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Light C_4 and C_5 in 3-polytopes with minimum degree 5

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ABSTRACT

Let $w_P(C_l)$ ($w_T(C_l)$) be the minimum integer k with the property that every 3-polytope (respectively, every plane triangulation) with minimum degree 5 has an l-cycle with weight, defined as the degree-sum of all vertices, at most k.

In 1998, O.V. Borodin and D.R. Woodall proved $w_T(C_4) = 25$ and $w_T(C_5) = 30$. We prove that $w_P(C_4) = 26$ and $w_P(C_5) = 30$.

1. Introduction

The *degree* $d(x)$ of a vertex or face x in a plane graph G is the number of its incident edges. A *k-vertex* (*k-neighbor*, *k-face*) is a vertex (neighbor, face) with degree k, a k^+*-vertex* has degree at least k, etc. The minimum vertex degree of G is $\delta(G)$. We will drop the arguments whenever this does not lead to confusion.

A normal plane map (NPM) is a plane pseudograph in which loops and multiple edges are allowed, but $d(x) \geq 3$ for every vertex and face x. As proved by Steinitz [27], the 3-connected plane graphs are planar representations of the convex 3-dimensional polytopes, called hereafter 3-polytopes.

In this note, we consider the class $\mathbf{M_5}$ of NPMs with $\delta = 5$ and its subclasses $\mathbf{P_5}$ of 3-polytopes and $\mathbf{T_5}$ of plane triangulations, where we define a triangulation to be simple (without loops or multiple edges), so that $\mathbf{T_5} \subset \mathbf{P_5} \subset \mathbf{M_5}$. A cycle on k vertices is denoted by C_k, and S_k stands for a k-star centered at a 5-vertex.

In 1904, Wernicke [28] proved that if $M_5 \in \mathbf{M_5}$ then M_5 contains a vertex of degree 5 adjacent to a vertex of degree at most 6. This result was strengthened by Franklin [15] in 1922 to the existence of a vertex of degree 5 with two neighbors of degree at most 6. In 1940, Lebesgue [22, p. 36] gave an approximate description of the neighborhoods of vertices of degree 5 in a $T_5 \in \mathbf{T_5}$.

Given a graph H, the *weight* $w_M(H)$ is the maximum over $M_5 \in \mathbf{M_5}$ of the minimum degree-sum of the vertices of H over subgraphs H of M_5. The *weights* $w_P(H)$ and $w_T(H)$ are defined similarly for $\mathbf{P_5}$ and $\mathbf{T_5}$, respectively.

The bounds $w_M(S_1) \leq 11$ (Wernicke [28]) and $w_M(S_2) \leq 17$ (Franklin [15]) are tight. It was proved by Lebesgue [22] that $w_M(S_3) \leq 24$ and $w_M(S_4) \leq 31$, which were improved much later to the following tight bounds: $w_M(S_3) \leq 23$ (Jendrol'–Madaras [17]) and $w_M(S_4) \leq 30$ (Borodin–Woodall [9]). Note that $w_M(S_3) \leq 23$ easily implies $w_M(S_2) \leq 17$ and immediately follows from $w_M(S_4) \leq 30$ (it suffices to delete a vertex of maximum degree from a star of the minimum weight).

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It follows from Lebesgue [22, p. 36] that $w_T(C_3) \leq 18$. In 1963, Kotzig [21] gave another proof of this fact and conjectured that $w_T(C_3) \leq 17$; the bound 17 is easily shown to be tight.

In 1989, Kotzig's conjecture was confirmed by Borodin [1] in a more general form, by proving $w_M(C_3) = 17$. Another consequence of this result is confirming a conjecture of Grünbaum [16] of 1975 that for every 5-connected planar graph the cyclic connectivity (defined as the minimum number of edges to be deleted to obtain two components each containing a cycle) is at most 11, which is tight (a bound of 13 was earlier obtained by Plummer [26]).

It also follows from Lebesgue [22, p. 36] that $w_T(C_4) \leq 26$ and $w_T(C_5) \leq 31$. In 1998, Borodin and Woodall [9] proved the following.

Theorem 1 (*Borodin–Woodall [9]*). *For the class of plane triangulations with minimum degree* 5, $w_T(C_4) = 25$ *and* $w_T(C_5) = 30$.

The *height* of a subgraph H of graph G is the maximum degree of vertices of H in G. Now let $\varphi_M(H)$ ($\varphi_P(H)$, $\varphi_T(H)$) be the minimum integer k with the property that every normal plane map (3-polytope, plane triangulation) with minimum degree 5 has a copy of H with all vertices of degree at most k.

