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Lower bounds on the number of edges in edge-chromatic-critical graphs with fixed maximum degrees

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ABSTRACT

In this article, we provide new lower bounds for the size of edge chromatic critical graphs with maximum degrees of 10, 11, 12, 13, 14, furthermore we characterize their class one properties.

1. Introduction

Let V and E be the vertex set and edge set of a graph G, while $|V|$ and $|E|$ represent the cardinality of V and E of G, respectively. For a vertex x, set $N(x) = \{v : xv \in E(G)\}$ and $d(x) = |N(x)|$, the degree of x in G. We use Δ and δ to denote the maximum and the minimum degrees of G, respectively. For a vertex set S of G, set $N(S) = \cup_{x \in S} N(x)$. A *$k$-edge-coloring* of a graph G is a function $\phi : E(G) \mapsto \{1, \ldots, k\}$ such that any two adjacent edges receive different colors. The *edge chromatic number*, denoted by $\chi_e(G)$, of a graph G is the smallest integer k such that G has a k-edge-coloring. Vizing's Theorem [13] states that the edge chromatic number of a simple graph G is either Δ or $\Delta + 1$. A graph G is *class one* if $\chi_e(G) = \Delta$ and is *class two* otherwise. A class two graph G is *critical* if $\chi_e(G - e) < \chi_e(G)$ for each edge e of G. A critical graph G is *Δ-critical* if it has maximum degree Δ.

The following conjecture was proposed by Vizing [13] concerning the sizes of critical graphs.

Conjecture 1.1. *If $G = (V, E)$ is a critical simple graph, then $|E| \geq \frac{1}{2}(|V|(\Delta - 1) + 3)$.*

Some best known lower bounds of size of critical graphs are listed below [7,5,16,15,10]. Let G be a Δ-critical graph with average degree q, where $q = \frac{\sum_{v \in V(G)} d(v)}{|V|}$.

$$\begin{array}{llllll} \text{If } \Delta = 7, & q \geq 6. & \text{If } \Delta = 8, & q \geq \dfrac{20}{3}. & \text{If } \Delta = 9, & q \geq 7.3. \\ \text{If } \Delta = 10, & q \geq 8. & \text{If } \Delta = 11, & q \geq 8.6. & \text{If } \Delta = 12, & q \geq 9.25. \\ \text{If } 8 \leq \Delta \leq 17, & q \geq \dfrac{4}{7}(\Delta + 3). & \text{If } \Delta \geq 8, & q \geq \dfrac{2}{3}(\Delta + 1). & & \end{array}$$

We improve some of the earlier results in the following theorem: main theorem.

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Theorem 1.2. *Let G be a Δ-critical graph with $\Delta \geq 8$. Then $|E(G)| \geq \frac{|V(G)|}{2}q$ where $q = 8.25, 9, \frac{126}{13}, \frac{134}{13}, \frac{142}{13}$ for $\Delta = 10, 11, 12, 13, 14$ respectively.*

We show some lemmas in Section 2, and then provide our proof of the main theorem in Section 3.

2. Adjacency lemmas

Throughout this paper, G is a Δ-critical graph with $\Delta \geq 10$. A k-vertex (or, $(\leq k)$-vertex, $(\geq k)$-vertex) is a vertex of degree k (or $\leq k$, $\geq k$, respectively). A vertex w is a *k-neighbor* of x if $w \in N(x)$ and $d(w) = k$. Let V_k (or $V_{\leq k}$) be the set of vertices with degree k (or $\leq k$). Let $d_{\leq k}(x)$ denote the number of $(\leq k)$-vertices adjacent to x. Similarly define $d_{\geq k}(x)$. Let ϕ be the Δ-edge coloring of $G - xw$, $\phi(v)$ be the set of colors of the edges adjacent to the vertex v under edge coloring ϕ. A vertex v sees color j if v is adjacent to an edge colored by j. Denote by $P_{j,k}(v)_\phi$ the (j, k)-bi-colored path starting at v under edge coloring ϕ, or by $P_{j,k}(v)$ if there is no confusion. The following one belongs to Vizing [14], which will be abbreviated as VAL in this article.

VAL: If xw is an edge of a Δ-critical graph G, then x has at least $(\Delta - d(w) + 1)$ Δ-neighbors. Any vertex of G has at least two Δ-neighbors.

Adjacency Condition [17,11]: Let G be Δ-critical, $xw \in E(G)$ and $d(x) + d(w) = \Delta + 2$. The following hold: (1) every vertex of $N(x, w) \setminus \{x, w\}$ is a Δ-vertex; (2) every vertex of $N(N(x, w)) \setminus \{x, w\}$ is of degree at least $\Delta - 1$; and (3) if $d(x), d(w) < \Delta$, then every vertex of $N(N(x, w)) \setminus \{x, w\}$ is a Δ-vertex.

Through this paper, without loss of generality, under coloring ϕ, edges incident with x in $G - xw$ are colored by $1, 2, \ldots, d - 1$, while those incident with w are colored by $\Delta - k + 2, \ldots \Delta$ where $d = d(x)$, $k = d(w)$.

Let C_1 be the set of colors present at only one of x, w and C_2 be the set of colors present at both. Further let C_{11} be the set of colors present only at x, and C_{12} be the set of colors present only at w. We may assume that $C_1 = C_{11} \cup C_{12} = \{1, \ldots, \Delta - k + 1\} \cup \{d, d + 1, \ldots, \Delta\}$ and $C_2 = \{\Delta - k + 2, \ldots, d - 1\}$, where $C_2 = \emptyset$ if $d + k = \Delta + 2$. $|C_1| = 2\Delta - d - k + 2$, $|C_2| = d + k - \Delta - 2$. Let $C_v = \{i : \text{ vertex } v \text{ misses color } i\}$.

Lemma 2.1 ([8]). *Let xw be an edge of G with $d(x) + d(w) = \Delta + 2$ and $d(x), d(w) < \Delta$. Then every vertex of $N(N(N(x, w))) \setminus \{x, w, N(x, w), N(N(x, w))\}$ (assume that it is not empty) is adjacent to all Δ-vertices.*

In order to give improved adjacency properties on the i-vertex, we provide some claims. First two claims are equivalent to Facts 1 and 2 in [9], and for the purpose of convenience of uniform discussion, we re-write them as Claims A and B.

Claim A. *For each neighbor w_j of w in $G - xw$ where $\phi(ww_j) = j$ present only at w, then w_j must see each color in C_1.*

Claim A will be often used in the discussion through this paper without notifying.

Claim B. *For each neighbor x^i of x where $\phi(xx^i) = i$ present only at x, then x^i must see each color in C_1. Note that x has at least $\Delta - k + 1$ such x^i.*

Due to Claim B, we call a swapping (i, j) a *nice* swapping if it does not change the *set* of colors of edges incident with x and w in $G - xw$.

Claim C. *For a neighbor w_b of w where $b \in C_2$, if one of such w_b's misses a color in C_1, then we could assume that one of those w_b's misses color 1. Note that we can only assure there is one such vertex w_b.*

We assume, without loss of generality, that w_b misses Δ but sees 1, then we swap color 1 with the missing color along the path starting at w_b, by Claim B, this swapping is a *nice* one because it does not affect the colors of edges that are incident with x, w. So w_b misses color 1.

Claim D which follows is similar to Fact 4 in [9] but it is slightly stronger. So the proof is provided in the appendix.

Claim D. *Let x and w be adjacent in Δ-critical graph G with $d(x) = d$, $d(w) = k$. $G - xw$ has a Δ-edge coloring ϕ. Let $xx^a y$ be a path in $G - xw$ where $\phi(xx^a) = a \in C_{11}$ and $y \neq w$ such that $\phi(x^a y) \in C_1$. Then y must see each color in C_1, that is, $d(y) \geq 2\Delta - d - k + 2$. Note that there are $2\Delta - d - k + 1$ such y's, and some of them may be adjacent to vertices in $N(x)$.*

Lemma 2.2. *For a Δ-edge coloring ϕ of $G - xw$ with $d(x) = d$, $d(w) = k$ (see Fig. 1), let $xx^\alpha y$ and $xx^r u$ be paths that start at x, where $\phi(xx^\alpha) = \alpha$ present only at x and $\phi(xx^r) = r$ is a color in C_2. If there is a vertex $w_j \in N(w)$, where $\phi(ww_j) = j \in C_{12}$, and w_j misses $r \in C_2$, or if there is a $w_r \in N(w)$ with $\phi(ww_r) = r \in C_2$, and w_r misses a color in C_1, then we have the following:*

(i) x^α must see $r \in C_2$. (ii) y sees each color in C_1 and r; further, if $\phi(x^\alpha y) = r \in C_2$, then y sees each color in C_1 and color $r'(\neq r)$ if there is a $w_{j'} \in N(w)$ $(j' \in C_{12})$ missing $r' \in C_2$, or there is a $w_{r'} \in N(w)$ with $\phi(ww_r') = r'$ and $w_{r'}$ misses a color in C_1. (iii) x^r must see each color in C_1 and also color r' as described in (ii). (iv) u sees each color in C_1 and also sees r' as described in (ii).

Proof. The proof consists of two parts: Part I and Part II. Part I: If there is a vertex $w_j \in N(w)$, where $\phi(ww_j) = j \in C_{12}$, and w_j misses a color $r \in C_2$, then our results hold. Part II: If there is a $w_r \in N(w)$ with $\phi(ww_r) = r \in C_2$, and w_r misses a color in C_1, then our results hold.

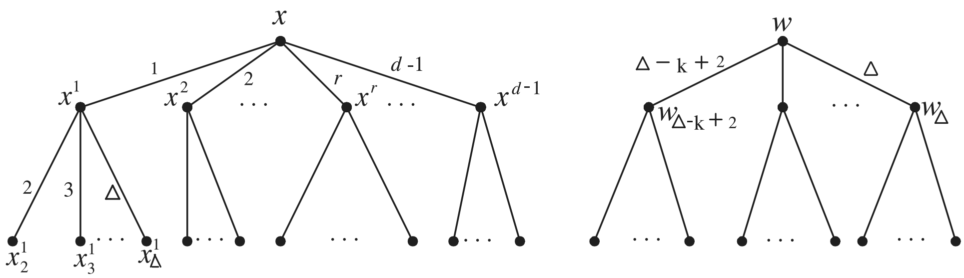

Fig. 1. Δ-edge coloring ϕ of $G - xw$ exhibited at $N(x) \cup N(w)$.

Proof of Part I. Note that $w_j (j \geq d)$ must see each color in C_1. Initially, we form two observations.

Observation 1. *$P_{\alpha,r}(w_j)$ must end at w where α ($\in C_{11}$) present only at x.*

Otherwise, we assume that $P_{\alpha,r}(w_j)$ does not end at w, so we swap (α, r) along the path starting at w_j because w_j sees α by Claim A. And note that the swapping does not affect colors of edges that are incident with x, so we recolor ww_j, xw with α, j respectively, which leads us a Δ-edge coloring of G, a contradiction.

