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Neighbor sum distinguishing index of planar graphs

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ABSTRACT

A proper $[k]$-edge coloring of a graph G is a proper edge coloring of G using colors from $[k] = \{1, 2, \ldots, k\}$. A neighbor sum distinguishing $[k]$-edge coloring of G is a proper $[k]$-edge coloring of G such that for each edge $uv \in E(G)$, the sum of colors taken on the edges incident to u is different from the sum of colors taken on the edges incident to v. By $\text{nsdi}(G)$, we denote the smallest value k in such a coloring of G. It was conjectured by Flandrin et al. that if G is a connected graph without isolated edges and $G \neq C_5$, then $\text{nsdi}(G) \leq \Delta(G) + 2$. In this paper, we show that if G is a planar graph without isolated edges, then $\text{nsdi}(G) \leq \max\{\Delta(G) + 10, 25\}$, which improves the previous bound ($\max\{2\Delta(G) + 1, 25\}$) due to Dong and Wang.

1. Introduction

The terminology and notation used but undefined in this paper can be found in [3]. Let $G = (V, E)$ be a simple, undirected graph. Let C be a set of colors where $C = [k] = \{1, 2, \ldots, k\}$ and let $\phi : E(G) \to C$ be a proper $[k]$-edge coloring of G. By $m_\phi(v)$ ($C_\phi(v)$), we denote the sum (set) of colors taken on the edges incident to v, i.e. $m_\phi(v) = \sum_{u \in N(v)} \phi(uv)$ ($C_\phi(v) = \{\phi(uv) \mid u \in N(v)\}$). If the coloring ϕ satisfies that $C_\phi(u) \neq C_\phi(v)$ for each edge $uv \in E(G)$, then we call such coloring a *neighbor distinguishing* $[k]$*-edge coloring* of G. We use $\text{ndi}(G)$ to denote the smallest value k such that G has a neighbor distinguishing $[k]$-edge coloring of G and we call it the *neighbor distinguishing index* of G. Sometimes, a neighbor distinguishing edge coloring is named an *adjacent vertex distinguishing edge coloring* [18,19]. If the coloring ϕ satisfies that $m_\phi(v) \neq m_\phi(u)$ for each edge $uv \in E(G)$, then we call such coloring a *neighbor sum distinguishing* $[k]$*-edge coloring* of G. By $\text{nsdi}(G)$, we denote the smallest value k such that G has a neighbor sum distinguishing $[k]$-edge coloring of G and we call it the *neighbor sum distinguishing index* of G.

It is known that to have a neighbor distinguishing or a neighbor sum distinguishing coloring, G cannot have an isolated edge (we call such graphs normal). If a normal graph G has connected components $G_1, \ldots, G_k$, then $\text{ndi}(G) = \max\{\text{ndi}(G_i) \mid i = 1, \ldots, k\}$ and $\text{nsdi}(G) = \max\{\text{nsdi}(G_i) \mid i = 1, \ldots, k\}$. Therefore, when analyzing the neighbor distinguishing index or the neighbor sum distinguishing index, we can restrict our attention to connected normal graphs. Apparently, for any normal graph G, $\Delta(G) \leq \chi'(G) \leq \text{ndi}(G) \leq \text{nsdi}(G)$, where $\chi'(G)$ is the chromatic index of G.

For neighbor distinguishing colorings, we have the following conjecture due to Zhang et al. [23].

Conjecture 1 (*[23]*). *If G is a connected normal graph with at least* 6 *vertices, then* $\text{ndi}(G) \leq \Delta(G) + 2$.

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Akbari et al. [1] proved that $\mathrm{ndi}(G) \leq 3\Delta(G)$ for any normal graph G. Hatami [10] has shown that if G is normal and $\Delta(G) > 10^{20}$, then $\mathrm{ndi}(G) \leq \Delta(G) + 300$. For more references, see [2,4,7,18,19,11].

