On general frameworks and threshold functions for multiple domination

Vadim Zverovich

University of the West of England, Bristol, BS16 1QY, UK

ARTICLE INFO

ABSTRACT

We consider two general frameworks for multiple domination, which are called $\langle \mathbf{r}, \mathbf{s} \rangle$-domination and parametric domination. They generalise and unify $\{k\}$-domination, k-domination, total k-domination and k-tuple domination. In this paper, known upper bounds for the classical domination are generalised for the $\langle \mathbf{r}, \mathbf{s} \rangle$-domination and parametric domination numbers. These generalisations are based on the probabilistic method and they imply new upper bounds for the $\{k\}$-domination and total k-domination numbers. Also, we study threshold functions, which impose additional restrictions on the minimum vertex degree, and present new upper bounds for the aforementioned numbers. Those bounds extend similar known results for k-tuple domination and total k-domination.

© 2015 Elsevier B.V. All rights reserved.

1. Introduction

All graphs will be finite and undirected without loops and multiple edges. If G is a graph of order n, then $V(G) = \{v_1, v_2, \ldots, v_n\}$ is the set of vertices in G and d_i denotes the degree of v_i. Let $N(x)$ denote the neighbourhood of a vertex x. Also let $N(X) = \cup_{x \in X} N(x)$ and $N[X] = N(X) \cup X$. Denote by $\delta = \delta(G)$ and $\Delta = \Delta(G)$ the minimum and maximum degrees of vertices of G, respectively. The following well-known definitions can be found in [13]. A set X is called a *dominating set* if every vertex not in X is adjacent to a vertex in X. The minimum cardinality of a dominating set of G is the *domination number* $\gamma(G)$. A set X is called a *k-dominating set* if every vertex not in X has at least k neighbours in X. Note that $d_i < k$ implies $v_i \in X$. The minimum cardinality of a k-dominating set of G is the *k-domination number* $\gamma_k(G)$. A set X is called a *k-tuple dominating set* of G if for every vertex $v \in V(G)$, $|N[v] \cap X| \geq k$. The minimum cardinality of a k-tuple dominating set of G is the *k-tuple domination number* $\gamma_{\times k}(G)$. The k-tuple domination number is only defined for graphs with $\delta \geq k - 1$. The 2-tuple domination number $\gamma_{\times 2}(G)$ is called the *double domination number* and the 3-tuple domination number $\gamma_{\times 3}(G)$ is called the *triple domination number*. A set X is called a *total k-dominating set* of G if for every vertex $v \in V(G)$, $|N(v) \cap X| \geq k$. The minimum cardinality of a total k-dominating set of G is the *total k-domination number* $\gamma_k^t(G)$. The total k-domination number is only defined for graphs with $\delta \geq k$. Note that $\gamma_1^t(G)$ is the well-known total domination number $\gamma_t(G)$. A function $f : V(G) \to \{0, 1, \ldots, k\}$ is called $\{k\}$-*dominating* if $\sum_{u \in N[v_i]} f(u) \geq k$ for all $i = 1, \ldots, n$. The $\{k\}$-*domination number* of a graph G, denoted by $\gamma_{\{k\}}(G)$, is the smallest weight of a $\{k\}$-dominating function of G. A survey of results devoted to k-domination and k-independence can be found in [7].

The following fundamental result was independently proved by Alon and Spencer [2], Arnautov [3], Lovász [14] and Payan [15].

E-mail address: vadim.zverovich@uwe.ac.uk.

Theorem 1 ([*[2,3,14,15]*]). *For any graph G,*

$$\gamma(G) \leq \frac{\ln(\delta + 1) + 1}{\delta + 1} n.$$

Alon [1] proved that the bound of Theorem 1 is asymptotically best possible. The bound in Theorem 2 is asymptotically the same, even though the latter is stronger for small values of δ.

Theorem 2 ([*[6]*], *[13] p. 48*). *For any graph G with $\delta \geq 1$,*

$$\gamma(G) \leq \left(1 - \frac{\delta}{(1 + \delta)^{1+1/\delta}}\right) n.$$

In this paper, we consider Cockayne's and Favaron's general frameworks for $\langle \mathbf{r}, \mathbf{s}\rangle$-domination and parametric domination, respectively. They generalise and unify $\{k\}$-domination, k-domination, total k-domination and k-tuple domination. In the next two sections, the aforementioned classical upper bounds are generalised for the $\langle \mathbf{r}, \mathbf{s}\rangle$-domination and parametric domination numbers. The generalisations imply new upper bounds for the $\{k\}$-domination and total k-domination numbers (see Section 2). These results are based on the probabilistic method, which is a further development of the proof technique from [11] and earlier papers [10,18].

In Section 4, we study threshold functions, which impose additional restrictions on the minimum vertex degree, and present new upper bounds for the above numbers. Those bounds extend similar known results for k-tuple domination and total k-domination.

Note that the probabilistic constructions used in the proofs of the theorems on $\langle \mathbf{r}, \mathbf{s}\rangle$-domination imply randomised algorithms for finding an $\mathbf{s}$-dominating $\mathbf{r}$-functions, whose weights satisfy the bounds of the corresponding theorems with positive probability. A similar statement is true for the theorems devoted to parametric domination.