It follows from Franklin [15] that $\varphi_M(S_2) = 6$. From $w_M(C_3) = 17$ (Borodin [1]), together with a simple example proving $\varphi_M(C_3) \geq 7$, we have $\varphi_M(C_3) = 7$. In 1996, Jendrol' and Madaras [17] proved $\varphi_M(S_4) = 10$ and $\varphi_T(C_4) = \varphi_T(C_5) = 10$. R. Soták (personal communication, see the surveys of Jendrol' and Voss [19, p.15], [20]) proved $\varphi_P(C_4) = 11$ and $\varphi_P(C_5) = 10$.

In 1999, Jendrol' et al. [18] obtained the following bounds: $10 \leq \varphi_T(C_6) \leq 11$, $15 \leq \varphi_T(C_7) \leq 17$, $15 \leq \varphi_T(C_8) \leq 29$, $19 \leq \varphi_T(C_9) \leq 41$, and $\varphi_T(C_p) = \infty$ whenever $p \geq 11$. Madaras and Soták [24] proved $20 \leq \varphi_T(C_{10}) \leq 415$.

For the broader class $\mathbf{P_5}$, it was known that $10 \leq \varphi_P(C_6) \leq 107$ due to Mohar et al. [25] (in fact, this bound is proved in [25] for all 3-polytopes with $\delta \geq 4$ in which no 4-vertex is adjacent to a 4-vertex). Recently, Borodin et al. [12] proved $\varphi_P(C_6) = \varphi_T(C_6) = 11$.

For C_7, besides the above mentioned result $15 \leq \varphi_T(C_7) \leq 17$, it was known that $\varphi_P(C_7) \leq 359$ (Madaras et al. [23]). Recently, Borodin and Ivanova [8] proved $\varphi_P(C_7) = \varphi_T(C_7) = 15$, which answers a question raised in Jendrol' et al. [18].

The purpose of this paper is to prove the following analogue of Theorem 1.

Theorem 2. *For the class of* 3*-polytopes with minimum degree* 5, $w_P(C_4) = 26$ *and* $w_P(C_5) = 30$.

As an easy corollary, we obtain the above-mentioned unpublished result by R. Soták (for one direction, it suffices to take a C_l with $4 \leq l \leq 5$ of smallest weight and subtract $l - 1$ smallest degrees of its vertices; the other direction follows from the examples in Section 2).

Corollary 3. *For the class of* 3*-polytopes with minimum degree* 5, $\varphi_P(C_4) = 11$ *and* $\varphi_P(C_5) = 10$.

In fact, instead of Theorem 2 we prove the following stronger statement, which extends Theorem 1.

Theorem 4. *Every* 3*-polytope with* $\delta = 5$ *has*

(i) *a* 4*-cycle of weight at most* 26,
(ii) *a* 5*-cycle of weight at most* 30,
(iii) *either a* 4*-cycle of weight at most* 25 *or a facial* 5*-cycle of weight* 25, *where all bounds* 26, 30 *and* 25 *are tight.*

In particular, Theorem 4(i +iii) says that $w_P(C_4)$ can reach its maximum of 26 only in the presence of a facial 5-cycle with weight 25, which is a 5-face completely surrounded by 5-vertices (as in Fig. 1). Theorem 4 refines Corollary 3 as follows.

Corollary 5. *Every* 3*-polytope with* $\delta = 5$ *has*

(i) *a* 4*-cycle of height at most* 11,
(ii) *a* 5*-cycle of height at most* 10,
(iii) *either a* 4*-cycle of height at most* 10 *or a facial* 5*-cycle of height* 5, *where all bounds* 11, 10 *and* 5 *are tight.*

At the second part of the proof of Theorem 4 we use some ideas from Borodin [1] and Borodin–Woodall [9].

Other related structural results on 3-polytopes, some of which have application to coloring, can be found in the already mentioned papers and in [2–8,10–14,24].

2. Proving the tightness of Theorem 4

The bounds in Theorem 4 and Corollary 5 are all precise, as the following examples show. Truncate all vertices of the dodecahedron and cap each 10-face by putting a new vertex inside it and joining it to all the boundary vertices. We have obtained a triangulation with $\delta = 5$ in which $w_T(C_4) = 25$, $w_T(C_5) = 30$, and $\varphi_T(C_4) = \varphi_T(C_5) = 10$.

