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# **Discrete Mathematics**

journal homepage: www.elsevier.com/locate/disc

# On independent doubly chorded cycles

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ABSTRACT

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#### ARTICLE INFO

Article history: Received 7 December 2012 Received in revised form 4 May 2015 Accepted 5 May 2015 Available online 6 June 2015

*Keywords:* Cycle Chord Vertex disjoint cycles

## 1. Introduction

We consider only simple graphs. Let *G* be a graph and  $P_t$  be a path on *t* vertices. For  $S \subseteq V(G)$ , let *G*[*S*] denote the subgraph of *G* induced by *S*. Let *A* and *B* be subgraphs of a graph *G*, then e(A, B) denotes the number of edges that have one end in *A* and the other end in *B*, so e(G) denotes the number of edges in *G*. Given a vertex set *W*, we say that a cycle *C* is a *proper* cycle if *C* does not span *W*. Let N(u) denote the set of neighbors of the vertex *u*, that is, the vertices adjacent to *u* in the graph. For a noncomplete graph *G*, we define

In a graph G, we say a cycle C :  $v_1, v_2, \ldots, v_k, v_1$  is chorded if its vertices induce an

additional edge (chord) which is not an edge of the cycle. The cycle C is doubly chorded

if there are at least two such chords. In this paper we show a sharp degree sum condition

that implies the existence of k vertex disjoint doubly chorded cycles in a graph.

 $\sigma_2(G) = \min \{ \deg(u) + \deg(v) \mid u \text{ and } v \text{ are nonadjacent} \},\$ 

with the convention that for the complete graph  $\sigma_2(G) = \infty$ . We say an edge that joins two vertices of a cycle *C* is a *chord* of *C* if the edge is not itself an edge of the cycle. We then say that *C* is a *chorded cycle*. We denote the adjacency of vertices *u* and *v* as  $u \sim v$  and nonadjacency as  $u \neq v$ . A *k*-degenerate graph is one in which every induced subgraph contains a vertex of degree at most *k*. We denote by  $K_4^-$  the graph obtained from  $K_4$  by removing one edge. For terms not defined here see [2].

The study of cycles and systems of vertex disjoint cycles in graphs is well established. Recently, there have been numerous papers considering cycles with additional properties such as containing a specific set of vertices, or containing a specific set of vertices in a specific order (see the survey [7]). Another natural additional property for cycles is that of containing at least one chord or at least some number  $t \ge 1$  of chords.

In 1961 Pósa [12] suggested the problem of finding, in a graph *G*, degree conditions that imply the existence of a cycle with at least one chord. J. Czipzer proved (see Lovász [10], problem 10.2) that any graph *G* with minimum degree  $\delta(G) \ge 3$  contains a chorded cycle, that is, a cycle with an additional edge. Corrádi and Hajnal [4] proved that any graph *G* with

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http://dx.doi.org/10.1016/j.disc.2015.05.010 0012-365X/© 2015 Elsevier B.V. All rights reserved.





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|V(G)| > 3r and  $\delta(G) > 2r$  contains r vertex disjoint cycles. Finkel [6] showed that if G is a graph with |V(G)| > 4r and  $\delta(G) > 3r$ , then G contains r vertex disjoint chorded cycles.

Bialostocki, Finkel and Gyárfás [1] made the following conjecture and verified it for the cases r = 0, s = 2 and for s = 1. Let r, s be nonnegative integers and let G be a graph with  $|V(G)| \ge 3r + 4s$  and  $\delta(G) \ge 2r + 3s$ . Then G contains a collection of r + s vertex disjoint cycles with s of these cycles chorded. This conjecture was settled completely in [3] where a slightly stronger  $\sigma_2$  condition was used.

**Theorem 1** ([3]). Let r and s be integers with r + s > 1, and let G be a graph of order at least 3r + 4s. If  $\sigma_2(G) > 4r + 6s - 1$ , then G contains a collection of r + s vertex disjoint cycles such that s of them are chorded.

More recently it was shown in [8] that a graph on at least 4k vertices such that  $|N(u) \cup N(v)| > 4k + 1$  for any pair of non-adjacent vertices *u* and *v*, contains *k* vertex disjoint chorded cycles.

Note that it is only slightly more difficult to show that  $\delta(G) \ge 3$  implies a cycle with at least two chords exist in *G*. Thus, in some sense, it is more natural to consider conditions implying the existence of such cycles (which we call doubly chorded cycles or DCC's for short). Since a spanning cycle of  $K_4$  has two chords, we are in some sense, seeking versions of  $K_4$  where the spanning cycle has been loosened. This point of view was taken in [9].