2. Cockayne's framework for $\langle \mathbf{r}, \mathbf{s}\rangle$-domination

Cockayne introduced in [8] an interesting framework for domination in graphs. Let $V(G) = \{v_1, \ldots, v_n\}$ denote the vertex set of a graph G, and let $\mathbf{r} = (r_1, \ldots, r_n)$ and $\mathbf{s} = (s_1, \ldots, s_n)$ be n-tuples of nonnegative integers, i.e. $r_i \in \mathbf{N}$ and $s_i \in \mathbf{N}$. A function $f : V(G) \to \mathbf{N}$ is called an $\mathbf{r}$-*function* of G if $f(v_i) \leq r_i$ for all $i = 1, \ldots, n$. Let $f[v_i] = \sum_{u \in N[v_i]} f(u)$. An $\mathbf{r}$-function f is $\mathbf{s}$-*dominating* if $f[v_i] \geq s_i$ for all $i = 1, \ldots, n$. The *weight* of a function f is denoted by $|f|$ and defined by $|f| = \sum_{i=1}^{n} f(v_i)$.

The $\langle \mathbf{r}, \mathbf{s}\rangle$-*domination number* of a graph G, denoted by $\gamma\langle \mathbf{r}, \mathbf{s}\rangle(G)$, is the smallest weight of an $\mathbf{s}$-dominating $\mathbf{r}$-function of G. As pointed out in [8], such functions exist if and only if $\sum_{v_j \in N[v_i]} r_j \geq s_i$ for all $i = 1, \ldots, n$. It is not difficult to see that $\langle \mathbf{r}, \mathbf{s}\rangle$-domination unifies and generalises the classical domination, k-tuple domination and $\{k\}$-domination if we put $r_i = s_i = 1$; $r_i = 1, s_i = k$; and $r_i = s_i = k$ for all $i = 1, \ldots, n$, respectively.

Let us denote

$$\tau = \min\{r_1, \ldots, r_n\}, \qquad s = \max\{s_1, \ldots, s_n\}, \qquad r = \left\lfloor \frac{s}{\delta + 1}\right\rfloor + 1,$$

$$\theta = (\delta + 1)r - s \quad \text{and} \quad B_t = \binom{(\delta + 1)r}{t}.$$

The following theorem provides an upper bound for the $\langle \mathbf{r}, \mathbf{s}\rangle$-domination number of a graph.

Theorem 3. *For any graph G of order n with $r \leq \tau$ and $\rho = 1/\theta$,*

$$\gamma\langle \mathbf{r}, \mathbf{s}\rangle(G) \leq \left(1 - \frac{(r\rho)^{\rho}}{(1 + \rho)^{1+\rho} B_{s-1}^{\rho}}\right) r n.$$

Proof. For each vertex $v \in V(G)$, we select δ vertices from $N(v)$ and denote the resulting set together with the vertex v by $N'[v]$. Thus, $|N'[v]| = \delta + 1$. For $i = 1, 2, \ldots, r$, let $a_i(v)$ be a $(0, 1)$-function on the set $V(G)$ such that it assigns "1" to every vertex of G independently with probability

$$p = 1 - \left(\frac{r}{(1 + \theta)B_{s-1}}\right)^{1/\theta}.$$

Let us define an $\mathbf{r}$-function $a(v)$ as follows: $a(v) = \sum_{i=1}^{r} a_i(v)$.

For $m = 0, 1, \ldots, s - 1$, we denote

$$C_m = \left\{v \in V(G) : \sum_{u \in N'[v]} a(u) = m\right\}.$$

Claim 1. *For each set C_m, there exists a function $c_m : V(G) \to \mathbf{N}$ such that*

$$|c_m| \leq (s - m)|C_m| \tag{1}$$

and for any vertex $v \in C_m$,

$$a(v) + c_m(v) \leq r \quad and \quad \sum_{u \in N'[v]} c_m(u) \geq s - m. \tag{2}$$

Proof. Let us initially put $c_m(v) = 0$ for all $v \in V(G)$. Then, for each vertex $v \in C_m$, we redefine c_m in the set $N'[v]$ as follows:

Case 1: Suppose that $c_m(u) = 0$ for any vertex $u \in N'[v]$. Note that because $\sum_{u \in N'[v]} a(u) = m$, the "spare capacity" in $N'[v]$ is $(\delta + 1)r - m > s - m$, i.e. the weight of c_m can be increased in $N'[v]$ by $s - m$ units. Thus, we can obviously redefine c_m in $N'[v]$ in such a way that

$$\sum_{u \in N'[v]} c_m(u) = s - m \tag{3}$$

and

$$a(u) + c_m(u) \leq r \quad \text{for any } u \in N'[v]. \tag{4}$$

In this case, we increased the weight of c_m in $N'[v]$ by $s - m$ units.

Case 2: Assume that $c_m(u) > 0$ for some $u \in N'[v]$, but

$$\sum_{u \in N'[v]} c_m(u) = \psi < s - m,$$

where $\psi \geq 1$. In this case, we can increase the weight of c_m in $N'[v]$ by $s - m - \psi$ units to make sure that (3) and (4) hold.

Case 3: Suppose now that $c_m(u) > 0$ for some $u \in N'[v]$ and

$$\sum_{u \in N'[v]} c_m(u) \geq s - m.$$

In this case, we do not change the weight of c_m in $N'[v]$.

Thus, when constructing the function c_m, we increased its weight at most $|C_m|$ times by at most $s - m$ units, and so (1) is true. The inequalities (2) are also true by construction. ∎

Let us define the function f on the set $V(G)$ as follows:

$$f(v) = a(v) + \max_{0 \leq m \leq s-1} c_m(v).$$

By Claim 1, the function f is an $(s, \ldots, s)$-dominating $(r, \ldots, r)$-function. Hence, it is also an **s**-dominating **r**-function because $s \geq s_i$ and $r \leq \tau \leq r_i$ for all $i = 1, \ldots, n$. Also,

$$f(v) \leq a(v) + \sum_{m=0}^{s-1} c_m(v).$$

The expectation of $|f|$ is as follows:

$$\mathbf{E}[|f|] \leq \mathbf{E}\left[|a| + \sum_{m=0}^{s-1} |c_m|\right] \leq \sum_{i=1}^{r} \mathbf{E}[|a_i|] + \sum_{m=0}^{s-1} (s - m)\mathbf{E}[|C_m|].$$

