose path L in some k-graph K with initial vertex x' and final vertex y' and disjoint sets $I, F \subseteq V(K) \setminus V(L)$ of size $k - 2, L^* = I \cup F \cup V(L)$ is a prepath. If we can find vertices $x, y \in V(K) \setminus L^*$ such that $\{x, x'\} \cup I, \{y, y'\} \cup F \in K$, then adding x and y to L^* gives a loose path; these $x \in V(K)$ are called possible initial vertices of L* and these $y \in V(K)$ are called possible final vertices of L*. We can use this idea to connect loose paths together: if L, L', L'' are disjoint loose paths, I, F, x, y as above, x is also the final vertex of L' and y is also the initial vertex of L'' then I and F together with L', L, L'' form a single loose path.

Corresponding to L_e in [15], we start by taking a short loose path L_e in H_1 from X_t to X_1 such that the initial vertex x_e lies in at least $|H_1[X_t]|/(2|X_t|)$ edges of $H_1[X_t]$ and y_e lies in at least $|H_1[X_1]|/(2|X_1|)$ edges of $H_1[X_1]$. Indeed, by utilizing the underlying tight path in R we can clearly find L_e that uses at most 2t vertices. We will extend L_e to a prepath L_e^* . (B3) and the above property of x_e imply that there is a set $I_0 \subseteq X_t$ for which X_t contains at least $c|X_t|$ vertices v which form an edge of $H_1[X_t]$ together with $I_0 \cup \{x_e\}$. Let $I'_0 \subseteq X_t$ be such a set of vertices. Similarly, there is a set $F_0 \subseteq X_1$ for which X_1 contains at least $c|X_1|$ vertices v which form an edge of $H_1[X_1]$ together with $F_0 \cup \{y_e\}$. Let $F'_0 \subseteq X_1$ be such a set of vertices. Let L_e^* be the prepath $I_0 \cup F_0 \cup L_e$. Then I'_0 is a set of possible initial vertices of L_e^* and F'_0 is a set of possible final vertices.

Then we apply Lemma 7 to each $e_i = (i, i + 1), 1 \le i \le t - 1$ in W in order to find a loose path L_i in H_1 connecting X_i and X_{i+1} and which we will extend to a prepath L_i^* with many possible initial vertices in X_i and with many possible final vertices of in X_{i+1} . More precisely, applying Lemma 7 for all $e_i = (i, i + 1), 1 \le i \le t - 1$ we find a loose path L_i and sets I_i , F_i extending L_i to a prepath L_i^* which satisfy the following properties:

- (C1) L_i lies in the sub-*k*-graph of H_1 induced by $X_i \cup X_{i+1}$ and contains at most $4k^3$ vertices.
- (C2) The initial vertex x_i of L_i lies in X_i and its final vertex y_i lies in X_{i+1} .
- (C3) $I_i \subseteq X_i$ and $F_i \subseteq X_{i+1}$.
- (C4) There is a set $I'_i \subseteq X_i$ of at least $c|X_i|$ possible initial vertices for L^*_i and there is a set $F'_i \subseteq X_{i+1}$ of at least $c|X_{i+1}|$ possible final vertices for L_i^* .

(C5) All the prepaths $L_e^*, L_1^*, \ldots, L_{t-1}^*$ are disjoint.

(C6) Let $t_i = |X_i \setminus (L_e \cup L_{i-1})|$. Then we have $|V(L_i) \cap X_i| \equiv t_i + 1 \mod (k-1)$.

These properties imply that for all i = 1, ..., t we have

$$|X_i \setminus (L_e \cup L_1 \cup \dots \cup L_{t-1})| = |X_i \setminus (L_e \cup L_{i-1} \cup L_i)| \equiv t_i - (t_i + 1) = -1 \mod (k-1)$$

(using the fact that only L_e , L_{i-1} and L_i may intersect X_i). Indeed, this is clearly true for i = 1, ..., t - 1 by (C5) and (C6), but then it is also true for t using the fact that the total number of vertices in $\bigcup_{i=1}^{t} X_i$ is divisible by k - 1.