We now construct a 3-polytope with $w_P(C_4) = 26$ (see Fig. 1). First, we replace each face of the icosahedron as shown in Fig. 1(a). The resulting dual "blue" graph G_1 is a cubic graph with only 5- and 6-faces such that the distance between 5-faces is at least two.

Then, with each 5-face of G_1, we perform the operation depicted in Fig. 1(b) to obtain a graph G_2 with only 3-faces and (very few) 5-faces, in which every vertex is of degree 5, 11, or 12. In particular, we truncate all vertices of G_1 not incident with 5-faces. It is easy to check that each 4-cycle of G_2 goes through an 11^+-vertex, and that $w_P(C_4) = 26$, $\varphi_P(C_4) = 11$, and every facial 5-cycle has weight 25 and hence consists of 5-vertices.

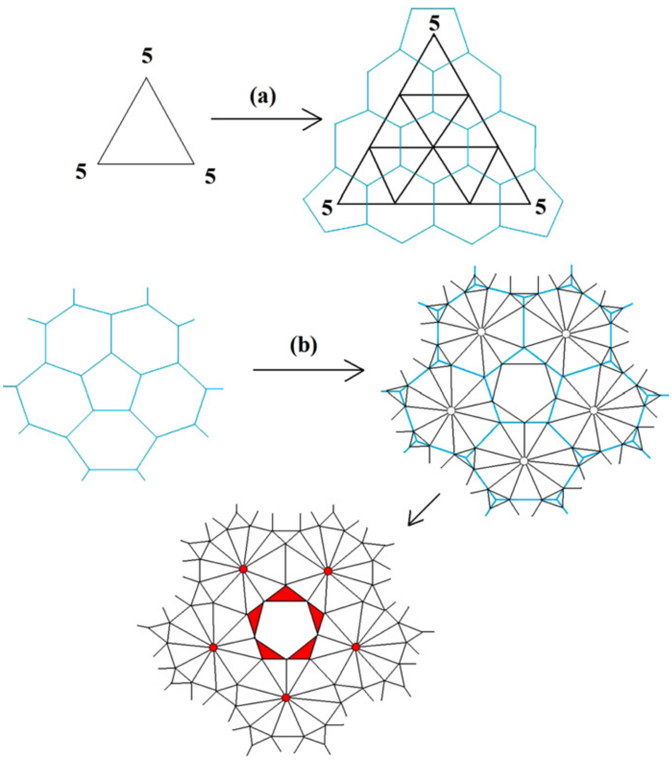

Fig. 1. A 3-polytope with $w_P(C_4) = 26$.

3. Proving the upper bounds in Theorem 4

Suppose G' is a counterexample to (some part of) Theorem 4. Let G be a maximal counterexample such that $V(G') = V(G)$ and $E(G') \subseteq E(G)$. Clearly G is 3-connected, since G' is. Denote the sets of vertices, edges, and faces of G by V, E and F, respectively. Euler's formula $|V| - |E| + |F| = 2$ for G yields

$$\sum_{v\in V}(d(v) - 6) + \sum_{f\in F}(2d(f) - 6) = -12. \tag{1}$$

We assign an *initial charge* $\mu(x)$ to x whenever $x \in V \cup F$ as follows: $\mu(v) = d(v) - 6$ if $v \in V$ and $\mu(f) = 2d(f) - 6$ if $f \in F$. Note that only 5-vertices have a negative initial charge.

Using the properties of G as a counterexample to Theorem 4, we will define a local redistribution of charges, preserving their sum, such that the *new charge* $\mu'(x)$ is non-negative whenever $x \in V \cup F$. This will contradict the fact that the sum of the new charges is, by (1), equal to -12, and this contradiction will prove Theorem 4.

3.1. Structural properties of G

Let $v_1, \ldots, v_{d(v)}$ denote the neighbors of a vertex v in cyclic order round v. The vertex v is *simplicial* if all its incident faces are 3-faces. If $d(v_i) = 5$ then v_i is a *strong*, *semiweak* or *weak* neighbor of v according as both, one or none of v_{i-1}, v_{i+1} are 6^+-vertices, and v_i is *twice weak* if $d(v_j) = 5$ whenever $|j - i| \le 2$ (modulo $d(v)$). If v_i is a strong, semiweak or weak neighbor of v then we say that v is a strong, semiweak or weak *donor* to v_i (even if in fact v gives nothing to v_i).