We have

$$\mathbf{E}[|C_m|] = \sum_{v \in V(G)} \mathbf{P}[v \in C_m] = \sum_{i=1}^{n} p^m (1 - p)^{(\delta+1)r - m} \binom{(\delta + 1)r}{m} = p^m (1 - p)^{(\delta+1)r - m} B_m n.$$

Thus,

$$\mathbf{E}[|f|] \leq pnr + \sum_{m=0}^{s-1} (s - m)p^m (1 - p)^{(\delta+1)r - m} B_m n$$

$$= pnr + (1 - p)^{(\delta+1)r - s + 1} n \sum_{m=0}^{s-1} (s - m)p^m (1 - p)^{s - m - 1} B_m.$$

Furthermore, for $0 \le m \le s - 1$,

$$(s - m)B_m = (s - m) \binom{(\delta + 1)r}{m} \le \prod_{j=1}^{\theta+1} \frac{(s - m + j - 1)}{j} \binom{(\delta + 1)r}{m}$$

$$= \binom{(\delta + 1)r - m}{\theta + 1} \binom{(\delta + 1)r}{m} = \binom{s - 1}{m} \binom{(\delta + 1)r}{s - 1} = \binom{s - 1}{m} B_{s-1}.$$

We obtain

$$\gamma \langle \mathbf{r}, \mathbf{s} \rangle (G) \le \mathbf{E}[|f|] \le pnr + (1 - p)^{\theta+1} n B_{s-1} \sum_{m=0}^{s-1} \binom{s - 1}{m} p^m (1 - p)^{s-m-1}$$

$$= pnr + (1 - p)^{\theta+1} n B_{s-1}$$

$$= \left(1 - \frac{(r\rho)^\rho}{(1 + \rho)^{1+\rho} B_{s-1}^\rho}\right) rn,$$

as required. The proof of the theorem is complete. ∎

The proof of Theorem 3 implies a weaker upper bound for the $\langle \mathbf{r}, \mathbf{s} \rangle$-domination number. This result generalises the classical bound in Theorem 1.

Corollary 1. *For any graph G of order n with $r \le \tau$,*

$$\gamma \langle \mathbf{r}, \mathbf{s} \rangle (G) \le \frac{\ln(\theta + 1) + \ln B_{s-1} - \ln r + 1}{\theta + 1} rn.$$

Proof. The proof easily follows if we use the inequality $1 - p \le e^{-p}$, and then minimise the following upper bound:

$$\gamma \langle \mathbf{r}, \mathbf{s} \rangle (G) \le pnr + e^{-p(\theta+1)} n B_{s-1}. \quad \blacksquare$$

It may be pointed out that the bound of Corollary 1 can be optimised with respect to r, where r is now any integer between $s/(\delta + 1)$ and τ:

$$\gamma \langle \mathbf{r}, \mathbf{s} \rangle (G) \le \min_{s/(\delta+1) \le r \le \tau} \left\{ \frac{\ln(\theta + 1) + \ln B_{s-1} - \ln r + 1}{\theta + 1} rn \right\}.$$

$\{k\}$-Domination is a particular case of $\langle \mathbf{r}, \mathbf{s} \rangle$-domination when $r_i = s_i = k$ for all $i = 1, \ldots, n$. Theorem 3 and Corollary 1 imply new upper bounds for the $\{k\}$-domination number. We have $\tau = s = k$, and hence

$$r = \left\lfloor \frac{k}{\delta + 1} \right\rfloor + 1, \qquad \theta = (\delta + 1)r - k \quad \text{and} \quad B_{k-1} = \binom{(\delta + 1)r}{k - 1}.$$

Corollary 2. *For any graph G with $\delta > 0$ and $\rho = 1/\theta$,*

$$\gamma_{\{k\}}(G) \le \left(1 - \frac{(r\rho)^\rho}{(1 + \rho)^{1+\rho} B_{k-1}^\rho}\right) rn \le \frac{\ln(\theta + 1) + \ln B_{k-1} - \ln r + 1}{\theta + 1} rn.$$

Similar to Corollary 1, the latter bound in Corollary 2 is weaker than the former bound, but it has a simpler formula and can be further optimised with respect to r for integers between $k/(\delta + 1)$ and k.

While $\langle \mathbf{r}, \mathbf{s} \rangle$-domination generalises the classical domination, k-tuple domination and $\{k\}$-domination, the following definition generalises total domination by considering open neighbourhoods.

An $\mathbf{r}$-function f is called *total $\mathbf{s}$-dominating* if

$$\sum_{u \in N(v_i)} f(u) \ge s_i \quad \text{for all } i = 1, \ldots, n.$$

The total $\langle \mathbf{r}, \mathbf{s} \rangle$-*domination number* $\gamma^t \langle \mathbf{r}, \mathbf{s} \rangle (G)$ of a graph G is the smallest weight of a total $\mathbf{s}$-dominating $\mathbf{r}$-function of G.