Recently, colorings and labelings related to sums of the colors have received much attention. The family of such problems includes e.g. vertex-coloring $[k]$-edge-weightings [13], total weight choosability [21,17], magic and antimagic labelings [12,22] and the irregularity strength [14,15]. As for neighbor sum distinguishing edge colorings, Flandrin et al. [8] completely determined the neighbor sum distinguishing indices for paths, cycles, trees, complete graphs and complete bipartite graphs. Based on these examples, they proposed the following conjecture.

Conjecture 2 *([8]). If G is a connected normal graph and $G \neq C_5$, then* $\mathrm{nsdi}(G) \leq \Delta(G) + 2$.

In the same paper, Flandrin et al. [8] gave an upper bound: $\lceil \frac{7\Delta(G)-4}{2} \rceil$. In [20], Wang and Yan improved it to $\lceil \frac{10\Delta(G)+2}{3} \rceil$. In [16], Przybyło proved that $\mathrm{nsdi}(G) \leq 2\Delta(G) + \mathrm{col}(G) - 1$, where $\mathrm{col}(G)$ is the coloring number of G. Dong et al. [6] studied neighbor sum distinguishing colorings of sparse graphs and proved that if G is a normal graph with maximum average degree at most $\frac{5}{2}$ and $\Delta(G) \geq 5$, then $\mathrm{nsdi}(G) \leq \Delta(G) + 1$. Dong and Wang [5] also considered the neighbor sum distinguishing colorings of planar graphs and proved the following result.

Theorem 1.1 *([5]). If G is a connected normal planar graph, then* $\mathrm{nsdi}(G) \leq \max\{2\Delta(G) + 1, 25\}$.

In this paper, we improve the result above and obtain the following result.

Theorem 1.2. *If G is a connected normal planar graph, then* $\mathrm{nsdi}(G) \leq \max\{\Delta(G) + 10, 25\}$.

2. Preliminaries

First we will introduce some notations. Let G be a graph. For a vertex $v \in V(G)$, let $N(v)$ denote the set of vertices adjacent to v and $d(v) = |N(v)|$ denote the degree of v. A vertex of degree k is called *k-vertex*. We write *k^+-vertex* for a vertex of degree at least k, and *k^--vertex* for that of degree at most k. Let $N_{k^-}(v) = \{x \in N(v) \mid d(x) \leq k\}$ and $n_{k^-}(v) = |N_{k^-}(v)|$. Similarly, $N_{k^+}(v) = \{x \in N(v) \mid d(x) \geq k\}$ and $n_{k^+}(v) = |N_{k^+}(v)|$.

Next we introduce a structural lemma about planar graphs, which was used in [9].

Lemma 2.1 *([9]). Let G be a planar graph. Then there exists a vertex v in G with exactly $d(v) = t$ neighbors $v_1, v_2, \ldots, v_t$ where $d(v_1) \leq d(v_2) \leq \cdots \leq d(v_t)$ such that at least one of the following is true:*

(A) $t \leq 2$,
(B) $t = 3$ *and* $d(v_1) \leq 11$,
(C) $t = 4$ *and* $d(v_1) \leq 7,\ d(v_2) \leq 9$,
(D) $t = 5$ *and* $d(v_1) \leq 6,\ d(v_2) \leq 7$.

Finally, we give a simple lemma, which will also be used in our proof.

Lemma 2.2 *([8]). Let z be an integer. For any two sets of integers X, Y, each of size at least 2, there exist (at least) $|X| + |Y| - 3$ pairs $(x_i, y_i) \in X \times Y$ with $x_i \neq y_i,\ i = 1, 2, \ldots, |X| + |Y| - 3$, such that all the sums $x_i + y_i$ are pairwise distinct and among them there are at most two pairs satisfying $x_i - y_i = z$.*

This lemma clearly holds. Indeed, it is sufficient to consider e.g. the pairs from the set

$$(\{x\} \times (Y \setminus \{x\})) \cup ((X \setminus (\{x\} \cup \{y\})) \times \{y\}),$$

where $x = \min X$ and $y = \max Y$.