In what follows, we will need the simple structural properties of G expressed by (SP1)–(SP4).

(SP1) *The boundary $\partial(f)$ of every face of G is an induced cycle; that is, $\partial(f)$ is a cycle, and no two nonconsecutive vertices of $\partial(f)$ are adjacent.*

This follows from the planarity and 3-connectedness of G.

(SP2) *Every 10^+-vertex v in G is simplicial.*

Otherwise, let f be a 4^+-face incident with v and let w be a vertex of $\partial(f)$ that is not adjacent to v, which exists by (SP1). Adding the edge vw preserves 3-connectedness and cannot create a 4-cycle of weight less than $11 + 6 + 5 + 5 = 27$ or a 5-cycle of weight less than $11 + 6 + 5 + 5 + 5 = 32$. So $G + vw$ is a counterexample to Theorem 4, which contradicts the maximality of G.

(SP3) *Any four neighbors of a simplicial* 5*-vertex have degree-sum at least* 26.

Otherwise, G would contain a 5-cycle of weight at most 30 and a 4-cycle of weight at most 25, and so would not be a counterexample to Theorem 4.

(SP4) *No* 10*-vertex can have a twice weak neighbor.*

Otherwise, by (SP2), G would have a 4-cycle of weight 25 and 5-cycle of weight 30.

An 11-vertex is *bad* or an 11_b*-vertex* if all its neighbors are 5-vertices. A 5-vertex v is *special* if it is incident with a 4-face vv_1xv_2 and four 3-faces, and v_3 is an 11_b-vertex (so that $d(v_2) = d(v_4) = 5$), and at least one of v_1 and x is a 5-vertex. A 5-vertex is *good* or a 5_g*-vertex* if it is incident with a 6^+-face, or with a 5-face that is incident with at least one 6^+-vertex, or with a 4-face that is incident with at least two 6^+-vertices, or with at least two 4^+-faces. Clearly the three adjectives *simplicial*, *special* and *good* are mutually exclusive: no 5-vertex can satisfy more than one of them.

3.2. Discharging on G

We use the following discharging rules (see Fig. 2).

Rule 0. *Let f be a 4^+-face.*

(a) *If $d(f) \geq 6$ then f gives* 1 *to every incident* 5*-vertex.*
(b) *If $d(f) = 5$ then f gives to every incident* 5*-vertex:*
  (i) 1 *if f is incident with at least one 6^+-vertex;*
  (ii) $\frac{4}{5}$ *otherwise.*
(c) *If $d(f) = 4$ then f gives to every incident 5-vertex:*
  (i) *1 if f is incident with at least two 6^+-vertices;*
  (ii) $\frac{2}{3}$ *if f is incident with precisely one 6^+-vertex, with the following exception. Suppose $f = vv_1v_2v_3$ where $d(v) = 9$, $d(v_3) = 5$ and v_1 and v_2 are special; then f gives $\frac{5}{12}$ to v_1, $\frac{3}{4}$ to v_2, and either $\frac{5}{12}$ or $\frac{2}{3}$ to v_3 according as v_3 is or is not special;*
  (iii) $\frac{1}{2}$ *otherwise.*

Rule 1.

(a) *Each vertex v of degree* 7 *sends $\frac{1}{3}$ to each strong* 5*-neighbor and $\frac{1}{6}$ to each semiweak* 5*-neighbor.*
(b) *Each vertex v with degree* 8, 9 *or* $\geq$ 12 *first gives a "basic" contribution of $\rho(v) = \frac{\mu(v)}{d(v)} = \frac{d(v)-6}{d(v)}$ to each neighbor. Then each 6^+-neighbor v_i shares the charge just received equally between v_{i-1} and v_{i+1}.*
(c) *Each* 10*-vertex or* 11*-vertex v first gives a "basic" $\frac{2}{5}$ to each neighbor. Then each 6^+-neighbor v_i transfers $\frac{1}{10}$ of v's donation to each 5-vertex in $\{v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}\}$.*

Rule 2. *If $d(v) = 11$ then v gives a "supplementary" $\frac{1}{10}$ to each twice weak neighbor.*

Rule 3. *If v is a simplicial* 5*-vertex adjacent to a bad* 11*-vertex w, say $w = v_5$, and if $d(v_1) = d(v_4) = 5$, then v gives back to v_5 the following:*
  (a) $\frac{1}{2}$ *if both v_2 and v_3 have degree $\geq$9;*
  (b) $\frac{1}{4}$ *if at least one of v_2, v_3 has degree exactly* 8.