Let

$$\tilde{r} = \left\lfloor \frac{s}{\delta} \right\rfloor + 1, \qquad \tilde{\theta} = \delta\tilde{r} - s, \qquad \tilde{B}_{s-1} = \binom{\delta\tilde{r}}{s - 1} \quad \text{and} \quad \tilde{\rho} = 1/\tilde{\theta}.$$

Theorem 4. *For any graph G of order n with $\tilde{r} \le \tau$ and $\delta > 0$,*

$$\gamma^t \langle \mathbf{r}, \mathbf{s} \rangle (G) \le \left(1 - \frac{(\tilde{r}\tilde{\rho})^{\tilde{\rho}}}{(1 + \tilde{\rho})^{1+\tilde{\rho}} \tilde{B}_{s-1}^{\tilde{\rho}}}\right) \tilde{r}n \le \frac{\ln(\tilde{\theta} + 1) + \ln \tilde{B}_{s-1} - \ln \tilde{r} + 1}{\tilde{\theta} + 1} \tilde{r}n.$$

Proof. For each vertex $v \in V(G)$, we select δ vertices from $N(v)$ and denote the resulting set $N'(v)$. Thus, $|N'(v)| = \delta$. The proof now follows immediately from the proofs of Theorem 3 and Corollary 1 if we replace $N'[v]$ by $N'(v)$. ∎

The following upper bounds for the total k-domination number follow from Theorem 4 if we put $r_i = 1$ and $s_i = k$ for all $i = 1, \ldots, n$, in which case $\tilde{r} = 1, k < \delta, \tilde{\theta} = \delta - k$ and $\tilde{B}_{k-1} = b_{k-1} = \binom{\delta}{k-1}$.

Corollary 3. *For any graph G with $\bar{\delta} = \delta - k > 0$,*

$$\gamma_k^t(G) \leq \left(1 - \frac{\bar{\delta}}{(1 + \bar{\delta})^{1 + 1/\bar{\delta}} \, b_{k-1}^{1/\bar{\delta}}}\right) n \leq \frac{\ln(\delta - k + 1) + \ln b_{k-1} + 1}{\delta - k + 1} n.$$

In particular, we obtain an upper bound for the total domination number for any graph G with $\delta \geq 1$: $\gamma_t(G) \leq \frac{\ln \delta + 1}{\delta} n$.

3. Favaron's framework for parametric domination

While Cockayne's framework is based on functions with prescribed properties, the focus of the generalisation considered in this section is on properties of vertex sets called (k, l)-dominating sets. These two frameworks complement each other because the former does not generalise k-domination, while the latter does not include $\{k\}$-domination.

The following definition with minor adaptations is due to Favaron et al. [9]. For integers $k \geq 1$ and $l \geq 1$, a set D is called a (k, l)-*dominating set* of G if for every vertex $v \notin D$, $|N[v] \cap D| \geq k$, and for every vertex $v \in D$, $|N[v] \cap D| \geq l$. The minimum cardinality of a (k, l)-dominating set of G is the *parametric domination number* $\gamma_{k,l}(G)$. Note that, using Favaron's terminology [9], the parametric domination number is called $(l-1)$-*total k-dominating number* $\gamma_{l-1,k}(G)$. Also, there is some similarity between this parameter and f-domination defined in [17].

It is natural to consider the parametric domination number for graphs with $\delta \geq \max\{k, l-1\}$. Since $V(G)$ is a (k, l)-dominating set of G, the parametric domination is well defined. It is easy to see that $\gamma_{1,1}(G)$ is the domination number $\gamma(G)$, $\gamma_{2,1}(G)$ is the 2-domination number $\gamma_2(G)$, $\gamma_{2,2}(G)$ is the double domination number $\gamma_{\times 2}(G)$ and $\gamma_{1,2}(G)$ is the total domination number $\gamma_t(G)$.

More generally, the parametric domination number unifies the following:

$l = 1$	$\gamma_{k,1}(G)$ is the k-domination number $\gamma_k(G)$
$l = k$	$\gamma_{k,k}(G)$ is the k-tuple domination number $\gamma_{\times k}(G)$
$l = k + 1$	$\gamma_{k,k+1}(G)$ is the total k-domination number $\gamma_k^t(G)$

Let $\varphi = \max\{k, l - 1\}$ and $b_t = \binom{\delta}{t}$.

Theorem 5. *For any graph G with $\bar{\delta} = \delta - \varphi > 0$,*

$$\gamma_{k,l}(G) \leq \left(1 - \frac{\bar{\delta}}{(1 + \bar{\delta})^{1 + 1/\bar{\delta}} \, b_{\varphi - 1}^{1/\bar{\delta}}}\right) n.$$

Proof. For each vertex $v \in V(G)$, we select δ vertices from $N(v)$ and denote the resulting set by $N'(v)$. Let A be a set formed by an independent choice of vertices of G, where each vertex is selected with probability

$$p = 1 - \left(\frac{1}{(1 + \bar{\delta}) b_{\varphi - 1}}\right)^{1/\bar{\delta}}.$$

For $m = 0, 1, \ldots, k - 1$, let us denote $B_m = \{v \in V(G) - A : |N'(v) \cap A| = m\}$. Also, for $m = 0, 1, \ldots, l - 2$, we denote $A_m = \{v \in A : |N'(v) \cap A| = m\}$. For each set A_m, we form a set A'_m in the following way. For every vertex $v \in A_m$, we take $l - m - 1$ neighbours from $N'(v) - A$ and add them to A'_m. Such neighbours always exist because $\delta \geq l - 1$. It is obvious that $|A'_m| \leq (l - m - 1)|A_m|$. For each set B_m, we form a set B'_m by taking $k - m$ neighbours from $N'(v) - A$ for every vertex $v \in B_m$. Such neighbours always exist because $\delta \geq k$. We have $|B'_m| \leq (k - m)|B_m|$.

Let us construct the set D as follows: $D = A \cup \left(\bigcup_{m=0}^{l-2} A'_m\right) \cup \left(\bigcup_{m=0}^{k-1} B'_m\right)$. The set D is a (k, l)-dominating set. Indeed, if there is a vertex v which is not (k, l)-dominated by D, then v is not (k, l)-dominated by A. Therefore, v would belong to A_m or B_m for some m, but all such vertices are (k, l)-dominated by the set D by construction.