3. Proof of Theorem 1.2

We prove the theorem by contradiction. Suppose that G is a minimal counterexample with respect to the number of edges. For simplicity, let $\Delta = \Delta(G)$ and $k = \max\{\Delta(G) + 10, 25\}$. Then $k \geq 25$. In the following, we will often delete two adjacent edges, say vv_1, vv_2 to get a subgraph H of G. If H has an isolated edge $e = wp$, then there must be an edge wp in G such that $d_G(w) = 3,\ d_G(p) = 1$ or $d_G(w) = d_G(p) = 2$ or $d_G(w) = 2,\ d_G(p) = 1$. Then $G - wp$ has a neighbor sum distinguishing $[k]$-edge coloring ϕ by the minimality of G. We can easily extend ϕ to the graph G, which is a contradiction. So in the following, we assume that the subgraph H obtained by deleting two adjacent edges from G has no isolated edges.

Claim 3.1. *Let $v \in V(G)$ and v_1, v_2 be the neighbors of v in G. If $d(v_1) \leq \frac{k+1-d(v)}{2}$ and $d(v_2) \leq \frac{k+1-d(v)}{2}$, then $d(v) \geq \frac{2k-2d(v_1)-2d(v_2)+5}{3}$.*

Proof. Let $H_1 = G - vv_1 - vv_2$. By the minimality of G, H_1 has a neighbor sum distinguishing $[k]$-edge coloring ϕ.

First suppose that v_1 is not adjacent to v_2. For vv_1, we surely cannot use the colors of its (already colored) at most $d(v_1) - 1 + d(v) - 2$ incident edges. Next, the colors in $\{m_\phi(v_2) - m_\phi(v)\} \cup \{m_\phi(u) - m_\phi(v_1) \mid uv_1 \in E(H_1)\}$ are also forbidden. Then we have at least $k - 2(d(v_1) - 1) - (d(v) - 2) - 1 \geq k - 2d(v_1) - d(v) + 3 \geq 2$ safe colors for vv_1. Similarly, we have at least $k-2d(v_2)-d(v)+3 \geq 2$ safe colors for vv_2. Let X, Y denote the sets of safe colors for vv_1 and vv_2 respectively. By Lemma 2.2, we have at least

$$k-2d(v_1)-d(v)+3+k-2d(v_2)-d(v)+3-3 = 2k-2d(v)-2d(v_1)-2d(v_2)+3$$

distinct pairs (x_i, y_i) with $x_i \neq y_i$ in $X \times Y$ such that all the sums $x_i + y_i$ are pairwise distinct. So we must have that $2k-2d(v)-d(v_1)-d(v_2)+3 \leq d(v)-2$, since otherwise we can choose a pair, say $(x, y) \in X \times Y$ with $x \neq y$, such that $x+y$ is not in $\{m_\phi(u)-m_\phi(v) \mid uv \in E(H_1)\}$, and thus we can get a neighbor sum $[k]$-edge coloring of G, which is a contradiction. Therefore $d(v) \geq \frac{2k-2d(v_1)-2d(v_2)+5}{3}$.

Next we assume that v_1 is adjacent to v_2. For vv_1, we cannot use the colors of its (already colored) at most $d(v_1)-1+d(v)-2$ incident edges. Next, the colors in $\{m_\phi(v_2)-m_\phi(v)\} \cup \{m_\phi(u)-m_\phi(v_1) \mid uv_1 \in E(H_1), u \neq v_2\}$ are also forbidden. Then we have at least $k-2(d(v_1)-1)-(d(v)-2) \geq k-2d(v_1)-d(v)+4$ safe colors for vv_1. Similarly, for vv_2, we cannot use the colors of its at most $d(v_2)-1+d(v)-2$ incident edges. In addition, the colors in $\{m_\phi(v_1)-m_\phi(v)\} \cup \{m_\phi(u)-m_\phi(v_2) \mid uv_2 \in E(H_1), u \neq v_1\}$ are also forbidden. So we have at least $k-2(d(v_2)-1)-(d(v)-2) \geq k-2d(v_2)-d(v)+4$ safe colors for vv_2. Let X, Y denote the sets of safe colors for vv_1 and vv_2 respectively. By Lemma 2.2, we have at least $k-2d(v_1)-d(v)+4+k-2d(v_2)-d(v)+4-3 = 2k-2d(v)-2d(v_1)-2d(v_2)+5$ distinct pairs (x_i, y_i) with $x_i \neq y_i$ in $X \times Y$ such that all the sums x_i+y_i are pairwise distinct. Moreover, among them there are at most two pairs such that $x_i-y_i = m_\phi(v_2)-m_\phi(v_1)$. So we must have that $2k-2d(v)-2d(v_1)-2d(v_2)+5-2 \leq d(v)-2$, since otherwise we can choose a pair, say $(x, y) \in X \times Y$ with $x \neq y$, such that $x+y$ is not in $\{m_\phi(u)-m_\phi(v) \mid uv \in E(H_1)\}$ and $x-y \neq m_\phi(v_2)-m_\phi(v_1)$, and thus we can get a neighbor sum $[k]$-edge coloring of G, which is a contradiction. Therefore $d(v) \geq \frac{2k-2d(v_1)-2d(v_2)+5}{3}$. □