In [13] the following was shown.

**Theorem 2.** If G is a graph of order  $n \ge 4k$  and minimum degree at least |7k/2|, then the graph G contains k vertex disjoint doubly chorded cycles.

The goal of this paper is to show the following stronger result.

**Theorem 3.** If G is a graph on n > 6k vertices with  $\sigma_2(G) > 6k - 1$ , then G contains k vertex disjoint doubly chorded cycles.

Theorem 3 is sharp in the sense that the degree sum cannot be lowered. The condition n > 6k is needed for the proof. The complete bipartite graph  $K_{3k-1,n-3k+1}$  has degree sum 6k - 2 and fails to contain k vertex disjoint doubly chorded cycles, since any such cycle in this graph must contain at least three vertices from each partite set.

### 2. Proof of Theorem 3

In this section we prove the main result, Theorem 3. Along the way we state several lemmas that are needed. We postpone the proofs of some of these lemmas until the next section.

**Proof.** Suppose k = 1. If  $\delta(G) > 3$ , the result follows for k = 1 from Theorem 2. If  $\delta(G) = 1$ , say the vertex x is adjacent only to y. Now all other vertices have degree at least 4 in order to satisfy the degree condition. Deleting x and y leaves a subgraph with minimum degree at least 3, and again we apply Theorem 2. If  $\delta(G) = 2$ , say x is adjacent to y and z. Now all other vertices have degree at least 3 by our degree condition. If y and z each have degree at least 3, then take a second copy of G, say G' and join x to its corresponding vertex in G' by an edge. The new graph has minimum degree 3 and so contains a doubly chorded cycle. Clearly, this cycle sits in one copy of G. Next suppose that one of y or z also has degree 2 (as both cannot), say y. Contract the edge xy and call the resulting vertex w and the resulting graph G". Take two copies of G" and join the corresponding copies of w by an edge. This new graph has minimum degree 3 (unless x, y and z form a triangle) and as before, contains a doubly chorded cycle. Note that upon expanding the vertex w back to an edge, it is easy to see we do not hurt our doubly chorded cycle. If x, y and z form a triangle, then remove x and y, leaving z of degree at least one. Now take 3 copies of  $G - \{x, y\}$  and join each of the copies of z, forming a triangle. This new graph has minimum degree at least three (since deg(z) > 3) and clearly the doubly chorded cycle must reside in one copy of  $G - \{x, y\}$ . So we may assume k > 2.

Our proof proceeds by contradiction. Let G be an edge maximal counterexample to the result. Thus, the addition of any edge to G will produce the desired system of k doubly chorded cycles. Hence, G must contain k-1 vertex disjoint doubly chorded cycles. Over all such possible collections of doubly chorded cycles, we choose one, say,  $C: C_1, \ldots, C_{k-1}$ , subject to the constraint that  $|\bigcup_{i=1}^{k-1} V(C_i)|$  is minimum. We assume that  $|C_1| \ge \cdots \ge |C_{k-1}|$ . Let  $H = G \setminus (\bigcup_{i=1}^{k-1} C_i)$ . The key to this proof is the following lemma, whose proof is rather involved and will be deferred until the next section.

**Lemma 1.** Suppose C and C' are two vertex disjoint doubly chorded cycles with  $\ell = |C| \ge |C'|$  and  $\ell \ge 7$ . If  $e(C, C') \ge 3\ell + 1$ , then the graph induced by  $C \cup C'$  contains two vertex disjoint doubly chorded cycles whose union is smaller than |C| + |C'|.

The following will also be useful.

**Lemma 2.** Let *C* be a cycle with at least 5 chords. Then *C* contains a proper doubly chorded cycle.

**Proof.** Suppose C is a cycle with at least 5 chords and suppose it has a clockwise ordering of vertices. Suppose e is a minimum length chord of C, and order the vertices of  $C = \{v_1, v_2, \dots, v_{|C|}\}$  so that e connects  $v_1$  and  $v_i$  so that i is as small as possible. If there are no chords incident to  $v_k$  for 1 < k < i, then clearly  $v_1, v_i, v_{i+1}, \ldots, v_1$  is a shorter doubly chorded cycle, so choose 1 < k < i minimum so that  $v_k$  is incident to a chord, e'. Since e is a shortest chord, the chord e' is of the form  $v_k v_i$ where j > i.