Let $Y_i = X_i \setminus (L_e^* \cup L_1^* \cup \cdots \cup L_t^*)$. Since by (C3) for each $i \in [t]$ there are exactly 2(k - 2) vertices of X_i which lie in $L_e^*, L_1^*, \ldots, L_{t-1}^*$ but not in $L_e, L_1, \ldots, L_{t-1}$, this in turn implies that

$$|Y_i| \equiv -1 - 2(k-2) \equiv 1 \mod (k-1).$$
(6)

Let $x_t = x_e$, $y_0 = y_e$, $L_0^* = L_e^*$, $I_t = I_0$ and $I'_t = I'_0$. Then we finish by finding spanning loose paths L^i in each $H_1[Y_i]$, $1 \le i \le t$ which 'connect' prepaths L_{i-1}^* and L_i^* . More precisely, we want to choose the spanning loose path L^i in $H_1[Y_i]$ in such a way that the initial vertex of L^i lies in F'_{i-1} and its final vertex lies in I'_i . To see that this can be done, first note that $|X_i \setminus Y_i| \ll \epsilon n_1$. So using Lemma 4 together with (B4) and (6) it is easy to check that the complete *k*-partite *k*-graph on Y_i contains such a loose spanning path. But then (B2) and (C4) together imply that $G_i[Y_i] \setminus M_i[Y_i]$ contains the *k*-complex induced by this path (i.e. $(L^i)^{\leq}$). But this means that we can find the required path L^i in each $H_1[Y_i]$.

Finally, for each L^i , $1 \le i \le t$ write x'_i for its initial and y'_i for its final vertex. To obtain our spanning loose cycle in $H_1[\bigcup_{i=1}^t X_i]$ we first traverse $L_0 = L_e$, then we use the edge $F_0 \cup \{y_0, x'_1\}$ in order to move to the initial vertex x'_1 of L^1 . (This is possible since $x'_1 \in F'_0$.) Now we traverse L^1 and use the edge $I_1 \cup \{y'_1, x_1\}$ to get to x_1 . (Again, this is possible since $y'_1 \in I'_1$.) Next we traverse L_1 and use the edge $F_1 \cup \{y_1, x'_2\}$ to move to the initial vertex x'_2 of L^2 . We continue in this way until we have reached the initial vertex $x_t = x_e$ of $L_0 = L_e$ again. (So in the last step we traversed L^t and used the edge $I_t \cup \{y'_t, x_t\}$.) This completes the proof of Lemma 6 (and thus the proof of Lemma 1).

5. Proof of Theorem 1

5.1. Step 1

We proceed exactly as in the proof of Lemma 1 but this time we stop just after applying Theorem 3 in each X'_i since here we need a slight strengthening of the proposition (B1)–(B4). Let again $X_{i,j} \subseteq X'_{i,j}$ and $X_i = \bigcup_{j=1}^k X_{i,j}$ denote the remaining vertices after the application of Theorem 3 and $G_i = G'_i[X_i]$, $M_i = M'_i[X_i]$. Thus again $G_i \setminus M_i$ is (c, ϵ') -robustly 2^k-universal.

Next we partition $X_{i,j}$ randomly into two parts $A_{i,j}$ and $A'_{i,j}$ by assigning each vertex to $A_{i,j}$ with probability $\frac{1}{4k}$ and to $A'_{i,j}$ with probability $(1 - \frac{1}{4k})$ independently of all other vertices. Let $A_i = \bigcup_{j=1}^k A_{i,j}$, $A'_i = \bigcup_{j=1}^k A'_{i,j}$ and $A = \bigcup_{i=1}^t A_i$. Then with high probability this partition satisfies the following properties:

(D1) For all *i*, *j* we have $n_1/8k \le |A_{i,j}| \le n_1/2k$.

- (D2) For all *i*, *j* and every $v \in X_{i,j}$ we have $|(G_i(v)[A'_i])| \ge 2c|G_i(v)||$.
- (D3) For all *i* we have $d(H_1[A'_i]) \ge c''/4$.

Indeed, (D1) is satisfied with high probability by a standard Chernoff bound. We get (D2) with high probability using (D1), Lemma 3 and $c \ll 1/k$:

$$|(G_i(v)[A'_i])_{=}| = d((G_i(v)[A'_i])_{=}) \prod_{j' \neq j} |A'_{i,j'}| \ge \frac{d(G_i(v)_{=})}{2} \prod_{j' \neq j} \frac{|X_{i,j'}|}{2} \ge 2c|G_i(v)_{=}|.$$

Finally (D3) is true with high probability from (5) and Lemma 3. Thus we may assume that our partition satisfies properties (D1)-(D3). In particular, (4) and (D1) imply that

$$|A| \ge \frac{n}{16rk^2} \ge \frac{n}{8(rk)^2}.$$
(7)

Here we will need slightly strengthened versions of the proposition (B1)–(B4) where we may delete an arbitrary number of vertices of each $A_{i,j}$ (this may not be allowed in (B1)–(B4) as $\epsilon \ll 1/k$):