Claim 3.2. *For each vertex $v \in V(G)$, if $n_{3^-}(v) \geq 2$, then $d(v) \geq \frac{2k-7}{3}$ and $n_{3^-}(v) \leq 7-2k+3d(v)$.*

Proof. Suppose that v_1 and v_2 are two neighbors of v such that $d(v_1), d(v_2) \leq 3$. By Claim 3.1, $d(v) \geq \frac{2k-2d(v_1)-2d(v_2)+5}{3} \geq \frac{2k-7}{3}$. Since $k \geq 25$,

$$\frac{d(v)(d(v)-1)}{2} \geq \frac{(k-5)(2k-7)}{9} > k+(k-1).$$

So in any proper coloring of G, the sum of the colors of the edges incident with v is different from its 3^- neighbors. By the same arguments as in Claim 3.1, we have that $2k-2d(v)-d(v_1)-d(v_2)+5 \leq d(v)-n_{3^-}(v)$. Thus $n_{3^-}(v) \leq 7-2k+3d(v)$. □

We have the following immediate corollary.

Corollary 3.1. *For each vertex $v \in V(G)$, if $n_{3^-}(v) \geq 2$, then $n_{4^+}(v) \geq 12$.*

Claim 3.3. *Let v be a vertex with $n_{3^-}(v) = 1$ and u be a neighbor of v with $d(u) \geq 4$. Then $d(u) \geq \min\{\frac{k-d(v)+2}{2}, \frac{2k-3d(v)-1}{2}\}$.*

Proof. We may assume that v_1 is the neighbor of v with $d(v_1) \leq 3$. If $2d(u) \leq k+1-d(v)$, then by Claim 3.1, $d(v) \geq \frac{2k-2d(v_1)-2d(u)+5}{3} \geq \frac{2k-2d(u)-1}{3}$. Thus $3d(v)+2d(u) \geq 2k-1$, which completes our proof. □

Next we will show that G contains the configuration (C) or (D). Let H be the graph obtained by deleting all 3^--vertices from G.

Claim 3.4. *For each vertex $v \in H$, $d_H(v) \geq 3$. Moreover, if $d_H(v) = 3$ and u is any neighbor of v in H, then $d_H(u) \geq 12$.*

Proof. Let v be a vertex in H. By the definition of H, $d_G(v) \geq 4$. If $n_{3^-}(v) \geq 2$, then by Corollary 3.1, $d_H(v) \geq n_{4^+}(v) \geq 12$. If $n_{3^-}(v) = 1$, then $d_H(v) \geq 3$. So for each vertex $v \in V(H)$, $d_H(v) \geq 3$.