The expectation of $|D|$ is

$$\mathbf{E}[|D|] \le \mathbf{E}\Big[|A| + \sum_{m=0}^{l-2} |A'_m| + \sum_{m=0}^{k-1} |B'_m| \Big]$$

$$\le \mathbf{E}[|A|] + \sum_{m=0}^{l-2}(l - m - 1)\mathbf{E}[|A_m|] + \sum_{m=0}^{k-1}(k - m)\mathbf{E}[|B_m|].$$

We have $\mathbf{E}[|A|] = \sum_{i=1}^n \mathbf{P}[v_i \in A] = pn$. Also,

$$\mathbf{E}[|A_m|] = \sum_{i=1}^n \mathbf{P}[v_i \in A_m] = \sum_{i=1}^n p \binom{\delta}{m} p^m (1 - p)^{\delta - m} = p^{m+1}(1 - p)^{\delta - m} b_m n$$

and

$$\mathbf{E}[|B_m|] = \sum_{i=1}^n \mathbf{P}[v_i \in B_m] = \sum_{i=1}^n (1 - p) \binom{\delta}{m} p^m (1 - p)^{\delta - m} = p^m (1 - p)^{\delta - m + 1} b_m n.$$

We obtain

$$\mathbf{E}[|D|] \le pn + \sum_{m=0}^{l-2}(l - m - 1)p^{m+1}(1 - p)^{\delta - m} b_m n + \sum_{m=0}^{k-1}(k - m)p^m (1 - p)^{\delta - m + 1} b_m n$$

$$\le pn + \sum_{m=0}^{\varphi - 1}(\varphi - m)p^{m+1}(1 - p)^{\delta - m} b_m n + \sum_{m=0}^{\varphi - 1}(\varphi - m)p^m (1 - p)^{\delta - m + 1} b_m n$$

$$= pn + \sum_{m=0}^{\varphi - 1}(\varphi - m)b_m n p^m (1 - p)^{\delta - m}.$$

Furthermore, for $0 \le m \le \varphi - 1$,

$$(\varphi - m)b_m = (\varphi - m) \binom{\delta}{m} \le (\varphi - m) \binom{\delta}{m} \prod_{j=2}^{\varphi - m} \frac{(\delta - \varphi + j)}{j}$$

$$= \frac{\delta!}{m!(\varphi - m - 1)!(\delta - \varphi + 1)!} = \binom{\varphi - 1}{m} \binom{\delta}{\varphi - 1} = \binom{\varphi - 1}{m} b_{\varphi - 1}.$$

Therefore,

$$\mathbf{E}[|D|] \le pn + n b_{\varphi - 1}(1 - p)^{\delta - \varphi + 1} \sum_{m=0}^{\varphi - 1} \binom{\varphi - 1}{m} p^m (1 - p)^{\varphi - 1 - m}$$

$$= pn + n b_{\varphi - 1}(1 - p)^{\delta - \varphi + 1}$$

$$= \left(1 - \frac{\bar{\delta}}{(1 + \bar{\delta})^{1 + 1/\bar{\delta}} \; b_{\varphi - 1}^{1/\bar{\delta}}} \right) n.$$

Since the expectation is an average value, there exists a particular (k, l)-dominating set of the above order, as required. The proof of Theorem 5 is complete. ∎

Corollary 4. *For any graph G with $\delta \ge \varphi$,*

$$\gamma_{k,l}(G) \le \frac{\ln(\delta - \varphi + 1) + \ln b_{\varphi - 1} + 1}{\delta - \varphi + 1} n.$$

Proof. Using the inequality $1 - p \le e^{-p}$, we obtain

$$\mathbf{E}[|D|] \le pn + n b_{\varphi - 1} e^{-p(\delta - \varphi + 1)}.$$

The proof easily follows if we minimise the right-hand side in the above inequality. ∎

Note that Theorem 5 and Corollary 4 imply the result formulated in Corollary 3 if we put $l = k + 1$.

The next result is similar to Theorem 5 and Corollary 4, which provide better bounds if $l \ge k + 1$. However, for small values of l, the bounds of Theorem 6 are better. In what follows, we put $b_{-1} = 0$.

Theorem 6. *For any graph G with* $\widehat{\delta} = \delta - \max\{k, l\} + 1 > 0$,

$$\gamma_{k,l}(G) \le \left(1 - \frac{\widehat{\delta}}{(1 + \widehat{\delta})^{1+1/\widehat{\delta}} \, (b_{k-1} + b_{l-2})^{1/\widehat{\delta}}}\right) n.$$

Also, for any graph G with $\delta \ge \max\{k, l - 1\}$,

$$\gamma_{k,l}(G) \le \frac{\ln(\widehat{\delta} + 1) + \ln(b_{k-1} + b_{l-2}) + 1}{\widehat{\delta} + 1} n.$$

Proof. Using the same construction as in the proof of Theorem 5, we obtain

$$\mathbf{E}[|D|] \le pn + \sum_{m=0}^{l-2}(l - m - 1)p^{m+1}(1 - p)^{\delta-m}b_m n + \sum_{m=0}^{k-1}(k - m)p^m(1 - p)^{\delta-m+1}b_m n$$

$$= pn + \sum_{m=1}^{l-1}(l - m)p^m(1 - p)^{\delta-m+1}b_{m-1} n + \sum_{m=0}^{k-1}(k - m)p^m(1 - p)^{\delta-m+1}b_m n$$

$$= pn + (1 - p)^{\delta-l+2}n\Theta_1 + (1 - p)^{\delta-k+2}n\Theta_2,$$

where

$$\Theta_1 = \sum_{m=1}^{l-1}(l - m)p^m(1 - p)^{l-m-1}b_{m-1}, \qquad \Theta_2 = \sum_{m=0}^{k-1}(k - m)p^m(1 - p)^{k-m-1}b_m.$$