Observation 2. *$P_{\beta,r}(w_j)$ must end at x where β ($\in C_{12}$) present only at w.*

Otherwise, swapping (β, r) along $P_{\beta,r}(w_j)$ does not affect colors of edges incident with x and w ($P_{\beta,r}(w_j)$ may pass though w). Under the current coloring, w_j sees r but not β. Note that β present only at w, and w_j should see β; thus we have a contradiction.

Now we are ready to show our results. Without loss of generality, we assume that w_Δ misses a color $r \in C_2$.

Proof of (i). We claim that x^α sees r.

By Claim A x^α sees each color in C_1. x^α also sees r if x^α is on one of the paths $P_{\alpha,r}(w_\Delta)$ and $P_{\beta,r}(w_\Delta)$. As a result, we assume that x^α is not on either one of them. Hence a nice swapping (β, r) along the path starting at x^α shows that x^α misses β, which is a contradiction.

Proof of (ii). We claim that y sees each color in C_1 and r.

(ii-1) We assume that $\phi(x^\alpha y)(=s) \in C_1$.

Through Claims A and D, y sees all colors in C_1. As a result, we only need to show that y sees r. If $s \in C_{11}$, we consider the $P_{\beta,r}(y)$ ($\beta \neq \Delta$). By Observation 2, $P_{\beta,r}(w_\Delta)$ ends at x, so $P_{\beta,r}(y)$ cannot end at x because two (β, r) paths cannot share a common ending edge. So a nice swapping (β, r) along $P_{\beta,r}(y)$ shows us that y sees r but not β, a contradiction. If $s \in C_{12}$, we consider the $P_{\alpha,r}(y)$ where α present only at x. By Observation 1, $P_{\alpha,r}(w_\Delta)$ ends at w, so the path $P_{\alpha,r}(y)$ cannot end at w. Thus, a nice swapping (α, r) along $P_{\alpha,r}(y)$ providing us that y sees r but not α, a contradiction.

(ii-2) We assume that $\phi(x^\alpha y) = r \in C_2$ which is missed by w_Δ.

Note that first we need to show y sees each color in C_1.

(ii-2-1) We claim that y must see each color $\beta(\beta \neq \Delta)$ present only at w.

Otherwise, we consider $P_{r,\beta}(y)$. Since $P_{\beta,r}(w_\Delta)$ ends at x through Observation 2, so $P_{r,\beta}(y)$ does not end at x. A nice swapping (r, β) along $P_{r,\beta}(y)$ gives us that the edge $x^\alpha y$ is colored by β which brings us to the case (ii-1).

(ii-2-2) We claim that *y sees each color present only at x.*

Otherwise, y misses a color $a \in C_{11}$, then a nice swapping (β, a) along $P_{\beta,a}(y)$ causes y to miss β, which is a contradiction.

(ii-2-3) We claim that *y sees Δ.*

Otherwise, a nice swapping (α, Δ) along $P_{\alpha,\beta}(y)$ causes y to miss α, which contradicts (ii-2-2).

(ii-2-4) We claim that *y sees r' if there is a $w_{j'} \in N(w)(j' \geq d)$ missing a color $r' \in C_2$.*

Note that $P_{\beta,r'}(w_\Delta)$ ends at x. If $r' \in C_y$, $P_{\beta,r'}(y)$ cannot end at x. A nice swapping (β, r') along $P_{\beta,r'}(y)$ causes y to see r', but not β, which contradicts (ii-2-1).

Proof of (iii). Consider that $\phi(xx^r) = r \in C_2$ where $r \in C_{w_\Delta}$. We show that x^r sees each color in C_1 and $r' \in C_2 (r' \neq r)$ if there is any $r' \in C_2 \cap C_{w_{j'}}$ where $w'_j \in N(w)$, $j' \in C_2$.

First, x^r sees each $\beta(\neq\Delta)$ in C_{12} through Observation 2. Second, x^r sees each $\alpha \in C_{11}$ through Claim A and nice swapping (β, α) argument. Third, x^r sees Δ by nice swapping (α, Δ) argument. Last, x^r sees $r' \in C_2$ other than r if there is any, where $r' \in C_{w_j}$ and $j \in C_{12}$. Otherwise, by Observation 2, x^r is not on the path $P_{\beta,r'}(w_j)$, so a nice swapping (β, r') starting at x^r results in a contradiction.

(iv) We claim that u sees each color in C_1 and color r, where $u \in N(x^r)$.

(iv-1) We assume that $\phi(x^r u)$ present at w, for example β.

Using Observation 2, u sees r because $x^r u$ must be on the path $P_{\beta,r}(w_\Delta)$.

First, u sees color $\beta' \in C_{12}$ ($\beta' \neq \beta$). Otherwise, by Observation 2, u is not on the path $P_{\beta',r}(w_\Delta)$, so a nice swapping (r, β') along $P_{r,\beta'}(u)$ causes u to see β' but not r, a contradiction to that u must see r. Second, u sees each color $a \in C_{11}$. Otherwise, a nice swapping (β', a) along $P_{\beta',a}(u)$ will result in u missing a color $\beta' \in C_{12}$, which is a contradiction. Third, u sees $r' \in C_2$ which is described in proof of (iii). Otherwise, apply Observation 2 and swapping argument similar to that in the proof of (iii), we will have a contradiction.

(iv-2) We assume that $\phi(x^r u) = \alpha \in C_{11}$.

Applying Observation 1 and swapping argument similar to (iv-1), we have that u sees each color in C_1 and color $r' \in C_{w_\Delta}$. The proof is omitted here.

(iv-3) We assume that $\phi(x^r u) = r' \in C_2$ missed by $w_j (j \geq d)$ which is described in (ii).

First, u sees each color $a \in C_{11}$. Otherwise, u is not on the path $P_{a,r'}(w_j)$ through Observation 1. On the other hand, a nice swapping (r', a) along the path starting at u causes that edge $x^r u$ is colored by a which is the case of (ii). That is, u sees all colors in C_1 and all colors missed by w_j, which is a contradiction. Second, u sees each color $\beta \in C_{12}$. Otherwise, a nice swapping (a, β) results u to miss color $a \in C_{11}$, which is a contradiction. Third, u sees each color $r'' \in C_2(\neq r)$ missed by $w_{j''}$ where $w_{j''} \in N(w)$ with $j'' \geq d$ if one exists. Otherwise, by Observation 2, a nice swapping (β, r'') along the path starts at u would result in a contradiction.

Proof of Part II. Recall that if $\phi(x^\alpha y) \in C_1$, through Claims B and D, y must see each color in C_1. The proof consists of two parts; Part A: w_r misses a color in C_{11}, and Part B: w_r misses a color in C_{12}.

Proof of Part A. We assume that w_r misses a color $a \in C_{11}$.

Case (A-1) $a \neq \alpha$.

We re-color ww_r by a, and denote the current coloring by ϕ^*. Now $r \in C_{11}$. Hence, (i) x^α sees r and (ii) y sees r. Further, (iii) x^r sees all color in C_1 except a. Now we show that x^r must see a. Otherwise, a nice (Δ, a) swapping along the path that starts at x^r which causes x^r to miss Δ, a contradiction. Next, consider $\phi(x^\alpha y) = r$. Under coloring ϕ^*, r present only at x, so y sees all colors in C_1 except a. Now uncolor edge ww_r and color it with its original color r. If y misses a, we could do a nice (Δ, a) swapping along the path that starts at y, which causes y to miss Δ, which is a contradiction. Last, we consider $\phi(x^r u) \in C_1$. Under coloring ϕ^*, r present only at x. So x^r and u play the same roles as that of x^α and y earlier. Hence, u sees all colors in C_2. Now we show that x^r sees r' if there exists a $w_{r'} \in N(w)$ with $r' \in C_2$ and $w_{r'}$ missing a color present only at x, say, s. We re-color $ww_{r'}$ by s. Then r' present only at x under current coloring. If x^r misses r', a nice swapping (Δ, r') along the path starting at x^r causes x^r to miss Δ. Now we re-color $ww_{r'}$ by its original color r'. But then x^r misses Δ which contradicts the previous result that x^r sees Δ. Finally, consider $\phi(x^r u) = r' \in C_2$, and we show that u sees each color in C_1 and r. We re-color ww_r by a, then r present only at x, and x^r and u play the same roles as that of x^α and y before, respectively. So the results hold for x^r and u.

Case (A-2) $a = \alpha$.

If $\phi(x^a y) \in C_1$, under ϕ^*, by nice (Δ, r) swapping arguments for y, x^a respectively, then clearly both y and x^a see r. Next, we show that u sees each color in C_1 where $xx^a u$ is a path with $\phi(x^a u) = r$. Under ϕ^*, r present only at x, if u misses a β which present only at w, then by Claim D, we could do a nice (r, β) swapping starting at u, which causes $x^a u$ to be colored by β; we then re-color ww_r by r, so now u plays the same role as y did before; that is, u sees all colors in C_1, which is a contradiction. So u sees each color in C_{12}. By a similar swapping argument, u sees each color present only at x and color $r' \in C_2$ if there exists a $w_{r'} \in N(w)$ missing a color present only at x. In order to avoid repetition, we omit the proof.

Proof of Part B. We assume that w_r misses a color $b \in C_{11}$.

We provide an observation first.

Observation 3. *$P_{r,b}(w_r)$ must end at x.*

Otherwise, a swapping (r, b) along the path starting at w_r does not affect colors of edges incident with x, ww_r is colored by b under current coloring and misses the color $r \in C_2$. By Part I, our result holds.

(B-1) We claim that y sees r where $\phi(x^\alpha y) \in C_1$.

Note that y sees all colors in C_1. If y misses r, then $P_{\Delta,r}(y)$ will not end at either x or w_r by Observation 3. So we do a nice swapping (b, r) along the path starting at y, it shows that y misses b, which is a contradiction.

(B-2) We claim that x^α sees r. Consider the path $P_{b,r}(x^\alpha)$, using similar argument as that for path $P_{b,r}(y)$ in (B-1), clearly x^α sees r.

(B-3) Let $\phi(x^\alpha v) = r \in C_2$, where $v \in N(x^\alpha)$. We claim that v sees all colors in C_1. Further, v sees $r' \in C_2$ if there exists a $w_{r'} \in N(w)(r' \neq r)$ that misses a color in C_1.

Through Observation 3, $P_{r,b}(v)$ does not end at x, so v sees the color b. And applying Claims A and B, v sees each color in C_1. Further, if $w_{r'}$ misses a color $a \in C_{11}$ (where $w_{r'} \in N(w)$ and $\phi(ww'_r) = r' \in C_2$), then we re-color $ww_{r'}$ by a, so v sees r'. If $w_{r'}$ misses a color $b' \in C_{12}$, by Observation 3, $P_{r',b'}(w_{r'})$ ends at x. Then $P_{b',r'}(v)$ will not end at either x or $w_{r'}$. Thus, we can perform a nice swapping (b', r') along the path that starts at v, which causes v to miss b', which is a contradiction.