Rule 4. *Suppose v is a bad* 11*-vertex where v_1 and v_2 are special* 5*-vertices incident with a* 4*-face v_1v_2xy with $d(x) = 5$ and $d(y) \leq 9$. Then each of v_1, v_2 gives back $\frac{1}{4}$ to v.*

Rule 5. *A good* 5*-vertex that is adjacent to a bad 11-vertex w gives back $\frac{1}{2}$ to w.*

3.3. Checking $\mu'(x) \geq 0$ for $x \in V \cup F$

Case 1. $x = f \in F$.

If $d(f) = 3$ then $\mu'(f) = \mu(f) = 2 \times 3 - 6 = 0$ since f does not participate in discharging.

If $d(f) = 4$ then f gives at most 2 in total to its 5-vertices by R0, so $\mu'(f) \geq 2 \times 4 - 6 - 2 = 0$.

If $d(f) = 5$ then f gives to its 5-vertices either $5 \times \frac{4}{5}$ or at most 4×1, hence $\mu'(f) \geq 2 \times 5 - 6 - 4 = 0$.

If $d(f) \geq 6$ then $\mu'(f) \geq 2d(f) - 6 - d(f) \times 1 \geq 0$ by R0.

In all cases, $\mu'(f) \geq 0$. From now on we assume that $x = v \in V$.

Case 2. $d(v) = 5$. Then $\mu(v) = d(v) - 6 = -1$, and v does not give charge to any other vertex, except that, by R3–R5, v may give back some charge to a bad 11-vertex w; note, however, that a bad 11-vertex gives $\frac{2}{5} + \frac{1}{10} = \frac{1}{2}$ to each of its neighbors by R1c and R2, and so v does not "give back" more charge to w than w has already given to v.

Subcase 2.1. v is simplicial. The amounts of charge received by v from its neighbors by Rules 1 and 2 are summarized in Table 1.

Suppose Rule 3(a) applies to v, so that v's neighbors $v_1, \ldots, v_5$ have degrees $(5, \geq 9, \geq 9, 5, 11)$. Then v is a semiweak neighbor of each of v_2 and v_3 and a weak neighbor of v_5, so that it receives at least $\frac{1}{2}$ from each of them by Table 1, and gives back exactly $\frac{1}{2}$ to v_5 by Rule 3(a) and nothing to v_2 or v_3. Hence $\mu'(v) \geq -1 + \frac{3}{2} - \frac{1}{2} = 0$.

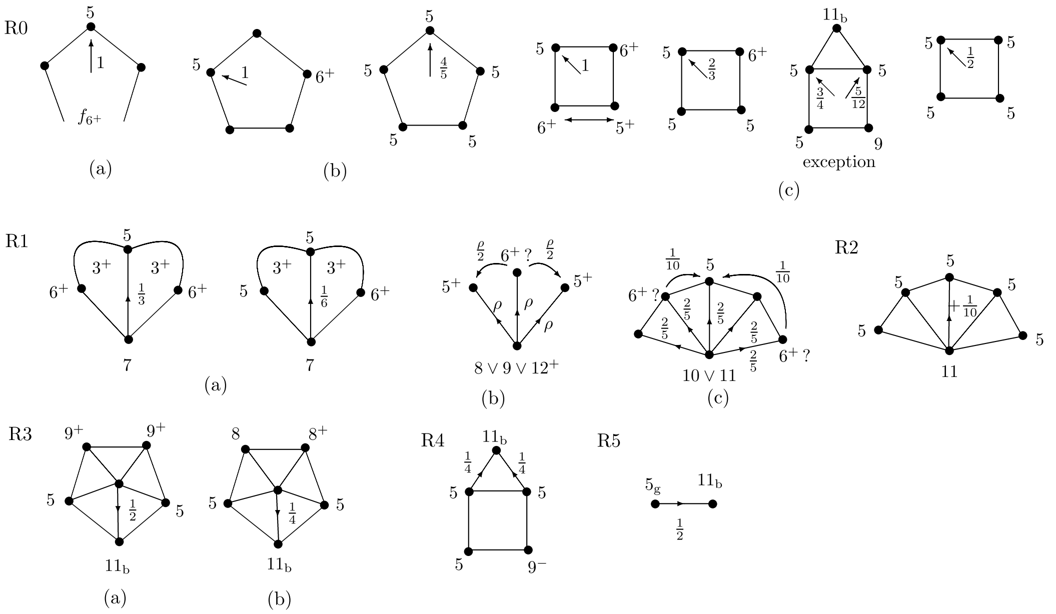

Fig. 2. Discharging rules.