Now, using an approach similar to that in the proof of Theorem 5, we can prove that

$$(l - m)b_{m-1} \le \binom{l-1}{m} b_{l-2} \quad \text{and} \quad (k - m)b_m \le \binom{k-1}{m} b_{k-1}.$$

Therefore, $\Theta_1 \le b_{l-2}$ and $\Theta_2 \le b_{k-1}$. We have

$$\mathbf{E}[|D|] \le pn + (1 - p)^{\delta-l+2}nb_{l-2} + (1 - p)^{\delta-k+2}nb_{k-1}$$

$$\le pn + (1 - p)^{\widehat{\delta}+1}n(b_{l-2} + b_{k-1}).$$

The first upper bound of the theorem is obtained by minimising the above function, while the second by minimising the following function: $\mathbf{E}[|D|] \le pn + e^{-p(\widehat{\delta}+1)}n(b_{l-2} + b_{k-1})$. ∎

The special case $l = 1$ in parametric domination is the *k-domination number* $\gamma_k(G)$, whereas the case when $l = k$ is the *k-tuple domination number* $\gamma_{\times k}(G)$. By Theorem 6, the following results from [11] are obtained: for any graph G with $\widehat{\delta} = \delta - k + 1 > 0$,

$$\gamma_k(G) \le \left(1 - \frac{\widehat{\delta}}{(1 + \widehat{\delta})^{1+1/\widehat{\delta}} \, b_{k-1}^{1/\widehat{\delta}}}\right) n \le \frac{\ln(\delta - k + 2) + \ln b_{k-1} + 1}{\delta - k + 2} n$$

and

$$\gamma_{\times k}(G) \le \left(1 - \frac{\widehat{\delta}}{(1 + \widehat{\delta})^{1+1/\widehat{\delta}} \, \tilde{b}_{k-1}^{1/\widehat{\delta}}}\right) n \le \frac{\ln(\delta - k + 2) + \ln \tilde{b}_{k-1} + 1}{\delta - k + 2} n,$$

where $\tilde{b}_{k-1} = b_{k-1} + b_{k-2} = \binom{\delta+1}{k-1}$.

4. Threshold functions for multiple domination

The bounds for multiple domination can be improved if we impose additional restrictions on graph parameters, i.e. by considering smaller graph classes. Such restrictions are called *threshold functions*. Caro and Roditty [4] and Stracke and Volkmann [17] were the first considering a threshold function for *k*-domination in the form $\delta \ge 2k-1$. For a slightly stronger threshold function Rautenbach and Volkmann [16] found an interesting upper bound for the *k*-tuple domination number:

Theorem 7 (*Rautenbach and Volkmann [16]*). *If* $\delta \ge 2k\ln(\delta + 1) - 1$, *then*

$$\gamma_{\times k}(G) \le \left(\frac{k\ln(\delta + 1)}{\delta + 1} + \sum_{i=0}^{k-1}\frac{k - i}{i!(\delta + 1)^{k-i}}\right) n.$$

It may be pointed out that similar threshold bounds for the k-domination number are considered in [12].

In the next theorem we consider a threshold function in the form $\delta \geq ck - 1$, where $c > 1$ is a constant. Although c is not restricted from above, for given k and δ the constant c should not be taken as large as possible. The best approach would be to optimise c for given k and δ in such a way that the bound (5) is minimised while $\delta \geq ck - 1$ holds. We will deal with this later.

Theorem 8. *For any graph G with $\delta \geq ck - 1$, where $c > 1$ is a constant,*

$$\gamma_{\times k}(G) < \left(\frac{c}{\delta + 1} + \frac{1}{e^{0.5k(c+1/c-2)}} \right) kn. \tag{5}$$

Proof. For each vertex $v \in V(G)$, we select δ vertices from $N(v)$ and denote the resulting set together with the vertex v by $N'[v]$. Thus, $|N'[v]| = \delta + 1$. Let A be a set formed by an independent choice of vertices of G, where each vertex is selected with probability $p = \frac{ck}{\delta+1} \leq 1$. For $m = 0, 1, \ldots, k - 1$, we denote $C_m = \{v \in V(G) : |N'[v] \cap A| = m\}$. For each set C_m, let us form a set C'_m in the following way: for every vertex $v \in C_m$ we take $k - m$ neighbours from $N'[v] - A$ and add them to C'_m. Such neighbours always exist because $\delta \geq k$. It is obvious that $|C'_m| \leq (k - m)|C_m| \leq k|C_m|$.

Let us construct the set D as follows:

$$D = A \cup \left(\bigcup_{m=0}^{k-1} C'_m \right).$$

It is easy to see that D is a k-tuple dominating set. The expectation of $|D|$ is

$$\mathbf{E}[|D|] \leq \mathbf{E}\left[|A| + \sum_{m=0}^{k-1} |C'_m| \right] \leq \mathbf{E}[|A|] + k \sum_{m=0}^{k-1} \mathbf{E}[|C_m|].$$

We have $\mathbf{E}[|A|] = pn$ and

$$\sum_{m=0}^{k-1} \mathbf{E}[|C_m|] = \sum_{m=0}^{k-1} \sum_{i=1}^{n} \mathbf{P}[v_i \in C_m] = \sum_{i=1}^{n} \sum_{m=0}^{k-1} \mathbf{P}[|N'[v_i] \cap A| = m] = \sum_{i=1}^{n} \mathbf{P}[|N'[v_i] \cap A| < k].$$

Let $X_1, \ldots, X_t$ be random variables, which are mutually independent with

$$\mathbf{P}[X_i = 1 - p] = p \quad \text{and} \quad \mathbf{P}[X_i = -p] = 1 - p.$$