- (B1') For each *i*, G_i is a *k*-partite sub-*k*-complex of *G* on the vertex set X_i . M_i is the *k*-partite *k*-graph $M \cap G_i$, and $G_i \setminus M_i \subseteq H_1$. Clearly these statements remain true after the deletion of an arbitrary number of vertices of $A_{i,j}$ and up to ϵn_1 vertices of $A'_{i,i}$.
- (B2') Even after the deletion of an arbitrary number of vertices of $A_{i,j}$ and up to ϵn_1 vertices of $A'_{i,j}$, the following statement holds. Let *L* be a *k*-partite *k*-complex on the vertex set $U = U_1 \cup \cdots \cup U_k$, where $|U_j| = |X_{i,j}|$ for each *j*, and let *L* have maximum vertex degree at most 2^k . Let $\ell \le 2t$ and suppose we have $u_1, \ldots, u_\ell \in U$ and sets $Z_s \subseteq X_{i,j(u_s)}$ with $|Z_s| \ge c|X_{i,j(u_s)}|$ for each $s \in [\ell]$ (where $j(u_s)$ is such that $u_s \in U_{j(u_s)}$). Then $G_i \setminus M_i$ contains a copy of *L*, in which for each *j* the vertices of U_i correspond to the vertices of $X_{i,j}$, and each u_s corresponds to a vertex in Z_s .

- (B3') For each *i*, $H_1[X_i]$ has density at least *c*', even after the deletion of an arbitrary number of vertices of $A_{i,j}$ and up to ϵn_1 vertices of $A'_{i,j}$.
- (B4') If we delete an arbitrary number of vertices of $A_{i,j}$ and up to ϵn_1 vertices of $A'_{i,j}$, and let $t_j = |X_{i,j}|$ for each $j \in [k]$ after these deletions, and let $n'_i = \frac{(\sum t_j) 1}{k-1}$, then $n'_i/2 + 1 \le t_j \le n'_i$ for all j.

Indeed, (B1') is clear again. (B2') follows from the fact that $G_i \setminus M_i$ was (c, ϵ') -robustly 2^k -universal. Note that (D2) and the fact that we have deleted only up to ϵn_1 vertices from each $A'_{i,j}$ imply that condition (i) is still satisfied in the definition of a robustly universal complex. (B3') follows from (D1), (D3) and again the fact that we have deleted only up to ϵn_1 vertices from each $A'_{i,j}$. Finally, for (B4') note that after the deletion of an arbitrary number of vertices of $A_{i,j}$ and up to ϵn_1 vertices of $A'_{i,j}$ using (D1) we still have

• $(1 - \frac{1}{2k} - 2\theta')n_1 \le t_j = |X_{i,j}| \le n_1,$

•
$$\frac{k}{k-1}(1-\frac{1}{2k}-2\theta')n_1 \le n'_i \le \frac{k}{k-1}n_1$$

But then

- $t_j \le n_1 \le \frac{k}{k-1}(1 \frac{1}{2k} 2\theta')n_1 \le n'_i$,
- $t_j \ge (1 \frac{1}{2k} 2\theta')n_1 \ge \frac{k}{2(k-1)}n_1 + 1 \ge \frac{n'_i}{2} + 1$,

and thus (B4') follows (using $\theta' \ll 1/k$ and $k \ge 3$).

As in Lemma 6, these properties imply that (after removing at most k - 2 vertices for divisibility reasons) there is a red loose cycle spanning the remaining vertices in $\bigcup_{i=1}^{t} X_i$. However, here we postpone the construction of this spanning red loose cycle since first we might have to use some of the vertices in A.

5.2. Step 2

Here we will use Lemma 1 repeatedly. We go back from the reduced hypergraph to the original hypergraph and we remove the vertices in *CM*, i.e. $\cup_{i=1}^{t} X_i$. We apply repeatedly Lemma 1 to the *r*-colored complete hypergraph induced by $K_n^{(k)} \setminus \bigcup_{i=1}^{t} X_i$. This way we choose *l* vertex disjoint monochromatic loose cycles in $K_n^{(k)} \setminus \bigcup_{i=1}^{t} X_i$. We wish to choose *l* such that the remaining set *B* of vertices in $K_n^{(k)} \setminus \bigcup_{i=1}^{t} X_i$ not covered by these *l* cycles has cardinality at most $n/8(rk)^4$. Since after *l* steps at most

$$\left(n - \left|\bigcup_{i=1}^{t} X_{i}\right|\right) \left(1 - \frac{1}{2rk}\right)^{l}$$

vertices are left uncovered, we have to choose *l* to satisfy

$$\left(n - \left|\bigcup_{i=1}^{t} X_{i}\right|\right) \left(1 - \frac{1}{2rk}\right)^{l} \le \frac{n}{8(rk)^{4}}$$

This inequality is certainly true if

$$\left(1-\frac{1}{2rk}\right)^l \le \frac{1}{8(rk)^4}$$

which in turn is true using $1 - x \le e^{-x}$ if

$$e^{-\frac{l}{2rk}}\leq \frac{1}{8(rk)^4}.$$

This shows that we can choose $l \leq 12rk \log(rk)$.