(B-4) Let $\phi(xx^r) = r$, we claim that x^r sees all colors in C_1; further, if there is a $w_{r'}$ $(r' \neq r)$ that misses a color in C_1, x^r also sees r'.

If x^r misses $b \in C_{12}$, by Observation 3, $P_{r,b}(x^r)$ does not end at x, so a nice swapping brings us that x^r sees each color in C_{12}. Then, through Claim A, Claim B and swapping method, x^r first sees all colors in C_{11}. Finally, by applying the same argument as that in (B-3), we have that x^r sees $r' \in C_2$.

(B-5) Let xx^ru be a path where $\phi(xx^r) = r \in C_2$. We claim that u sees all colors in C_1 and color $r' \in C_2$ if there exists a $w_{r'} \in N(w)(r' < d)$ that misses a color in C_1.

First, let $\phi(x^ru) = \beta \in C_{12}$. By Claim B and the swapping method, clearly we see that (1) u sees all colors in C_{11}; (2) u sees all colors in C_{12}. Now we show that u sees $r \in C_2$. If $\beta = b$ where b is missing by w_r, applying Observation 3, u sees r. If $\beta \neq b$, since u sees b, applying Observation 3 again, $P_{b,r}(u)$ does not end at either x or w_r. Hence, a nice swapping (b, r) along the path that starts at u causes u to miss b, a contradiction. Second, we assume that $\phi(x^ru) \in C_{11}$. By Claim B and similar swapping methods as seen in the previous paragraph for the path $P_{b,r}(u)$, clearly we have that (1) u sees all colors in C_{12}; (2) u sees all colors in C_{11}; and (3) u must see $r \in C_2$. Third, we assume that $\phi(x^ru) = r'$ ($r' \in C_2$ where $\phi(ww_{r'}) = r'$ and $w_{r'}$ misses a color in C_1). If $w_{r'}$ misses $a \in C_{11}$, we simply re-color $ww_{r'}$ with a; r' now present only at x. u sees each color in C_1, and sees r by similar discussion in the previous discussion. If $w_{r'}$ misses b' which present at w only, applying Observation 3, $P_{r',b'}(w_{r'})$ must end at x, a contradiction. If u misses $b \in C_{12}$, note that x^r sees b, so we can perform a nice swapping (r', b) along the path that starts at u, using discussion from the first two lines of (B-5), we have a contradiction. Now we show that u sees r. Otherwise, we consider $P_{b,r}(u)$. Applying Observation 3, the path will not end either at x or at w_r. So a nice swapping (b, r) could be performed along the path starting at u, which shows that u misses b, which is a contradiction.

Hence we finish the proof of Lemma 2.2. □

The following lemma uses vertex sequence rotation method to generalize the adjacency lemma by Sanders and Zhao [12].

Lemma 2.3. *For a Δ-edge coloring ϕ of G-xw (see Fig. 1), $d(x) = d$, $d(w) = k$ and $|C_2| = d + k - \Delta - 2$. If the number of $(\leq \Delta - \frac{|C_2|}{2})$-neighbors of w is $|C_2| - 1$ or $|C_2|$, then there are $\Delta - k + 1 + \lfloor \frac{1}{2}|C_2| \rfloor$ neighbors x^α of x satisfying: $x^\alpha \neq w$; x^α is adjacent to at least $2\Delta - d - k + 1 + \lfloor \frac{1}{2}|C_2| \rfloor$ vertices y different from x with degree at least $2\Delta - d - k + 2 + \lfloor \frac{1}{2}|C_2| \rfloor$.*

Proof. The set of $(\leq \Delta - \frac{|C_2|}{2})$-neighbor of w could be categorized as below. Let

$$R^1 = \left\{ w_j : \phi(ww_j) = j \in C_2, d(w_j) \leq \Delta - \frac{1}{2}|C_2|, \mathcal{C}_{w_j} \cap [\phi(w)\Delta\phi(x)] \neq \emptyset \right\}$$

where $\phi(w)\Delta\phi(x)$ is symmetrical difference of $\phi(x)$ and $\phi(w)$. In other words, each vertex w_j in R^1 misses at least $\frac{1}{2}|C_2|$ colors including at least one color in C_1. Let

$$R^2 = \left\{ w_j : \phi(ww_j) = j \in C_{12}, d(w_j) \leq \Delta - \frac{1}{2}|C_2|, \mathcal{C}_{w_j} \cap [\phi(w)\Delta\phi(x)] = \emptyset \right\}.$$

In other words, each vertex w_j ($j \in C_{12}$) in R^2 misses at least $\frac{1}{2}|C_2|$ colors in C_2. Let

$$R^* = \left\{ w_j : \phi(ww_j) = j \in C_2, d(w_j) \leq \Delta - \frac{1}{2}|C_2|, \mathcal{C}_{w_j} \subseteq [\phi(x) \cap \phi(w)] \right\}.$$

In other words, each vertex $w_j \in C_2$ in R^* misses at least $\frac{1}{2}|C_2|$ colors in C_2. Note that R^1, R^2, R^* are vertex pairwise disjointed and $|R^1 \cup R^2 \cup R^*| = |C_2| - 1$ or $|C_2|$.

Let $xx^\alpha y$ and xx^ru be two paths that starts at x, where $\phi(xx^\alpha) = \alpha \in C_{11}$, and $\phi(xx^r) = r$, where $r \in C_2 \cap C_{w_j}$ if $w_j \in R^2$ or $w_r \in R^1 \cup R^*$. In order to prove the results, we need to prove following equivalent results:

(i) If $\phi(x^\alpha y) \in C_1$, then y sees each color in C_1 and also sees r. (ii) x^α must see color $r \in C_2$, and let $\phi(x^\alpha y) = r$, then y sees each color in C_1 and also sees r' if there exists one $w_{r'} \in R^1 \cup R^2 \cup R^*$ with $\phi(ww'_r) = r'$. (iii) x^r must see each color in C_1 and $r' \in C_2$ described in (ii). (iv) u sees each color in C_1 and $r' \in C_2$ described in (ii).

By Lemma 2.2, if $|R^1 \cup R^2| \geq \lfloor \frac{|C_2|}{2} \rfloor$, then our results hold. Now we assume that $|R^1 \cup R^2| < \lfloor \frac{|C_2|}{2} \rfloor$, so that $|R^*| \geq \frac{|C_2|}{2}$. Without loss of generality, let $w_r (\in R^*)$ miss at least $\lfloor \frac{|C_2|}{2} \rfloor$ colors in C_2.

Proof of (i). We consider the path $xx^\alpha y$ where $\phi(x^\alpha y) = s$, and $s \in C_1$.

Note that y sees each color in C_1. *We show that y sees color r*. We prove it by contradiction. We assume that y misses color r. The procedure showed below is called *vertex sequence rotation method*. Be aware that w_r misses at least $\frac{1}{2}|C_2|$ colors in C_2. So we can find a color $r_1 \in C_2$ that is free at vertex w_r such that the corresponding vertex w_{r_1} is also in R^*. Since $w_{r_1} \in R^*$, and surely, w_{r_1} misses at least $\frac{1}{2}|C_2|$ colors in C_2, then there is a color of C_2, for example, r_2, which is free at vertex w_{r_1} such that the corresponding vertex w_{r_2} is still in R^*. By repeating this procedure up to $|R^*|$ times, we obtain a vertex sequence $[w_r, w_{r_1}, w_{r_2}, \ldots, w_{r_s}]$ of R^* where w_{r_i} misses color r_{i+1}, and w_{r_s} misses color r. We claim that the $P_{\Delta,r}(y)$ passes through w and ends at x. That is, three vertices y, w, and x must be in the same (Δ, r) component of $G - xw$. Otherwise, if $P_{\Delta,r}$ does not pass w, we swap (r, Δ) on an (r, Δ)-bi-colored component of $G - xw$ containing w which shows edge ww_r is colored by Δ, by the proof of Lemma 2.2(ii), y sees r, which is a contradiction. If $P_{\Delta,r}(y)$ does not end at x, we swap (r, Δ) along the path starting at x which causes y to see color r by Claim D under current coloring, which is contradiction. Thus the claim holds.

Since the path $P_{\Delta,r}(y)$ passes w and ends at x, first, we assume w_r is a successor of w on the path $P_{\Delta,r}(y)$, that is, $P_{\Delta,r}(y) = yz_1z_2 \dots w_\Delta ww_r \dots x$. We re-color $ww_r, ww_{r_1}, ww_{r_2}, \dots, ww_{s-1}, ww_{r_s}$ by $r_1, r_2, r_3, \dots, r_s, r$ respectively. We denote the current coloring by ϕ^*. Under ϕ^*, $P_{\Delta,r}(x)$ must end at w_{r_1} because w_r sees r_1 but not r. Then swapping (r, Δ) along the path that starts at x does not affect colors of edges incident with y, w under ϕ^*, so y must see r, as r present only at w under ϕ^*, which is a contradiction. Next, we assume w_r is a predecessor of w, that is, $P_{\Delta,r}(y) = yz_1 \dots w_r ww_\Delta \dots x$. Under ϕ^*, so $P_{\Delta,r}(y)$ must end at w_r. We perform a nice swap (r, Δ) along $P_{r,\Delta}(y)$, then y misses Δ under ϕ^*, which is a contradiction. Thus $d(y) \geq 2\Delta - d - k + 2 + |R^*| \geq 2\Delta - d - k + 2 + \lfloor \frac{1}{2}|C_2| \rfloor$.

Proof of (ii). We claim that x-neighbor x^α must see at least $\lfloor \frac{|C_2|}{2} \rfloor$ r where $w_r \in R^*$ and $\phi(xx^\alpha) = \alpha$ present only at x.

We assume that x^α misses a color $r \in C_2$ where $w_r \in R^*$, $\phi(ww_r) = r$. We perform the same vertex sequence rotation operation as that in (i), then applying swapping argument with respect to the path $P_{\Delta,r}(x^\alpha)$ as that in (i), we have a contradiction. In order to avoid repetition, we omit the detail.

Next, let $y \in N(x^\alpha)$ and $\phi(x^\alpha y) = r$. First, y sees each color in C_{12} by using a similar swapping argument on the path $P_{r,\Delta}(y)$ as that of $P_{\Delta,r}(y)$ in (i). Second, y sees each color in C_{11} by applying the same argument as in (i) on path $P_{\Delta,a}(y)$. Finally, y sees $r' \in C_2 (r' \neq r)$ if there exists a $\phi(ww_{r'}) = r'$ and $w_{r'} \in R^*$. Otherwise by using similar argument on $P_{\Delta,r'}(y)$ as that of $P_{\Delta,r}(y)$ in (i), we have that y sees r'. Hence $d(y) \geq 2\Delta - d - k + 2 + |R^*| \geq 2\Delta - d - k + 2 + \lfloor \frac{1}{2}|C_2| \rfloor$.