Also, let $X = X_1 + \cdots + X_t$, i.e. X has distribution $B(t, p) - np$. By Alon–Spencer's theorem,

$$\mathbf{P}[X < -a] < e^{-a^2/2pt},$$

where $a > 0$ (Theorem A.1.13 in [2]). The above random set A can be seen as a set of vertices labelled by 1, where each vertex is assigned 1 with probability p and 0 with probability $1 - p$. Let us now subtract p from all the labels, i.e. for each vertex v_j we have a random variable τ_j such that

$$\mathbf{P}[\tau_j = 1 - p] = p \quad \text{and} \quad \mathbf{P}[\tau_j = -p] = 1 - p.$$

For a vertex v_i, we define a random variable $\tau_i^* = \sum_{v_j \in N'[v_i]} \tau_j$. Taking into account that $k - (\delta + 1)p = k(1 - c) < 0$, we obtain by Alon–Spencer's theorem:

$$\mathbf{P}[|N'[v_i] \cap A| < k] = \mathbf{P}[\tau_i^* < k - (\delta + 1)p] < e^{-\frac{[k-(\delta+1)p]^2}{2p(\delta+1)}} = e^{-\frac{[k-ck]^2}{2ck}} = e^{-0.5k(c+1/c-2)}.$$

Thus,

$$\sum_{m=0}^{k-1} \mathbf{E}[|C_m|] < ne^{-0.5k(c+1/c-2)}$$

and

$$\gamma_{\times k}(G) \leq \mathbf{E}[|D|] < pn + kne^{-0.5k(c+1/c-2)} = \left(\frac{c}{\delta + 1} + \frac{1}{e^{0.5k(c+1/c-2)}} \right) kn,$$

as required. The proof of the theorem is complete. ∎

Let us consider a particular case of Theorem 8 when $c = 3$, and compare it to Theorem 7 for graphs with $\delta \geq 20$. We have

$$\gamma_{\times k}(G) < \left(\frac{3}{\delta + 1} + \frac{1}{e^{2k/3}} \right) kn \qquad (6)$$

for any graph G with $\delta \geq 3k - 1$. This bound is better than the bound of Theorem 7 if the former is less than the first term of the latter, i.e.

$$\left(\frac{3}{\delta + 1} + \frac{1}{e^{2k/3}} \right) k < \frac{k \ln(\delta + 1)}{\delta + 1},$$

which is equivalent to

$$k > 1.5 \ln(\delta + 1) - 1.5 \ln[\ln(\delta + 1) - 3] = 1.5 \ln(\delta + 1)(1 - o(1)).$$

Since Theorem 7 is applicable for $k \leq (\delta + 1)/(2 \ln(\delta + 1))$, we conclude that (6) provides a better upper bound than Theorem 7 if

$$1.5 \ln(\delta + 1)(1 - o(1)) < k \leq \frac{\delta + 1}{2 \ln(\delta + 1)},$$

which is the largest part of the applicable interval.

For example, if $\delta(G) = 1000$, then Theorem 7 is applicable for $k \leq 72$, whereas (6) is applicable for $k \leq 333$. Since $1.5 \ln(1001) - 1.5 \ln[\ln(1001) - 3] = 8.3$, we obtain that the bound (6) is stronger than the bound of Theorem 7 if $9 \leq k \leq 72$. If $k \leq 8$, then Theorem 7 provides a better upper bound than (6). However, we can try to optimise the constant c in Theorem 8 for given δ and k as follows. The right-hand side of the bound (5) is minimised for c satisfying the following equation:

$$0.5k \left(c + \frac{1}{c} - 2 \right) - \ln(0.5k(\delta + 1)) = \ln \left(1 - \frac{1}{c^2} \right).$$

Now, replacing $\ln(1 - \frac{1}{c^2})$ by $-1/c^2$, we obtain the following cubic equation:

$$kc^3 - 2 (k + \ln[0.5k(\delta + 1)]) c^2 + kc + 2 = 0.$$

The real root $c > 1$ of this equation, which satisfies the condition $\delta \geq ck - 1$, can be used in Theorem 8. For example, if $k = 5$ and $\delta = 1000$, then the above cubic equation becomes

$$c^3 - 5.13c^2 + c + 0.4 = 0.$$

For this equation, the largest real root is $c = 4.910$ (3 dp). Using this value of c in Theorem 8, we obtain $\gamma_{\times 5}(G) < 0.027n$, whereas Theorem 7 produces the bound $\gamma_{\times 5}(G) < 0.035n$.

It is not difficult to generalise Theorem 8 for parametric domination and $\langle \mathbf{r}, \mathbf{s} \rangle$-domination. Let us denote $\mu = \max\{k, l\}$.

Theorem 9. *For any graph G with $\delta \geq c\mu - 1$, where $c > 1$ is a constant,*

$$\gamma_{k,l}(G) < \left(\frac{c}{\delta + 1} + \frac{1}{e^{0.5\mu(c+1/c-2)}} \right) \mu n.$$

Theorem 10. *For any graph G with $(\delta + 1)\tau \geq cs$, where $c > 1$ is a constant,*

$$\gamma \langle \mathbf{r}, \mathbf{s} \rangle (G) < \left(\frac{c}{\delta + 1} + \frac{1}{e^{0.5s(c+1/c-2)}} \right) sn.$$

Caro and Yuster [5] proved an important asymptotic result that if δ is much larger than k, then the upper bound for the total k-domination number is 'close' to the bound of Theorem 1. More precisely, they proved the following:

Theorem 11 (*Caro and Yuster [5]*)**.** *If $k < \sqrt{\ln \delta}$, then*

$$\gamma_k^t(G) \leq \frac{\ln \delta}{\delta} n(1 + o_\delta(1)).$$

The same upper bound is therefore true for the k-tuple domination and k-domination numbers. The threshold function $k < \sqrt{\ln \delta}$ in Theorem 11 is indeed very strong, but the corresponding bound is similar to that of Theorem 1, which is best possible in the class of all graphs. Let us consider a weaker but similar threshold function $k \leq (1 - c) \ln \delta$, where $0 < c < 1$ is a constant. The following explicit and asymptotic bounds are obtained:

Theorem 12. *For any graph G with $k \leq (1 - c) \ln \delta$, where $0 < c < 1$ is a constant,*

$$\gamma_{\times k}(G) < \left(\frac{\ln \delta}{\delta + 1} + \frac{k}{\delta^{0.5c^2}} \right) n \leq \frac{(1 - c) \ln \delta}{\delta^{0.5c^2}} n (1 + o_\delta(1)).$$

Proof. The proof is similar to that of Theorem 8 if we put $p = \frac{\ln \delta}{\delta + 1} \leq 1$. Now,

$$p \geq \frac{k}{(1 - c)(\delta + 1)} > \frac{k}{(\delta + 1)},$$

i.e. $k - (\delta + 1)p < 0$ and

$$\mathbf{P}[|N'[v_i] \cap A| < k] = \mathbf{P}[\tau_i^* < k - (\delta + 1)p] < e^{-\frac{[k-(\delta+1)p]^2}{2p(\delta+1)}} = e^{-\frac{[k-\ln\delta]^2}{2\ln\delta}} \leq e^{-0.5c^2 \ln \delta} = \delta^{-0.5c^2}.$$

Thus,

$$\gamma_{\times k}(G) \leq \mathbf{E}[|D|] < pn + kn\delta^{-0.5c^2} = \left(\frac{\ln \delta}{\delta + 1} + \frac{k}{\delta^{0.5c^2}} \right) n,$$

as required. ∎

Theorem 12 can be generalised for parametric domination and $\langle \mathbf{r}, \mathbf{s} \rangle$-domination as follows.

Theorem 13. *For any graph G with $\mu \leq (1 - c) \ln \delta$, where $0 < c < 1$ is a constant,*

$$\gamma_{k,l}(G) < \left(\frac{\ln \delta}{\delta + 1} + \frac{\mu}{\delta^{0.5c^2}} \right) n.$$

Similar to Theorem 10 the upper bound in the next result does not depend on τ. Note also that $s \leq (\delta + 1)\tau$ holds because $s \leq (1 - c) \ln \delta < \ln \delta \leq (\delta + 1)\tau$, i.e. $\gamma \langle \mathbf{r}, \mathbf{s} \rangle (G)$ is well defined.

Theorem 14. *For any graph G with $s \leq (1 - c) \ln \delta$, where $0 < c < 1$ is a constant,*

$$\gamma \langle \mathbf{r}, \mathbf{s} \rangle (G) < \left(\frac{\ln \delta}{\delta + 1} + \frac{s}{\delta^{0.5c^2}} \right) n.$$

Acknowledgements

References

[1] N. Alon, Transversal numbers of uniform hypergraphs, Graphs Combin. 6 (1990) 1–4.
[2] N. Alon, J.H. Spencer, The Probabilistic Method, Second Edition, John Wiley & Sons Inc., New York, 2000.
[3] V.I. Arnautov, Estimation of the exterior stability number of a graph by means of the minimal degree of the vertices, Prikl. Mat. i Programmirovanie 11 (1974) 3–8. in Russian.
[4] Y. Caro, Y. Roditty, A note on the k-domination number of a graph, Int. J. Math. Math. Sci. 13 (1990) 205–206.
[5] Y. Caro, R. Yuster, Dominating a family of graphs with small connected subgraphs, Combin. Probab. Comput. 9 (2000) 309–313.
[6] Y. Caro, Y. Roditty, Improved bounds for the product of the domination and chromatic numbers of a graph, Manuscript, 1997, 6 pages [Theorem 2.3]. (A reduced version of this paper appeared in: Ars Combin. 56 (2000) 189–191).
[7] M. Chellali, O. Favaron, A. Hansberg, L. Volkmann, k-Domination and k-independence in graphs: a survey, Graphs Combin. 28 (1) (2012) 1–55.
[8] E.J. Cockayne, Towards a theory of $\langle \mathbf{r}, \mathbf{s} \rangle$-domination in graphs, Util. Math. 80 (2009) 97–113.
[9] O. Favaron, M.A. Henning, J. Puech, D. Rautenbach, On domination and annihilation in graphs with claw-free blocks, Discrete Math. 231 (2001) 143–151.
[10] A. Gagarin, Improved upper bounds for the k-tuple domination number, Australas. J. Combin. 41 (2008) 257–261.
[11] A. Gagarin, A. Poghosyan, V. Zverovich, Randomized algorithms and upper bounds for multiple domination in graphs and networks, Discrete Appl. Math. 161 (2013) 604–611.
[12] A. Hansberg, L. Volkmann, Upper bounds on the k-domination number and the k-Roman domination number, Discrete Appl. Math. 157 (7) (2009) 1634–1639.
[13] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New York, 1998.
[14] L. Lovász, On the ratio of optimal integral and fractional covers, Discrete Math. 13 (1975) 383–390.
[15] C. Payan, Sur le nombre d'absorption d'un graphe simple, Cahiers Centre Études Recherche Opér. 17 (1975) 307–317. in French.
[16] D. Rautenbach, L. Volkmann, New bounds on the k-domination number and the k-tuple domination number, Appl. Math. Lett. 20 (2007) 98–102.
[17] C. Stracke, L. Volkmann, A new domination conception, J. Graph Theory 17 (1993) 315–323.
[18] V.E. Zverovich, The k-tuple domination number revisited, Appl. Math. Lett. 21 (2008) 1005–1011.