5.3. Step 3

The complete *bipartite* k-uniform hypergraph $K^{(2,k-2)}(B, A)$ contains all edges of type (2, k - 2), i.e. edges that contain exactly two vertices from B and k - 2 vertices from A (we assumed $k \ge 3$). The key to this step is the following lemma about r-colored complete unbalanced bipartite hypergraphs.

Lemma 8. For all integers $r \ge 2$ and $k \ge 3$ there exists a constant $n_0 = n_0(r, k)$ such that if the edges of the complete bipartite hypergraph $K^{(2,k-2)}(B, A)$ are colored with r colors and $n_0 \le |B| \le \frac{|A|}{2r(k-2)^2}$, then there are at most 100r log r pairwise disjoint monochromatic loose cycles whose link vertices cover B.

Recall the definitions of $A = \bigcup_{i=1}^{t} \bigcup_{j=1}^{k} A_{i,j}$ and B (the set of remaining vertices). Consider the *r*-colored complete bipartite *k*-uniform hypergraph $K^{(2,k-2)}(B, A)$. We apply Lemma 8 in $K^{(2,k-2)}(B, A)$. The conditions of the lemma are satisfied by the above since (using (7))

$$|B| \le \frac{n}{8(rk)^4} \le \frac{|A|}{(rk)^2} \le \frac{|A|}{2r(k-2)^2}.$$

Let us remove the at most 100*r* log *r* vertex disjoint monochromatic loose cycles covering *B* in $K^{(2,k-2)}(B, A)$. Then we delete an additional at most k - 2 vertices to make sure that the total number of remaining vertices in $\bigcup_{i=1}^{t} X_i$ is divisible by k - 1. In the next step we have to verify that there is still a red loose cycle in H_1 spanning all the remaining vertices in $\bigcup_{i=1}^{t} X_i$.

5.4. Step 4

To verify that there is still a red loose cycle in H_1 spanning all the remaining vertices in $\bigcup_{i=1}^{t} X_i$ we note that the proof of Lemma 6 still goes through. The only difference is that now we may have deleted in Step 3 an arbitrary number of vertices of $A_{i,j}$ but this is taken into account in the strengthened properties (B1')–(B4').

Thus the total number of vertex disjoint monochromatic loose cycles we used to partition the vertex set of K_n^k in the various steps is at most (using $k \ge 3$)

$$12rk\log(rk) + 100r\log r + (k-2) + 1 \le 12rk\log(rk) + \frac{100}{3}rk\log r + k \le 50rk\log(rk),$$

finishing the proof. \Box

5.5. Proof of Lemma 8

We define an *r*-edge-colored complete graph *G* on the vertex set *B* as follows: $u, v \in B$ are adjacent by an edge of color *i* if at least $\frac{\binom{|A|}{r}}{r}$ edges of $K^{(2,k-2)}(B, A)$ containing uv are colored with color *i*. Applying the main result of [7] the vertex set of *G* can be covered by at most 100*r* log *r* vertex disjoint monochromatic cycles. We try to make loose cycles from these graph cycles by extending each edge of these cycles with k - 2 vertices to form a hyperedge of the same color. To achieve this we have to make the extension so that the (k - 2)-sets of *A* used are pairwise disjoint. The definition of the edge colors allows to perform this extension greedily. Indeed, assume that we have the required extension for some number of edges and *e* is the next edge to be extended. Since the cycle partition of *G* has at most |B| edges, if the (k - 2)-subsets of *A* used so far cover $U \subset A$, then |U| < |B|(k - 2). However, at least $\frac{\binom{|A|}{k-2}}{r}$ edges of $K^{(2,k-2)}(B, A)$ containing *e* are colored with the

color of *e*. Overestimating the number of (k - 2)-subsets of *A* intersecting *U* by $|U| \binom{|A|}{k-3}$, we get

$$U\left(\frac{|A|}{k-3}\right) < |B|(k-2)\binom{|A|}{k-3}.$$

We claim that

$$|B|(k-2)\binom{|A|}{k-3} \leq \frac{\binom{|A|}{k-2}}{r},$$

i.e. we have an extension that is disjoint from U, as desired. Indeed, otherwise we get

$$\frac{|A|}{2r(k-2)} \le \frac{|A|-k+3}{r(k-2)} < |B|(k-2),$$

contradicting the assumptions of the lemma. \Box

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