Suppose now that Rule 3(b) applies. By (SP3), v's neighbors have degrees $(5, 8, \geq 8, 5, 11)$. Now v receives at least $\frac{3}{8}$ from each of v_2 and v_3 and at least $\frac{1}{2}$ from v_5 by Table 1, and gives back exactly $\frac{1}{4}$ to v_5 by Rule 3(b) and nothing to v_2 or v_3. Hence $\mu'(v) \geq -1 + \frac{3}{8} + \frac{3}{8} + \frac{1}{2} - \frac{1}{4} = 0$.

So we may assume that Rule 3 does not apply to v at all, and the amount that v receives from its neighbors is at least that given in Table 1. By (SP3), if three neighbors of v have degree-sum at most 16 then the remaining two neighbors each have degree at least 10 and so each gives v at least $\frac{1}{2}$ by Table 1. We may assume this does not happen; in particular, v has at most two 5-neighbors. If it has exactly two, then two of v's 6^+-neighbors are semiweak donors and the third, say w, is either a strong or a weak donor, according as the two 5-neighbors are adjacent or not; for the present purpose we may assume w is a weak donor to v. If v has exactly one 5-neighbor then v has two semiweak donors and two strong donors. If v has no 5-neighbors then it has five strong donors.

By these remarks, the degree-sequence of v's neighbors, in nondecreasing order, must be one of the following. In each case, we use Table 1 to check that v receives at least 1 in total from its neighbors.

$(5, 5, 7, \geq 9, \geq 9)$: v receives at least $0 + \frac{1}{2} + \frac{1}{2} = 1$ (if its weak donor has degree 7) or $\frac{1}{6} + \frac{1}{3} + \frac{1}{2} = 1$ (if its weak donor has degree 9).

$(5, 5, \geq 8, \geq 8, \geq 8)$: v receives at least $\frac{1}{4} + \frac{3}{8} + \frac{3}{8} = 1$.

$(5, 6, 6, \geq 9, \geq 9)$: v receives at least $\frac{1}{2} + \frac{1}{2} = 1$.

$(5, 6, \geq 7, \geq 8, \geq 8)$: v receives at least $\frac{1}{3} + \frac{3}{8} + \frac{3}{8} > 1$ (if a strong donor has degree 7) or $\frac{1}{6} + \frac{1}{2} + \frac{3}{8} > 1$ (if a strong donor has degree ≥ 8).

$(5, \geq 7, \geq 7, \geq 7, \geq 7)$: v receives at least $\frac{1}{6} + \frac{1}{6} + \frac{1}{3} + \frac{1}{3} = 1$.

$(6, 6, 6, \geq 8, \geq 8)$: v receives at least $\frac{1}{2} + \frac{1}{2} = 1$.

$(\geq 6, \geq 6, \geq 7, \geq 7, \geq 7)$: v receives at least $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$.

Subcase 2.2. v is special. Suppose v is surrounded by faces $f = vv_1xv_2$, vv_2v_3, vv_3v_4, vv_4v_5, and vv_5v_1, where v_3 is a bad 11-vertex (so that $d(v_2) = 5$), and at least one of v_1 and x is a 5-vertex; then the other of these vertices is a 9^--vertex, by (SP2). Note that v receives $\frac{2}{5} + \frac{1}{10} = \frac{1}{2}$ from v_3 by R1c and R2, and may give back $\frac{1}{4}$ to v_3 by R4, so that v retains at least $\frac{1}{4}$ from v_3.

If $\max\{d(v_1), d(x)\} \leq 8$, then $d(v_5) \geq 8$, for otherwise we have a 4-cycle of weight at most 23 and 5-cycle of weight at most 30 and G is not a counterexample to Theorem 4, which is a contradiction. Hence v_5 gives at least $\frac{1}{4}$ to v by Table 1. Also v receives at least $\frac{1}{2}$ from f by R0c and retains at least $\frac{1}{4}$ from v_3, as remarked above. So $\mu'(v) \geq 0$, as desired.