Proof of (iii). We claim that x^r, where $\phi(xx^r) = r \in C_2$ and $w_r \in R^*$, has the same property as that of x^α in (i) and (ii).

(iii-1) We claim that x^r sees each color in C_1.

We assume that x^r misses a color $a \in C_{11}$. We consider $P_{r,a}(x^r)$. If edge $w_r w$ is not on $P_{r,a}(x^r)$, swapping (r, a) along $P_{r,a}(x^r)$ does not affect colors seen by w. Now x^r plays the same role as x^α was in (i), so our results hold. Now we assume that $P_{r,a}(x^r)$ ends at w and passes through edge $w_r w$. Note that $w_r \in R^*$, by applying vertex sequence rotation operation as described in (i), current coloring brings us to the previous case since $P_{r,a}(x^r)$ does not use edge $w_r w$ any more. So x^r sees each color in C_{11}. If x^r misses a color $b \in C_{12}$, by Claim D and swapping (a, b) along the path that starts at x^r, this process causes a contradiction because x^r misses a.

(iii-2) We claim that x^r sees r' if there exists a $w_{r'} \in R^*$ where $ww_{r'} = r' \in C_2$.

The argument is similar to that in the case of y seeing r' in (ii), so we omit the proof.

Proof of (iv). Let u be a neighbor of x^r other than x, we have that u sees all colors in C_1 and color r' if there is a $w_{r'} \in R^*$.

(iv-1) If $\phi(x^r u) \in C_1$. Then u plays a similar role to that of y in (ii). By similar argument as y in (ii), clearly u sees all colors in C_1. Now we claim that u sees r. Otherwise, we consider $P_{\Delta,r}(u)$. Note that the path $P_{\Delta,r}(u)$ plays the same role as the path $P_{\Delta,r}(y)$ in (i), so we perform vertex-sequence rotation operation on w_r, by similar argument as that in (i), and we have that u sees r, and furthermore u sees a color $r'' \in C_2 (r'' \neq r)$ if there exists a $w_{r''} \in R^*$.

(iv-2) If $\phi(x^r u) = r' \in C_2$, then by argument used in (iii-2) we have that u sees each color in C_1 and color $r \in C_2$ where $w_r \in R^*$. Furthermore u also sees $r^* \in C_2$ if there is $w_{r^*} \in R^*$ where $\phi(ww_{r^*}) = r^*$. Note that vertex u plays the same role as vertex x^r in (iii). r^* plays same role as r' in (iii). The argument is similar, so we omit the proof here.

Thus we complete the proof of Lemma 2.3. □

Corollary 2.4. *Let x be a 3-vertex of a critical graph G which is adjacent to three Δ-vertices: y, z, w. If one of three Δ-neighbors of x, say w, is adjacent to one $(\leq \Delta - 1)$-vertex other than x, then there are at least two Δ-neighbors of x, say y and z, such that $d_{<\Delta}(y) = 1$, $d_{<\Delta}(z) = 1$.*

Proof. We provide the proof by contradiction. Let w be adjacent to one $(\leq \Delta - 1)$-neighbor, say w_j. We have that j either present only at w, or present at both x, w (see Fig. 2), so by Lemma 2.3, where $|C_1| = \Delta - 1$, $|C_2| = 1$, our result holds. □

Denote that $\delta_1(x) = \min\{d(y), y \in N(x)\}$.

Lemma 2.5 ([9,7]). *Let x be a d-vertex with $4 \leq d \leq 6$ and w be a $\delta_1(x)$-neighbor of x.*

(i) *$d(w) = \Delta$. If w is adjacent to at least $d - 2$ $(\leq \Delta - d + 2)$-vertices other than x, then each of the rest Δ-neighbors y of x has $d_{\leq \Delta-d+2}(y) = 1$.*
(ii) *$d(w) = \Delta - 1$.*
  (ii-1) *If w is adjacent to $(d - 3)$ $(\leq \Delta - d + 3)$-vertices other than x, the remaining neighbors y of x are all Δ-vertices and $d_{<\Delta-d+4}(y) = 1$.*
  (ii-2) *If w (where $d(x) \neq 6$) is adjacent to $(d-4)$ $(\leq \Delta - d + 3)$ vertices other than x, then there are $(d-2)$ $(\geq \Delta - d + 4)$-neighbors y of x including at least one Δ-neighbor that satisfy the following situations: if y is a Δ-vertex, then $d_{\leq \Delta-1}(y) \leq 2$; if y is a $(\Delta - 1)$-vertex, then $d_{\leq \Delta-1}(y) = 1$.*
  (ii-3) *For the case of $d(x) = 6$, if (ii-1) does not happen, then each $(\Delta - 1)$-neighbor y of x has $d_{\leq \Delta-3}(y) \leq 3$.*

Lemma 2.6. *Let x be a 4-vertex in Δ-critical graph G and w be a $\delta_1(x)$-neighbor of x.*

(i) [17] *If $|N(x) \cap V_\Delta| = 2$, then $N(N(x) \cap V_\Delta) \subset V_{\Delta-1} \cup V_\Delta \cup \{x\}$.*
(ii) [8] *If $d(w) = \Delta$ and w is adjacent to two $(\leq \Delta - 2)$ vertices including x, then x has two Δ-neighbors y that satisfy the following:*
  (ii-1) *y is adjacent to all $(\geq \Delta - 1)$-vertices other than x including at least $(\Delta - 4)\Delta$-neighbors in $N(y) \setminus N(x)$.*
  (ii-2) *And y has at least $(\Delta - 5)\Delta$-neighbors t $(t \notin N(x))$ such that $d_{<\Delta-1}(t) = 0$ (see Fig. 3).*

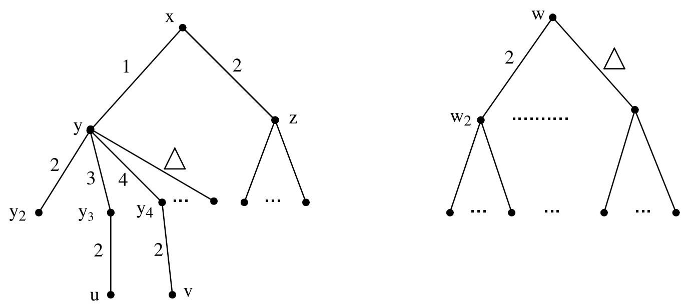

Fig. 2. $d(x) = 3, d(w) = \Delta$.

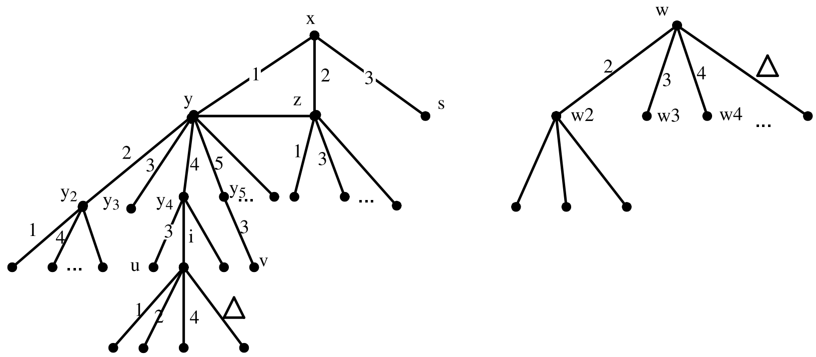

Fig. 3. $d(x) = 4, d(w) = \Delta$.

Lemma 2.7 ([7]). *Let x be a 5-vertex in G which has $(\Delta - 2)$-neighbor w. If w is adjacent to only one $(\leq \Delta - 2)$-vertex which is x, then there are three $(\geq \Delta - 1)$-neighbors of x including at least two Δ-neighbors y satisfying: if it is a Δ-vertex, then $d_{\leq \Delta-2}(y) \leq 2$; if y is a $(\Delta - 1)$-vertex, then $d_{\leq \Delta-2}(y) = 1$.*

Using the same method of Lemma 2.7 in [8], we have following result.

Corollary 2.8. *Let x be a 5-vertex having a $(\Delta - 2)$-neighbor w and x has at least three Δ-neighbors. Then there exist at least three Δ-neighbors y of x such that each of them has at least $(\Delta - 6)$ Δ-neighbors u with $d_{<\Delta-1}(u) = 0$.*

Next, we consider a special case of 5-vertex.

Lemma 2.9. *Let x be a 5-vertex of a critical graph G and x is also adjacent to exactly three $(\Delta - 1)$-vertices and two Δ-vertices. Then each of two Δ-neighbors y of x has $d_{\leq \Delta-2}(y) = 2$. Furthermore, if there is one $(\Delta - 1)$-neighbor, say w, which has $d_{\leq \Delta-2}(w) = 2$, then there exists a $(\Delta - 1)$-neighbor y of x with $d_{<\Delta-1}(y) = 1$.*

Proof. The proof is almost the same as that of Lemma 2.10 of [8] except for a restriction on $\Delta = 8, 9$. But the restriction on the maximum degree Δ does not affect the proof at all. Release the restriction on Δ and the result is still valid for all Δ. In order to avoid repetition, the proof is omitted. □

Adjacency Condition gives us some information on two adjacent vertices of a critical graph whose sum of degrees is $\Delta + 2$. The following Lemma summarizes adjacency conditions for two adjacent vertices of a critical graph G whose sum of degrees is $\Delta + 2 + p$ where $p = 1, 2, 3, 4$. The following lemma generalizes results of Lemma 2.9 in [6].

Lemma 2.10. *Let x be a d-vertex $(d \geq 5)$ of a critical graph G which is adjacent to a k-vertex w such that $d(x) + d(w) = \Delta + 2 + p$ where $p = 1, 2, 3, 4$. If $|(N(x) \setminus \{w\}) \cap V_{\leq \Delta-s}| \geq 1 (s \geq 1)$, then there are at least $\Delta - k + 1 (= d - p - 1)$ Δ-vertices $z \in N(x)$ satisfying: $z \neq w$; z is adjacent to at least K vertices of degree at least $\Delta - p + 1$ where $K = (\Delta - 1) - p + s$ if $s < p$ and $K = \Delta - 1$ if $s \geq p$.*

Proof. The proof is similar to Lemma 2.9 in [6], so we omit the detail of proof here. □

3. The proof of main results

In this section, we will prove our main theorem. A vertex x is called small if $d(x) < q$. Suppose to the contrary, the theorem is not true, then $\sum_{x \in V}(d(x) - q) < 0$. Note that $\delta_1(x) = \min\{d(y) : y \in N(x)\}$.

We perform charge–discharge method to obtain a contradiction. We call $C(x) = d(x) - q$ the *initial charge* of the vertex x and will assign a new charge to each vertex x according to the following rules. Let $C'(x)$ be the new charge of each vertex x of G, and $C'(x)$ will be calculated for each x-vertex following discharge rules that are described below.