Now suppose that u is a neighbor of v in H and $d_H(v) = 3$. We know that $d_G(v) = 4$ and $n_{3^-}(v) = 1$. If $n_{3^-}(u) \geq 2$, then by Corollary 3.1, $d_H(u) \geq n_{4^+}(u) \geq 12$. If $n_{3^-}(u) = 1$, we claim that $d_G(u) \geq 13$. Otherwise, $d_G(u) \leq 12$, and by Claim 3.3, $d_G(v) \geq \min\{\frac{k-d_G(u)+2}{2}, \frac{2k-3d_G(u)-1}{2}\} \geq \min\{\frac{k-10}{2}, \frac{2k-37}{2}\} \geq 5$, which is a contradiction. Hence $d_G(u) \geq 13$ and $d_H(u) = d_G(u)-1 \geq 12$. If $n_{3^-}(u) = 0$, then $d_G(u) = d_H(u)$. By Claim 3.3, we have $d_H(u) = d_G(u) \geq \min\{\frac{k-d_G(v)+2}{2}, \frac{2k-3d_G(v)-1}{2}\} \geq \min\{\frac{k-2}{2}, \frac{2k-13}{2}\}$. So $d_H(u) \geq 12$. □

Claim 3.5. *Let u be any neighbor of v in H. If $4 \leq d_H(v) \leq 5$ and $d_H(v) < d_G(v)$, then $d_H(u) \geq 10$.*

Proof. Since $d_H(v) < d_G(v)$, $n_{3^-}(v) \geq 1$. By Corollary 3.1, we may assume that $n_{3^-}(v) = 1$ or else we have $d_H(v) \geq 12$. If $n_{3^-}(u) \geq 2$, then $d_H(u) \geq 12$. If $n_{3^-}(u) = 1$, then $d_H(u) \geq 10$. Otherwise, $d_G(u) = d_H(u)+1 \leq 10$, and by Claim 3.3, $6 \geq d_G(v) \geq \min\{\frac{k-d_G(u)+2}{2}, \frac{2k-3d_G(u)-1}{2}\} \geq \min\{\frac{k-8}{2}, \frac{2k-31}{2}\} \geq 7$, which is a contradiction. If $n_{3^-}(u) = 0$, then $d_H(u) = d_G(u)$. By Claim 3.3, $d_H(u) = d_G(u) \geq \min\{\frac{k-d_G(v)+2}{2}, \frac{2k-3d_G(v)-1}{2}\} \geq \min\{\frac{k-4}{2}, \frac{2k-19}{2}\} \geq 10$. □

By Lemma 2.1 and Claim 3.4, there exists a 5^--vertex v in H such that v belongs to one of the configurations (B), (C), (D). However, if $d_H(v) = 3$, then by Claim 3.4, each neighbor u of v has $d_H(u) \geq 12$. We must have that $4 \leq d_H(v) \leq 5$ and H contains the configuration (C) or (D). By Claim 3.5, if $d_H(v) < d_G(v)$, then for any edge $uv \in E(H)$, we have $d_H(u) \geq 10$. So it must hold that $d_H(v) = d_G(v)$. We claim that v belongs to the configuration (C) or (D) in G. Otherwise, v has a neighbor u in H such that $d_H(u) \leq 9$ and $d_H(u) < d_G(u)$. Clearly, $n_{3^-}(u) = 1$ or else $n_{3^-}(u) \geq 2$, and then $d_H(u) \geq 12$ by Corollary 3.1, a contradiction. Then $d_G(u) \leq 10$. By Claim 3.3, $d_G(v) \geq \min\{\frac{k-d_G(u)+2}{2}, \frac{2k-3d_G(u)-1}{2}\} \geq \min\{\frac{k-8}{2}, \frac{2k-31}{2}\} \geq 6$ since $k \geq 25$. This contradiction proves that v belongs to the configuration (C) or (D) in G.

Suppose that v has neighbors $v_1, v_2, \ldots, v_t$, where $t = 4, 5$, with $d(v_1) \leq d(v_2) \leq \cdots \leq d(v_t)$. If $t = 4,\ d(v_1) \leq 7$ and $d(v_2) \leq 9$, then by Claim 3.1, it holds that $4 = d(v) \geq \frac{2k-2d(v_1)-2d(v_2)+5}{3} \geq 7$, which is a contradiction. If $t = 5,\ d(v_1) \leq 6$ and $d(v_2) \leq 7$, by Claim 3.1, it holds that $5 = d(v) \geq \frac{2k-2d(v_1)-2d(v_2)+5}{3} \geq 9$, which is a contradiction. This completes the whole proof of Theorem 1.2.

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