Now suppose $\max\{d(v_1), d(x)\} = 9$. If $d(v_1) = 9$, then v receives at least $\frac{1}{3}$ from v_1 by R1b and $\frac{5}{12}$ from f by the exception to R0c, a total of at least $\frac{3}{4}$; otherwise, if $d(x) = 9$, v receives $\frac{3}{4}$ from f by the exception to R0c. In each case, adding in the $\frac{1}{4}$ retained from v_3 shows that $\mu'(v) \geq 0$.

Subcase 2.3. v is good. Then v receives 1 by R0a from a 6^+-face, or by R0b from a 5-face that is incident with at least one 6^+-vertex, or by R0c from a 4-face that is incident with at least two 6^+-vertices, or else v receives at least two donations of at least $\frac{1}{2}$ from 4- or 5-faces by R0b and R0c. Thus $\mu'(v) \geq -1 + 1 = -1 + \frac{1}{2} + \frac{1}{2} = 0$, as desired.

Table 1
Donations to a 5-vertex by Rules 1 and 2, using (SP4).

Donor:	Strong	Semiweak	Weak
7:	1/3	1/6	0
8:	1/2	3/8	1/4
9:	2/3	1/2	1/3
$10 \vee 11$:	≥3/5	≥1/2	≥1/2
≥12:	≥1	≥3/4	≥1/2

So from now on we will assume that v is neither simplicial nor special nor good, which implies that v gives back no charge to bad 11-vertices by Rules 3, 4 and 5. It also implies that v is incident with exactly one 4^+-face f, which is a 5-face incident with five 5-vertices or a 4-face incident with at most one 6^+-vertex.

Subcase 2.4. $d(f) = 5$, say $f = v_1vv_2xy$, where all five vertices have degree 5. Thus G is not a counterexample to Theorem 4(ii) or (iii), and so must be a counterexample to Theorem 4(i). Note that at least one of v_3 and v_4 is an 8^+-vertex, because otherwise the 4-cycle $vv_2v_3v_4$ has weight at most 24. This vertex gives at least $\frac{1}{4}$ to v by Table 1, and v receives $\frac{4}{5}$ from f, and so $\mu'(v) \geq -1 + \frac{4}{5} + \frac{1}{4} > 0$.

Subcase 2.5. $d(f) = 4$, say $f = v_1vv_2x$, where at most one vertex has degree ≥6, and this vertex has degree at most 9 by (SP2). Here, the boundary of f is a 4-cycle of weight at most $3 \times 5 + 9$; thus G must be a counterexample to Theorem 4(ii).

Now we have two cases to consider. First, suppose $d(v_1) = d(v) = d(v_2) = d(x) = 5$. Note that v_3 and v_5 are 11^+-vertices, for otherwise we have a 5-cycle of weight at most 30 and G is not a counterexample to Theorem 4(ii). So $\mu'(v) \geq -1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} > 0$ by R0c and Table 1.

So we may suppose by symmetry that $d(v_2) = 5$ and either $6 \leq d(v_1) \leq 9$ or $6 \leq d(x) \leq 9$. Note that v_3 and v_5 are 7^+-vertices, for otherwise we have a 5-cycle of weight at most 30.

If $d(v_4) \geq 6$, then v_3 and v_5 are at least semiweak donors to v and $\mu'(v) \geq -1 + \frac{2}{3} + \frac{1}{6} + \frac{1}{6} = 0$ by R0c and Table 1. So we may suppose that $d(v_4) = 5$, which means that $d(v_3) + d(v_5) \geq 16$ since otherwise the 5-cycle $vv_2v_3v_4v_5$ has weight at most 30. Now v_3 and v_5 are at least weak donors to v. If $d(v_3) \geq 9$ or $d(v_5) \geq 9$, then $\mu'(v) \geq -1 + \frac{2}{3} + \frac{1}{3} = 0$; otherwise $d(v_3) = d(v_5) = 8$, and $\mu'(v) \geq -1 + \frac{2}{3} + \frac{1}{4} + \frac{1}{4} > 0$.

Case 3. $d(v) \notin \{5, 11\}$. If v is the vertex v_i in the last sentence of Rule 1(b) or 1(c), then v may pass on to 5-neighbors some or all of the charge that v has received from a 6^+-vertex w. But since v never passes on more than it receives from w, we may ignore these two sentences, provided that we also ignore any charge that v receives from 6^+-vertices.

Subcase 3.1. $d(v) = 6$. Then $\mu(v) = d(v) - 6 = 0$, and $\mu'(v) = 0$ also, since v does not participate in discharging.