R0 If $\Delta = 10$, each Δ-vertex sends $\frac{1}{8}$ to each of its 8-neighbor. If $\Delta = 13$, each Δ-vertex sends 0.15 to each of its 10-neighbor.

R1 Let x be a 2-vertex adjacent to u, v. Each of u, v sends $\Delta - q$ to x. By the Adjacency Condition, there are at least $(\Delta - 2)$ Δ-vertices z adjacent to either u or v, and each sends $\frac{\Delta-q}{\Delta}$ to x through u, v. So those z send $2 \times (\Delta - 2)\frac{\Delta-q}{\Delta}$ to x. Hence $C'(x) = 2 - q + 2(\Delta - q) + 2 \times (\Delta - 2)\frac{\Delta-q}{\Delta}$. It is straightforward to check that $C'(x) \geq 0.05, 0.2, 0.7, 1.6, 2.5$ for $\Delta = 10, 11, 12, 13, 14$, respectively.

R2 Let x be a 3-vertex.

(R2-1) If x is adjacent to a $(\Delta - 1)$-vertex, by Adjacency Condition and Lemma 2.1, then two Δ-neighbors y of x are adjacent to one $(\leq \Delta - 2)$-vertex which is x and there are at least $(\Delta - 3)$ Δ-neighbors z of y with $d_{\leq \Delta - 2}(z) = 0$. Thus each y sends $\Delta - q$ to x and each z sends $\frac{\Delta-q}{\Delta}$ to x by passing through each of two y. Hence $C'(x) = (3 - q) + 2(\Delta - q) + 2 \times (\Delta - 3)\frac{\Delta-q}{\Delta} \geq 0.7, 0.9, 1.3, 2.2, 3$ for $\Delta = 10, 11, 12, 13, 14$, respectively.

(R2-2) If x is adjacent to three Δ-vertices, and one of Δ-neighbors y is adjacent to two $(\Delta - 1)$-vertices including x, by VAL and Corollary 2.4, two Δ-neighbors z other than y are adjacent to no small vertices except x, and there are at least $(\Delta - 2)$ Δ-vertices $u \in N(z)$ which are adjacent to no small vertices. Hence, y sends $\frac{\Delta-q}{2}$ to x, two Δ-neighbors z send $2(\Delta - q)$ to x, $(\Delta - 2)$ Δ-vertices u send $(\Delta - 2)\frac{\Delta-q}{\Delta}$ to x through each of two Δ-neighbors z of x. Thus $C'(x) = 3 - q + \frac{\Delta-q}{2} + 2(\Delta - q) + 2 \times (\Delta - 2)\frac{\Delta-q}{\Delta} \geq 1, 2, 2, 3, 5$, for $\Delta = 10, 11, 12, 13, 14$, respectively.

(R2-3) If x is adjacent to three Δ-vertices and each of them is adjacent to only one $(\leq \Delta - 1)$ vertex x, then each Δ-neighbor sends $(\Delta - q)$ to x. Thus, $C'(x) = 3 - q + 3(\Delta - q) \geq 0, 0, 0.2, 0.7, 1$ for $\Delta = 10, 11, 12, 13, 14$, respectively.

R3 Let $d(x) + \delta_1(x) = \Delta + 2$ and $d(x) \geq 4$. We first consider the case of $d(x) \geq 5$. Note that both x and its $\delta_1(x)$-neighbor may be small vertices. By the Adjacency Condition, let $(d(x) - 1)$ Δ-neighbors of x send half of $(d(x) - 1)(\Delta - q)$ to x if $d(x) \geq 5$. Furthermore, and by Lemma 2.1, there are $(\Delta - 4)\Delta$-vertices in $N^2(x, w) \setminus \{x, w\}$ send half of $(\Delta - 4) \times \frac{\Delta-q}{\Delta}$ to x through each of $(d(x) - 1)$ Δ-neighbors of x. So (≥ 5)-vertex x receives $\frac{1}{2}(d(x) - 1)(\Delta - q) + (d(x) - 1)\frac{1}{2}(\Delta - 4)\frac{\Delta-q}{\Delta}$ totally if $d(x) \geq 5$. It is straightforward to check that $C'(x) \geq 0$ for $10 \leq \Delta \leq 14$.

Now we consider the case of $d(x) = 4$. Let each Δ-neighbor send $\frac{1}{3} \times (0.25)$ to $\delta_1(x)$ $(=8)$-neighbor if $d(x) = 4$ and $\Delta = 10$. Each Δ-neighbor sends $(\Delta - q)$ to x if $\Delta = 11, 12, 13, 14$. Hence,

$$C'(x) \geq \begin{cases} (4 - 8.25) + 3 \times \left(1.75 - \dfrac{1}{3} \times 0.25\right) = 0.75 & \text{if } d(x) = 4, \delta_1(x) = 8, \Delta = 10. \\ (4 - 9) + 3 \times 2 = 1 & \text{if } d(x) = 4, \delta_1(x) = 9, \Delta = 11. \\ \left(4 - \dfrac{126}{13}\right) + 3 \times \left(12 - \dfrac{126}{13}\right) \geq 1.2 & \text{if } d(x) = 4, \delta_1(x) = 10, \Delta = 12. \\ \left(4 - \dfrac{134}{13}\right) + 3 \times \left(13 - \dfrac{134}{13}\right) \geq 1.7 & \text{if } d(x) = 4, \delta_1(x) = 11, \Delta = 13. \\ \left(4 - \dfrac{142}{13}\right) + 3 \times \left(14 - \dfrac{142}{13}\right) > 2.3 & \text{if } d(x) = 4, \delta_1(x) = 12, \Delta = 14. \end{cases}$$

From now on we consider $d(x) \geq 4$ and $d(x) + \delta_1(x) \geq \Delta + 3$.

R4 Let x be a 4-vertex and $d(x) + \delta_1(x) \geq \Delta + 3$.

(R4-1) If x is adjacent to two $(\Delta - 1)$-vertices and two Δ-vertices, by Lemmas 2.5(ii) and 2.6, each Δ-neighbor y of x is adjacent to only one small vertex which is x, thus x receives $2(\Delta - q)$ from its two Δ-neighbors y and $2 \times \lfloor\frac{\Delta-4}{2}\rfloor\frac{\Delta-q}{\Delta}$ from neighbors of those y. So $C'(x) \geq (4 - q) + 2(\Delta - q) + 2\lfloor\frac{\Delta-4}{2}\rfloor \times \frac{\Delta-q}{\Delta} \geq 0.3, 0.09, 0.4, 0.7, 1.42$ for $\Delta = 10, 11, 12, 13, 14$, respectively.

(R4-2) If x is adjacent to one $(\Delta - 1)$-vertex w and three Δ-vertices y, for $(\Delta - 1)$-neighbor w may be adjacent to only one $(\leq \Delta - 2)$-vertex which is x, we consider Δ-neighbors of x. There are two cases: Either w is adjacent to two $(\leq \Delta - 1)$-vertices or there is one Δ-neighbor y that is adjacent to three $(\leq q)$-vertices, by Lemma 2.6 and VAL, x receives $\min\{\frac{\Delta-q}{3} + 2(\Delta - q) + 2 \times \lfloor\frac{\Delta-4}{2}\rfloor \times \frac{\Delta-q}{\Delta}, 3 \times (\Delta - q)\} \geq 2(\Delta - q) + 2\lfloor\frac{\Delta-4}{2}\rfloor\frac{\Delta-q}{\Delta}$ which is the same charge received in (R4-1), so $C'(x) > 0$ for $10 \leq \Delta \leq 14$.

(R4-3) 4-vertex x is adjacent to four Δ-vertices y. If there is one Δ-neighbor w that is adjacent to two $(\leq \Delta - 2)$-vertices other than x, then by VAL and Lemma 2.6(ii), three remaining Δ-neighbors of x send $3 \times (\Delta - q)$ to x, by straightforward checking, $C'(x) \geq (4 - q) + 3 \times (\Delta - q) > 0$ for $10 \leq \Delta \leq 14$. If each Δ-neighbor of x is adjacent to one $(\leq \Delta - 2)$-vertex other than x, by Lemma 2.6, four Δ-neighbors of x send $4 \times \frac{\Delta-q}{2}$ to x; furthermore, there are $2 \times \lfloor\frac{\Delta-4}{2}\rfloor$ vertices in $N(N(x))$, which send $2 \times \lfloor\frac{\Delta-4}{2}\rfloor\frac{\Delta-q}{\Delta}$ to x by passing through those Δ-neighbors. So x receives charges at least as much as that in (R4-1), hence $C'(x) > 0$ for $10 \leq \Delta \leq 14$.

R5 Let x be a 5-vertex and $d(x) + \delta_1(x) \geq \Delta + 3$.

(R5-1) If x is adjacent to $(\Delta - 2)$-vertex w, by VAL, x has at least three Δ-neighbors. By Lemmas 2.2, 2.7 and 2.9 and **(R0)**, x receives following charge from its neighbors: $\min\{4(\Delta - q - \theta_\Delta), 3(\Delta - q - \theta_\Delta), 3 \times \frac{1}{2}(\Delta - q - \theta_\Delta)\} = 3 \times \frac{1}{2} \times (\Delta - q - \theta_\Delta)$, where $\theta_\Delta = \frac{1}{8}, 0.15$ if $\Delta = 10, 13$ otherwise $\theta_\Delta = 0$. Furthermore, by Corollary 2.8, there are $3 \times \lfloor \frac{\Delta-5}{2} \rfloor$ Δ-vertices u, which send $3 \times \lfloor \frac{\Delta-5}{2} \rfloor \frac{\Delta-q}{\Delta}$ to x by passing through *its* three Δ-neighbors. Hence,

$$C'(x) \geq \begin{cases} (5-8.25) + 3 \times \dfrac{1}{2}\left(\dfrac{7}{4} - \dfrac{1}{8}\right) + 3 \times 2 \times \dfrac{\frac{7}{4}}{10} = \dfrac{17}{40} & \text{if } \Delta = 10. \\ (5-9) + 3 \times \dfrac{1}{2}(11-9) + 3 \times 3 \times \dfrac{11-9}{11} = \dfrac{7}{11} & \text{if } \Delta = 11 \\ \left(5 - \dfrac{126}{13}\right) + 3 \times \dfrac{1}{2}\left(12 - \dfrac{126}{13}\right) + 3 \times 3 \times \dfrac{12 - \frac{126}{13}}{12} = 0.5 & \text{if } \Delta = 12. \\ \left(5 - \dfrac{134}{13}\right) + 3 \times \dfrac{1}{2}\left(\left(13 - \dfrac{134}{13}\right) - 0.15\right) + 3 \times 4 \times \dfrac{13 - \frac{134}{13}}{13} > 0.9 & \text{if } \Delta = 13. \\ \left(5 - \dfrac{142}{13}\right) + 3 \times \dfrac{1}{2}\left(14 - \dfrac{142}{13}\right) + 3 \times 4 \times \dfrac{14 - \frac{142}{13}}{14} > 1.3 & \text{if } \Delta = 14. \end{cases}$$

(R5-2) If x is adjacent to a $(\Delta-1)$-vertex w, to avoid repetition, we consider the worst case, that is, x is adjacent to *two* Δ-vertices and three $(\Delta-1)$-vertices. By Lemma 2.9, two Δ-neighbors send $2 \times \frac{\Delta-q}{2}$ to x. By Lemma 2.9 again, either there are two $(\Delta-1)$ neighbors which send $2 \times (\Delta - q - 1)$ to x, or there is one $(\Delta-1)$-neighbor which sends $(\Delta - 1 - q)$ to x and remaining two $(\Delta-1)$-neighbors send $2 \times \frac{1}{2} \times (\Delta - 1 - q)$ to x. Thus x receives $2 \times \frac{\Delta-q}{2} + \min\{2(\Delta-1-q) + \frac{\Delta-1-q}{3}, (\Delta-1-q) + 2 \times \frac{\Delta-1-q}{2}\}$ totally. So $C'(x) \geq (5-q) + 2 \times \frac{1}{2}(\Delta-q) + 2 \times (\Delta-1-q) = 0, 0, 0.2, 0.7, 1.3$ for $\Delta = 10, 11, 12, 13, 14$, respectively.