Subcase 3.2. $d(v) = 7$. Then $\mu(v) = d(v) - 6 = 1$. By R1a, the amount given out by v does not exceed $\frac{1}{3}$ times the number of 5-neighbors of v. Also, v does not give more than $\frac{1}{3}$ times the number of 6^+-neighbors of v; for, if each 5-neighbor v_i that receives $\frac{1}{3}$ from v transfers $\frac{1}{6}$ to each of v_{i-1} and v_{i+1}, and each v_i that receives $\frac{1}{6}$ from v transfers it to whichever of v_{i-1} and v_{i+1} is a 6^+-vertex, then all the charge given by v now resides with its 6^+-neighbors, and each has received at most $\frac{1}{3}$. But a 7-vertex must have either at most three 5-neighbors or at most three 6^+-neighbors, and so $\mu'(v) \geq \mu(v) - 3 \times \frac{1}{3} = 0$.

Subcase 3.3. $8 \leq d(v) \leq 10$ or $d(v) \geq 12$. By R1b and R1c, $\mu'(v) \geq d(v) - 6 - d(v) \times \frac{d(v)-6}{d(v)} = 0$.

Case 4. $d(v) = 11$. Then $\mu(v) = d(v) - 6 = 5$. If v has a 6^+-neighbor v_i then none of $v_{i-2}, \ldots, v_{i+2}$ is twice weak and so none of them receives a supplementary $\frac{1}{10}$ from v by Rule 2; thus $\mu'(v) \geq 5 - 11 \times \frac{2}{5} - 6 \times \frac{1}{10} = 0$. So we may assume that all neighbors of v have degree 5, i.e. v is bad, and v gives $\frac{2}{5} + \frac{1}{10} = \frac{1}{2}$ to each neighbor by R1c and R2. Thus $\mu'(v) \geq -\frac{1}{2}$. We must find an extra $\frac{1}{2}$ for v in order to show that $\mu'(v) \geq 0$.

First suppose that some neighbor, say v_1, is not simplicial. If v_1 is good, then v_1 gives back $\frac{1}{2}$ to v by R5, and so $\mu'(v) \geq 0$ as desired. Thus we may assume that v_1 is not good. This means that v_1 is incident with exactly one 4^+-face f, which is either a 5-face incident with five 5-vertices or a 4-face incident with at most one 6^+-vertex. However, if f is a 5-face incident with five 5-vertices then G cannot be a counterexample to Theorem 4(ii) or (iii), and it cannot be a counterexample to Theorem 4(i) either as we have a 4-cycle of weight $5 + 5 + 5 + 11 = 26$. (Recall that v is simplicial, by (SP2).)

This contradiction shows that f is a 4-face, say $f = v_1xyz$, which is incident with at most one 6^+-vertex, which must have degree at most 9 by (SP2). Thus the boundary of f is a 4-cycle with weight at most 24. This means that G cannot now be a counterexample to Theorem 4(i) or (iii), and so it must be a counterexample to Theorem 4(ii).

If $\{x, z\} \cap \{v_2, v_{11}\} = \emptyset$ then we have a 5-cycle of weight at most $24 + 5 < 30$, and G is not a counterexample to Theorem 4(ii), which is a contradiction. So by symmetry we can assume that $x = v_2$, which means that both v_1 and v_2 are special. Thus v receives $\frac{1}{4}$ from each of them by R4, and $\mu'(v) \geq 0$ as desired.

From now on we can assume that all neighbors of v are simplicial. Each edge v_iv_{i+1} lies in two triangles, say $v_iv_{i+1}v$ and $v_iv_{i+1}w_i$, and $v_iw_{i-1}w_i$ is also a triangle since v_i is simplicial. For each i, $d(w_{i-1}) + d(w_i) \geq 16$ by (SP3). If $d(w_{i-1}) \geq 9$ and $d(w_i) \geq 9$ for some i, then v_i gives back $\frac{1}{2}$ to v by Rule 3(a), and $\mu'(v) \geq 0$. If $d(w_i) = 8$ for some i, then by Rule 3(b) v

receives $\frac{1}{4}$ from each of v_i and v_{i+1}, and again $\mu'(v) \geq 0$. So we may assume that, for each i, one of $d(w_{i-1})$ and $d(w_i)$ is at most 7 and the other is at least 9. But this cannot hold for all i modulo 11, since 11 is odd.

We have shown that $\mu'(x) \geq 0$ whenever $x \in V \cup F$. This contradiction with (1) completes the proof of Theorem 4.

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