(R5-3) If x is adjacent to five Δ-vertices, by Lemma 2.5 and VAL, x receives $\min\{4(\Delta-q) + \frac{1}{4}(\Delta-q), 3(\frac{1}{2})(\Delta-q) + 2(\frac{1}{3})(\Delta-q), 5(\frac{1}{2})(\Delta-q)\} = 3(\frac{1}{2})(\Delta-q) + 2(\frac{1}{3})(\Delta-q) = \frac{13}{6}(\Delta-q)$. Hence $C'(x) \geq (5-q) + \frac{13}{6}(\Delta-q) \geq 0.54, 0.33, 0.3, 0.5, 0.7$ if $\Delta = 10, 11, 12, 13, 14$, respectively.

R6 Let x be a 6-vertex.

(R6-1) First consider that $6 \leq d(x) < q$ with $d(x) + \delta_1(x) \geq \Delta + 3$. Let $y \in N(x)$ with $d(x) + d(y) = \Delta + 3$. By Lemma 2.10 for $p = 1$, x receives $\min\{(d(x)-2)\frac{\Delta-q}{2}, (d(x)-1)\frac{\Delta-q}{2}, (d(x)-1)(\Delta-q)\} = (d(x)-2)\frac{\Delta-q}{2}$ from its neighbors. Thus $C'(x) \geq (6-q) + (6-2)\frac{\Delta-q}{2} > 1, 0.9, 0.8, 1, 1$ for $\Delta = 10, 11, 12, 13, 14$, respectively.

From now on we consider that $d(x) + \delta_1(x) \geq \Delta + 4$.

(R6-2) $\delta_1(x) = \Delta - 2$.

Let w be the $\delta_1(x)$-vertex of $N(x)$. By VAL, x has at least three Δ-neighbors. If x is adjacent to two $(\Delta-2)$-vertices, then by Lemma 2.10 for $p = 2, s = 2$, there are three Δ-neighbors z such that z is adjacent to $\Delta - 1$ vertices with degree $\geq \Delta - 1$. That is, z is adjacent to one small vertex x. x receives $3 \times (\Delta - q)$. $C'(x) = (6-q) + 3(\Delta-q) \geq 2, 2, 3, 3, 4$ for $\Delta = 10, 11, 12, 13, 14$, respectively. If x is adjacent to one $(\Delta-1)$-vertex other than w and four Δ-vertices, by Lemma 2.10 for $p = 2, s = 1$, there are three Δ-neighbors z such that z is adjacent to $\Delta - 2$ vertices with degree $\geq \Delta - 1$. That is, z is adjacent to two small vertices, so each z sends $\frac{\Delta-q}{2}$ to x together with remaining Δ-neighbor sending $\frac{\Delta-q}{5}$ to x and one $(\Delta-1)$-neighbor sending $\frac{\Delta-1}{4}$ to x.

$C'(x) \geq (6-q) + 3 \times \frac{1}{2} \times (\Delta-q) + \frac{1}{5}(\Delta-q) + \frac{1}{4}(\Delta-1-q) > 0.8, 0.5, 0.1, 0.6, 0.7$ for $\Delta = 10, 11, 12, 13, 14$, respectively.

If x is adjacent to two $(\Delta-1)$-vertices other than w and three Δ-vertices, similar to previous discussion, we have that three Δ-neighbors send $3 \times \frac{1}{2} \times (\Delta-q)$ to x and two $(\Delta-1)$-neighbors send $2 \times \frac{1}{4}(\Delta-1-q)$ to x. So $C'(x) \geq (6-q) + 3 \times \frac{1}{2}(\Delta-q) + \frac{1}{2} \times (\Delta-1-q) > 0.6, 0.4, 0.3, 0.5, 0.7$ if $\Delta = 10, 11, 12, 13, 14$, respectively.

(R6-3) $\delta_1(x) = \Delta - 1$.

In order to avoid repetition, we consider that x has at least two $(\Delta-1)$-neighbors.

If each $(\Delta-1)$-neighbor x is adjacent to all $(\geq \Delta-2)$-vertices other than x, then x receives at least $2 \times \frac{1}{3}(\Delta-q) + 4(\Delta-1-q-3\theta_\Delta)$ (note that $\theta_\Delta = \frac{1}{8}, 0.15$ if $\Delta = 10, 13$ respectively, and $\theta_\Delta = 0$ otherwise). If there is one $(\Delta-1)$-neighbor which is adjacent to a $(\leq \Delta-3)$-vertex other than x and two $(\Delta-2)$-vertices, then x receives $3 \times \frac{1}{3}(\Delta-q)$ and $i \times \frac{1}{2}(\Delta-1-q-2\theta_\Delta) + j(\Delta-1-q-3\theta)$ where $i+j = 3$. If x has a $(\Delta-1)$-neighbor which is adjacent to two $(\leq\Delta-3)$-vertices other than x and one $(\geq\Delta-2)$-vertex, then x receives at least $4 \times \frac{1}{2} \times (\Delta-q) + i \times \frac{1}{3}(\Delta-1-q-\theta_\Delta + j(\Delta-1-q-3\theta_\Delta))$ where $i+j = 2$. If x has a $(\Delta-1)$-neighbor which is adjacent to three $(\leq\Delta-3)$-vertices other than x, then x receives at least $5 \times (\Delta-q)$. For the sake of convenience, let K be the smallest charge that x receives from its neighbors. By straightforward calculation, $K = 3 \times \frac{1}{3}(\Delta-q) + 3 \times \frac{1}{2}(\Delta-1-q-2\theta_\Delta)$. $C'(x) \geq (6-q) + K > 0.1, 1.4, 0.4, 0.4, 1$ for $\Delta = 10, 11, 12, 13, 14$, respectively.

(R6-4) $\delta_1(x) = \Delta$.

Let w be the Δ-neighbor of x with $\max\{t : t = d_{\leq\Delta-2}(y) : y \in N(x)\}$ which denoted by k. Note that $k \leq 4$. If each Δ-neighbor of x is adjacent to at most three $(\leq\Delta-3)$-vertices, then it is straightforward to check that $C'(x) \geq (6-q) + 6 \times$

$\frac{1}{3}(\Delta - q) > 1, 1, 0.9, 1, 1$ for $\Delta = 10, 11, 12, 13, 14$ respectively. Now we assume that one Δ-neighbor, say w, is adjacent to $i(i = 4, 5)$ $(\leq \Delta - 3)$-vertices. By Lemma 2.3, where $|C_2| = 4$ and $|R^1 \cup R^2 \cup R^*| = 3$ or 4, there are $3(=1 + \lfloor \frac{|C_2|}{2} \rfloor)$ Δ-neighbors $x^b \in N(x)$ and each x^b has $2\Delta - 6 - \Delta + 1 + 2 (= \Delta - 3)$ vertices $y \in N(x^b)$ such that $d(y) \geq \Delta - 2$ $(= 2\Delta - d - \Delta + 2 + \lfloor \frac{|C_2|}{2} \rfloor)$ which means that x^b is adjacent to at most three small vertices. Thus x receives $3 \times \frac{1}{3} \times (\Delta - q)$ from those three Δ-neighbors and $3 \times \frac{1}{5}(\Delta - q)$ from rest three Δ-neighbors. So $C'(x) \geq (6 - q) + 3 \times \frac{1}{3}(\Delta - q) + 3 \times \frac{1}{5}(\Delta - q) \geq 0.15, 0.2, 0, 0, 0$ for $\Delta = 10, 11, 12, 13, 14$, respectively.

R7 Let x be a 7-vertex.

(R7-1) From **(R6-1)**, we first consider $d(x) + \delta_1(x) = \Delta + 4$, and let w be the $\delta_1(x)$-vertex.

Subcase 1. $|(N(x) \cap V_{\leq \Delta - s}) \setminus \{w\}| > 1$ where $0 < s \leq 2$. Applying Lemma 2.10 for $p = 2$, $0 < s \leq p$, there are $(d(x) - 2 - 1)$ Δ-neighbors in $N(x) \setminus \{w\}$, and each of which is adjacent to all $(\geq \Delta - p + 1)(\geq q)$-vertices. Thus, x receives at least $\frac{(d-3)}{2}(\Delta - q)$ from those Δ-neighbors. It is straightforward to check that $C'(x) \geq (d(x) - q) + (d(x) - 3)\frac{1}{2}(\Delta - q) \geq 7 - q + (7 - 3)\frac{1}{2}(\Delta - q) \geq 2, 1, 1.5, 2, 2$ for $\Delta = 10, 11, 12, 13, 14$, respectively.

Subcase 2. $|(N(x) \cap V_{\leq \Delta - 1}) \setminus \{w\}| = 1$.

Note that $d(x) + \delta_1(x) = \Delta + 4$. That is, there are *four* Δ-neighbors having at most two $(\leq \Delta - 2)$-neighbors other than x. Thus x receives $4 \times \frac{1}{3}(\Delta - q)$ from those Δ-vertices and receives $2 \times \frac{1}{6}(\Delta - q)$ from the rest two Δ-neighbors. Hence $C'(x) \geq 7 - q + 4\frac{\Delta - q}{3} + 2\frac{\Delta - q}{6} \geq 7 - q + \frac{5}{3}(\Delta - q) \geq 1, 1, 1, 1, 1$ for $\Delta = 10, 11, 12, 13, 14$, respectively.

From now on, we consider cases of $d(x) + \delta_1(x) \geq \Delta + 5$, and $d(x) \geq 7$.

(R7-2) $\delta_1(x) = \Delta - 3, \Delta - 2$.

Let w be the $\delta_1(x)$-neighbor of x. We discuss three cases below.

Subcase 1. $|(N(x) \cap V_{\Delta - 2}) \setminus \{w\}| \geq 1$. By Lemma 2.10 where $p = 3$, $s = 2$, there are 3 Δ-neighbors z of x satisfying: $z \neq w$, which are adjacent to $\Delta - 2(\geq \Delta - 1)$-vertices. So x receives $3 \times \frac{\Delta - q}{2}$ from those three Δ-neighbors, thus we have $C'(x) \geq (7 - q) + 3 \times \frac{\Delta - q}{2} \geq 1, 1, 0.7, 0.7, 0.6$ for $\Delta = 10, 11, 12, 13, 14$ respectively.

Subcase 2. x has no $(\leq \Delta - 2)$-neighbor other than w, and x has at least two $(\Delta - 1)$-neighbors. By Lemma 2.10 for $p = 2$ (if $\delta_1(x) = \Delta - 3$), there are three Δ-neighbors z satisfying: $z \neq w$, which are adjacent to $(\Delta - 2)$ $(\geq \Delta - 2)$-vertices. Thus x receives at least the same amount charges from its neighbors as the previous case, so $C'(x) \geq 0$ for $10 \leq \Delta \leq 14$. For the case of $p = 3$ (if $\delta_1(x) = \Delta - 2$), similarly, x receives at least the same amount of charges as that in subcase 1 **(R7-2)**, we have $C'(x) \geq 0$.

Subcase 3. x has no $(\leq \Delta - 2)$-neighbor other than w and x has one $(\Delta - 1)$-neighbor, for example y. Using the same discharge argument as in Subcase 1 above, there are three Δ-neighbors having at most 3 $(\leq \Delta - 2)$-neighbors. So x receives $3 \times \frac{\Delta - q}{3}$ from those three Δ-neighbors, and by VAL, x also receives $2 \times \frac{\Delta - q}{6}$ from the rest two Δ-neighbors and $\frac{1}{5}(\Delta - 1 - q)$ from one $(\Delta - 1)$-neighbor. So $C'(x) \geq (7 - q) + 3 \times \frac{\Delta - q}{3} + 2 \times \frac{\Delta - q}{6} + \frac{\Delta - 1 - q}{5} = \frac{23}{15}(\Delta - q) + 7 - q - 0.2 \geq 1, 0.7, 0.5, 0.5, 0.5$ for $\Delta = 10, 11, 12, 13, 14$, respectively.

(R7-3) $\delta_1(x) = \Delta - 1$.

In order to avoid repetition, we consider the worst case, that is, x has two Δ-neighbors and five $(\Delta - 1)$-neighbors. If each $(\Delta - 1)$-neighbor of x is adjacent to at most *three* $(\leq \Delta - 2)$ vertices, then $C'(x) \geq (7 - q) + 5 \times \frac{1}{3}(\Delta - 1 - q) + 2 \times \frac{1}{6}(\Delta - q) = 2(\Delta - q) + 7 - q - \frac{5}{3} > 0.5, 0.3, 0.2, 0.3, 0.5$ for $\Delta = 10, 11, 12, 13, 14$, respectively. If there is one $(\Delta - 1)$-neighbor w of x is adjacent to at least *four* $(\leq \Delta - 3)$ vertices, through Lemma 2.3 where $|C_2| = 4$, there are $2 + \lfloor \frac{|C_2|}{2} \rfloor (=4)$ $(\geq \Delta - 1)$ neighbors of x, and each of which is adjacent to $\Delta - 7 + 4$ $(= \Delta - 3)$ vertices of degree $\geq \Delta - 7 + 5$ $(= \Delta - 2)$. So x receives $\min_{i=0,1,2}\{(4 - i)\frac{1}{3}(\Delta - 1 - q) + (2 + i)\frac{1}{6}(\Delta - q) + \frac{1}{5}(\Delta - 1 - q)\}$, so $C'(x) \geq 7 - q + 4\frac{1}{3}(\Delta - 1 - q) + 2\frac{1}{6}(\Delta - q) + \frac{1}{5}(\Delta - 1 - q) = \frac{28}{15}(\Delta - q) + 7 - q - \frac{23}{15} \geq 0.4, 0.2, 0.08, 0.18, 0.28$ for $\Delta = 10, 11, 12, 13, 14$, respectively.

(R7-4) $\delta_1(x) = \Delta$.

By VAL, we have $C'(x) \geq 7 - q + 7\frac{1}{6}(\Delta - q) \geq \frac{19}{24}, \frac{1}{3}, 0$ for $\Delta = 10, 11, 12$, respectively.

For the cases of $\Delta = 13$ and 14, it needs to be considered more sophisticated. Let w be Δ-neighbor of x with $k = \max\{t : t = d_{\leq q}(y), y \in N(x)\}$ where $k \leq 5$. If each Δ-neighbor of x is adjacent to at most *five* $(\leq \Delta - 3)$-vertices, then x receives $7 \times \frac{1}{5}(\Delta - q)$, so $C'(x) \geq 7 - q + \frac{7}{5}(\Delta - q) \geq 0$ for $\Delta = 13, 14$. Now we assume that one Δ-neighbor of x, say w, is adjacent to 6 $(\leq \Delta - 3)$-vertices including x. Note that $|C_2| = 5$. Through Lemma 2.3, where $|R^1 \cup R^2 \cup R^*| = 5$ or 4, there are 3 $(1 + \lfloor \frac{|C_2|}{2} \rfloor)$ Δ-neighbor x^b of x such that x^b has at least $\Delta - 4 (= 2\Delta - 7 - \Delta + 1 + 2)$ neighbors $y \in N(N(x))$ with $d(y) \geq \Delta - 3$. Hence, $C'(x) \geq 7 - q + 3 \times \frac{1}{4}(\Delta - q - \theta_\Delta) + 4 \times \frac{1}{6}(\Delta - q) \geq 0.39, 0.43$ for $\Delta = 13, 14$ respectively.

R8 Let x be a 8-vertex.

In the case of $\Delta = 10$, each vertex has at least two Δ-neighbors, and by **(R0)**, each Δ-neighbor sends $\frac{1}{8}$ to x. So $C'(x) \geq (8 - \frac{33}{4}) + 2 \times \frac{1}{8} = 0$.

Next we consider cases of $\Delta = 11, 12, 13, 14$. Note that the arguments in previous cases discussion in **R7** could be used for the case of $\delta_1(x) = \Delta - 3$ here. To avoid repetition, we consider $\delta_1(x) \geq \Delta - 2$.

(R8-1) $\delta_1(x) = \Delta - 2$.

Note that $d_\Delta(x) \geq 3$. Without loss of generality, let $d_\Delta(x) = 3$ and $N(x) \cap V_{\Delta-2} = 5$. By Lemma 2.10 for $p = 4$, there are three Δ-neighbors such that each of which is adjacent to at most five $(\leq \Delta - 3)$-vertices and each of five $(\Delta-2)$-neighbors is adjacent to at most 5 $(\leq \Delta - 3)$-vertices, so $C'(x) \geq 8 - q + 3\frac{1}{5}(\Delta - q) + 5 \times \frac{1}{5}(\Delta - 2 - q) \geq 0.5, 0.2, 0, 0, 0$ for $\Delta = 11, 12, 13, 14$, respectively.

(R8-2) $\delta_1(x) = \Delta - 1$.

Without loss of generality, let $d_\Delta(x) = 2$, $|N(x) \cap V_{\Delta-1}| = 6$.

If each of $(\Delta - 1)$-neighbors is adjacent to at most five $(\leq \Delta - 3)$-vertices, then x receives $6\frac{1}{5}(\Delta - 1 - q) + 2\frac{1}{7}(\Delta - q)$. Thus $C'(x) \geq 8 - q + (\frac{2}{7} + \frac{6}{5})(\Delta - q) - 1.2 > 1, 0.7, 0.5, 0.4, 0.4$ for $\Delta = 11, 12, 13, 14$, respectively.

If there exists a $(\Delta - 1)$-neighbor of w which is adjacent to six $(\leq \Delta - 3)$-vertices, note that $d(x) = 8$, $d(w) = \Delta - 1$ and $|C_2| = 5$, then there exist at least one $(\leq \Delta - 3)$-neighbor w_j of w with $j \in C_{12}$. Applying Lemma 2.2, there are at least 4 $(\geq \Delta - 1)$-neighbors x^α of x which are adjacent to at most 2 $(\leq \Delta - 3)$-vertices, so x receives $4\frac{1}{2}(\Delta - 1 - q)$ from 4 $(\geq \Delta - 1)$-neighbors, $2\frac{1}{7}(\Delta - q)$ from two Δ-neighbors, and $2\frac{1}{6}(\Delta - 1 - q)$ from the rest three $(\Delta - 1)$-neighbors. It is straightforward to check that $C'(x) \geq (8 - q) + (4 \times \frac{1}{2}(\Delta - 1 - q) + \frac{2}{7})(\Delta - q) + 2 \times \frac{1}{6}(\Delta - 1 - q) > 0$ for $11 \leq \Delta \leq 14$.

(R8-3) $\delta_1(x) = \Delta$.

By VAL, eight Δ-neighbors send $8\frac{1}{7}(\Delta - q)$ to x, it is straightforward to check that $C'(x) \geq 0$ for $11 \leq \Delta \leq 14$.

R9 Let x be a 9-vertex.

(R9-1) For $\Delta = 10, 11$, we perform the discharge rules from **(R1)**–**(R8)**, x sends at most $(9 - q)$ out, so $C'(x) \geq 0$. Now we consider the cases of $\Delta = 12, 13, 14$.

(R9-2) If $\delta_1(x) \leq \Delta - 3$, x has at least 4 Δ-neighbors. By VAL, x receives st least $4\frac{1}{8}(\Delta - q)$ from its 4 Δ-neighbors, thus $C'(x) \geq 9 - q + 4\frac{1}{8}(\Delta - q) \geq 0.46, 0.03$, for $\Delta = 12, 13$.

For $\Delta = 14$, more sophisticated discussion is needed. If $\delta_1(x) \leq \Delta - 4$, then x has at least 5 Δ-neighbors, and by VAL x receives at least $5\frac{1}{8}(\Delta - q)$ from those Δ-neighbors, thus $C'(x) \geq 9 - \frac{142}{13} + 5 \times \frac{1}{8}(14 - \frac{142}{13}) = 0$ for $\Delta = 14$. Now consider the case of $\delta_1(x) = \Delta - 3$. x has at least 4 Δ-neighbors. To avoid repetition, we provide detail discussion on the worst case, that is, x has *four* Δ-neighbors and *five* $(\Delta - 3)$-neighbors. Firstly, if there is a $(\Delta - 3)$-vertex denoted by w, which has at least 3 $(\leq \Delta - 4)$-neighbors w_j, so each of these neighbors either misses at least one x, w-color with $j \leq 8$ or misses 4 trouble colors with $j \geq 9$. Note that there are 4 x-colors, and by Lemma 2.3, there exist $7(=4+3)$ neighbors x^c of x including at least 2 Δ-neighbors of x, such that each of x^c is adjacent to at least $\Delta - 3(=4+6+1 = 11)$ vertices y and $d(y) \geq \Delta - 1(=4+6+3 = 13)$, thus, x could receive $\frac{1}{2}(\Delta - q)$ from those two Δ-neighbors, so $C'(x) \geq (9 - \frac{142}{13}) + 2 \times \frac{1}{3}(14 - \frac{142}{13}) + 2 \times \frac{1}{8}(14 - \frac{142}{13}) + 5 \times \frac{1}{3}(14 - 3 - \frac{142}{13}) > 0$.

Secondly, we assume that each of 5 $(\Delta - 3)$-neighbors is adjacent to at most 2 $(\leq \Delta - 4)$-vertices including x. If each of four Δ-neighbors have at most 7 $(\Delta - 4)$-vertices, then $C'(x) \geq (9 - \frac{142}{13}) + 4 \times \frac{14 - \frac{142}{13}}{7} + 5 \times \frac{11 - \frac{142}{13}}{2} > 0$. If there is a Δ-neighbor of x, say w, which is adjacent to 8 $(\leq \Delta - 4)$-vertices, note that here $L = 7$, there exist at least one w_j $(j \geq 9)$ missing all trouble colors. By Lemma 2.3, there are at least $5(=\Delta - \Delta + 1 + \lfloor \frac{8}{2} \rfloor)$ vertices x^c of x such that x^c has $\Delta - 3(=2\Delta - 9 - \Delta + 1 + \lfloor \frac{8}{2} \rfloor)$ neighbors y, and $d(y) \geq \Delta - 1(=2\Delta - 9 - \Delta + 2 + \lfloor \frac{8}{2} \rfloor)$. This information implies that if some of such *five* vertices x^c is $(\Delta - 3)$-neighbor of x, then such x^c incidents only one $(\leq \Delta - 4)$-vertex which is x, or if some of such *five* vertices is Δ-vertex, say x^c, then x^c has at most 3 $(\leq \Delta - 4)$-vertices including x. Hence x receives at least $\min_{0 \leq i \leq 4, 0 \leq j \leq 5}\{i \times \frac{1}{3}(14 - \frac{142}{13}) + (4 - i) \times \frac{1}{8}(14 - \frac{142}{13}) + j \times (11 - \frac{142}{13}) + (5 - j) \times \frac{1}{2}(11 - \frac{142}{13})\}$. It is straightforward to check that $C'(x) \geq 0$.

(R9-3) If $\delta_1(x) \geq \Delta - 2$, then x has at least 3 Δ-neighbors. By VAL, x receives at least $3\frac{1}{8}(\Delta - q)$ from its 3 Δ-neighbors and at least $6\frac{1}{6}(\Delta - 2 - q)$ from six of its $(\geq \Delta - 2)$-neighbors. Thus $C'(x) \geq 9 - q + 3\frac{1}{8}(\Delta - q) + 6 \times \frac{1}{6}(\Delta - 2 - q) \geq 0.4, 0.3, 0.2$ for $\Delta = 12, 13, 14$, respectively.

If $\delta_1(x) = \Delta - 1$, or Δ, without loss of generality, we assume that x has two Δ-neighbors and seven $(\Delta - 1)$-neighbors, or simply nine Δ-neighbors. By VAL, x receives at least $\min\{2\frac{1}{8}(\Delta - q) + 7\frac{1}{7}(\Delta - 1 - q), 9\frac{1}{8}(\Delta - q)\}$. $C'(x) \geq 9 - q + 2 \times \frac{1}{8}(\Delta - q) + 7\frac{1}{7}(\Delta - 1 - q) \geq 1, 1, 0.8$ for $\Delta = 12, 13, 14$, respectively.

(R10) x is a 10-vertex.

Note that charge of x keeps unchanged when $\Delta = 12$. So we consider the case of $\Delta = 13, 14$. If $\Delta = 13$, x receives 0.15 from its at least two Δ-neighbors, so $C'(x) \geq 0$. Next we consider the case of $\Delta = 14$. If $\delta_1(x) \leq \Delta - 2$, then x has at least three Δ-neighbors by VAL, so $C'(x) \geq 10 - \frac{142}{13} + 3\frac{1}{9}(14 - \frac{142}{13}) > 1$. If $\delta_1(x) = \Delta - 1$ or Δ, so x receives at least $2\frac{1}{9}(14 - \frac{142}{13}) + 8 \times \frac{1}{8}(14 - 1 - \frac{142}{13}) \geq 1$. $C'(x) \geq 0$.

Final step. x is an i-vertex where $i = 11, 12, 13, 14$, by the discharge rules **(R1)**–**(R9)**, it is clear that x sends at most $(d(x) - q)$ out. Hence $C'(x) \geq 0$.

From **(R1)**–**(R9)**, $C'(x) \geq 0$ for each vertex x, and therefore, $\sum_{x \in V(G)} C'(x) \geq 0$. Since the discharge rules only move charge around and do not change the sum, we have $0 \leq \sum_{x \in V(G)} C'(x) = \sum_{x \in V(G)} c(x) < 0$. This contradiction completes the proof.

4. Class one graphs with $c_S = -4, -5, -6, -7, -8$

Theorem 4.1. *Let G be a simple graph that is embeddable in a surface S of characteristic* $c_S = -4, -5, -6, -7, -8$*, then G is class one if* $\Delta \geq 10, 11, 12, 13, 14$ *respectively.*

Before we proceed our proof of the Theorem, we need following results on critical graphs with small orders.

Lemma 4.2 (*Beineke and Fiorini [1], Brinkmann and Steffen [2,4,3]*)**.**

(i) *There are no critical graphs of even order up to* 14*;*
(ii) *there are only two critical graphs of order* 11*, both of which are* 3*-critical;*
(iii) *Petersen graph minus a vertex is the only non-trivial critical graph on up to* 10 *vertices, which is* 3*-critical;*
(iv) *There are only three critical graphs of order* 13*, which are* 3*-critical.*

Proof of Theorem 4.1. By Theorem 1.2 and Lemma 4.2, we only need to prove it when $\Delta = 10, 11, 12, 13, 14$ respectively. Let V and F be vertex set and face set of G respectively. Suppose to the contrary, let G be the smallest counterexample with respect to edges. Then G is Δ-critical where $\Delta = 10, 11, 12, 13, 14$, respectively. By Euler's Formula, we have

$$\begin{cases} \sum_{x\in V}(d(x)-6)+\sum_{f\in F}(d(f)-3)=24 & \text{if } c_S=-4,\ \Delta=10.\\ \sum_{x\in V}(d(x)-6)+\sum_{f\in F}(d(f)-3)=30 & \text{if } c_S=-5,\ \Delta=11.\\ \sum_{x\in V}(d(x)-6)+\sum_{f\in F}(d(f)-3)=36 & \text{if } c_S=-6,\ \Delta=12.\\ \sum_{x\in V}(d(x)-6)+\sum_{f\in F}(d(f)-3)=42 & \text{if } c_S=-7,\ \Delta=13.\\ \sum_{x\in V}(d(x)-6)+\sum_{f\in F}(d(f)-3)=48 & \text{if } c_S=-8,\ \Delta=14.\end{cases}$$

By Theorem 1.2, we have

$$\begin{cases} 2.25\times|V|\leq 24 & \text{if } c_S=-4,\ \Delta=10.\\ 3\times|V|\leq 30 & \text{if } c_S=-5,\ \Delta=11.\\ \frac{48}{13}\times|V|\leq 36 & \text{if } c_S=-6,\ \Delta=12.\\ \frac{56}{13}\times|V|\leq 42 & \text{if } c_S=-7,\ \Delta=13.\\ \frac{64}{13}\times|V|\leq 48 & \text{if } c_S=-8,\ \Delta=14.\end{cases}$$

Hence, $|V| \leq 10.67$ or $|V| \leq 10$ for $\Delta = 10$ or 11 respectively. And $|V| \leq 9.75$ for $\Delta = 12, 13, 14$. By Lemma 4.2, we have contradictions.

References

[1] L.W. Beineke, S. Fiorini, On small graphs critical with respect to edge-colourings, Discrete Math. 16 (1976) 109–121.
[2] D. Bokal, G. Brinkmann, S. Grünewald, Chromatic-index-critical graphs of orders 13 and 14, Discrete Math. 300 (2005) 16–29.
[3] G. Brinkmann, E. Steffen, 3- and 4-critical graphs of small even order, Discrete Math. 169 (1997) 193–197.
[4] G. Brinkmann, E. Steffen, Chromatic-index-critical of orders 11 and 12, European J. Combin. 19 (1998) 889–900.
[5] X. Li, Lower bounds on edge critical graphs with maximum degree of 7, 8 and 9, Adv. Appl. Discrete Math. 5 (2) (2010) 97–113. India.
[6] X. Li, A new lower bound on critical graphs with maximum degree of 8 and 9, Ars Combin. 98 (2011) 241–257.
[7] S. Li, X. Li, Edge coloring of graphs with small maximum degrees, Discrete Math. 309 (2009) 4843–4852.
[8] X. Li, B. Wei, An improved lower bond on edge critical graphs with maximum degree of 8 and 9, Adv. Appl. Discrete Math. 12 (1) (2013) 1–38. India.
[9] R. Luo, L. Miao, Y. Zhao, The size of edge chromatic critical graphs with maximum degree 6, J. Graph Theory 60 (2009) 149–171.
[10] L.-Y. Miao, W.-Y. Song, S.-Y. Pang, Z.-K. Miao, On the size of critical graphs with small maximum degree, Int. J. Comput. Math. (2014) http://dx.doi.org/10.1080/00207160.2014.883068. March.
[11] D. Sanders, Y. Zhao, Planar graphs of maximum degree seven are class I, J. Combin. Theory Ser. B 83 (2) (2001) 201–212.
[12] D. Sanders, Y. Zhao, Coloring edges of graphs embedded in a surface of characteristic zero, J. Combin. Theory Ser. B 87 (2003) 254–263.
[13] V.G. Vizing, On an estimate of the chromatic class of a p-graph, Metody Diskret. Anal. 3 (1964) 25–30.
[14] V.G. Vizing, Some unsolved problems in graph theory, Uspekhi Mat. Nauk 23 (1968) 117–134. (in Russian); English translation in Russian Math. Surveys, 23 (1968) 125–141.
[15] D. Woodall, The average degree of an edge-chromatic critical graph. II, J. Graph Theory 56 (3) (2007) 194–218.
[16] D. Woodall, The average degree of an edge-chromatic critical graph, Discrete Math. 308 (5–6) (2008) 803–819.
[17] L. Zhang, Every planar graph with maximum degree 7 is of class 1, Graphs Combin. 16 (4) (2000) 467–495.