Consider the three segments  $S_1 = [v_k, v_i]$ ,  $S_2 = [v_i, v_j]$  and  $S_3 = [v_j, v_1]$ . First suppose that k > 2. We have three proper cycles which are constructed from these segments:  $S_1$ ,  $S_2$ , e';  $S_2$ ,  $S_3$ , e and  $S_3$ , e,  $S_1^-$ , e'. Since there are no edges incident to the segment  $(v_1, v_k)$ , all of the at least three remaining chords are contained within or between these segments. Further note that none of the above three cycles contain two or more chords or, since they are proper, we are done.

We have three chords remaining. If one chord lies within a segment, then no other chord can enter that segment or a DCC is formed. This implies that the other two chords lie between segments or one lies in each of the other two segments. But in each case, a shorter DCC exists, since each segment lies in a cycle with one of the other two segments. Thus, there is exactly one chord joining each pair of segments. In each case, however, we can still find a shorter doubly chorded cycle, as we now exhibit. Let  $x = x_1x_2$  denote the chord from  $S_1$  to  $S_2$  (so that  $x_1 \in [v_k, v_i]$  and  $x_2 \in [v_i, v_j]$ ). Likewise, let  $y = y_1y_2$  denote the chord from  $S_1$  to  $S_3$  and  $z = z_1z_2$  denote the chord from  $S_2$  to  $S_3$ .

We now look at possibilities for how *x*, *y* and *z* intersect:

#### **Case 1.** Suppose $x_2 \leq z_1$ .

Consider the cycle:  $x_1$ , x,  $x_2$ , C,  $x_1$ . This contains the chords z and e', and is proper since  $v_i$  is not included.

## **Case 2.** Suppose $z_2 \leq y_2$ .

The cycle  $z_1, z, z_2, C, z_1$  contains the chords y and e, and is proper as  $v_i$  is not included.

#### **Case 3.** Suppose $y_1 \leq x_1$ .

The cycle  $x_1, x, x_2, C, x_1$  contains the chords y and e', and is proper as  $v_i$  is not included.

**Case 4.** Suppose  $x_1 < y_1, z_1 < x_2$ , and  $y_2 < z_2$ .

Then the cycle  $y_2$ ,  $C^-$ ,  $z_1$ ,  $z_2$ , C,  $y_1$ ,  $y_2$  has e' and x as chords and avoids  $v_i$  unless  $y_1 = v_i$  or  $z_1 = v_i$ . If  $y_1 = v_i$  then the cycle  $v_1$ , C,  $x_1$ ,  $x_2$ ,  $C^-$ ,  $v_i$ ,  $y_2$ , C,  $v_1$  has e and z as chords and avoids  $v_j$ , so is proper. Note this still works if  $z_1 = y_1 = v_i$ . Finally, if  $z_1 = v_i$  and  $y_1 < v_i$ , then the cycle  $v_1$ , C,  $y_1$ ,  $y_2$ ,  $C^-$ ,  $v_i$ ,  $v_1$  has e' and x as chords and avoids  $z_2$ , so it is proper. This completes the k > 2 case.

Next suppose k = 2 (and j = i + 1). Now the cycle  $S_3$ , e,  $S_1^-$ , e' is not proper, so the above argument does not hold. However, we can consider e' as the shortest chord and repeat the above argument. This works unless the minimum chord is of the form  $e'' = v_3 v_{i+2}$ . Then, we can consider e'' as the shortest chord and try to repeat the original argument. Again this works unless the minimum chord is of the form  $e''' = v_4 v_{i+3}$ . But now, the cycle  $v_1$ ,  $v_2$ ,  $v_3$ ,  $v_4$ ,  $v_{i+3}$ ,  $v_{i+2}$ ,  $v_{i+1}$ ,  $v_i$ ,  $v_1$ is proper (unless the cycle has order exactly 8) and has chords e' and e''. In this case, the fifth chord lies between  $[v_1, v_4]$ and  $[v_5, v_8]$  and no matter how it is placed, a proper doubly chorded cycle is easy to find. This completes the proof of the Lemma.  $\Box$ 

One additional lemma will be useful, as when it applies it will allow us to transfer our  $\sigma_2$  condition into a more applicable degree condition.

**Lemma 3.** Suppose that *C* is a doubly chorded cycle with  $|C| \ge 7$  and containing no proper doubly chorded cycle. Then the complement  $\overline{C}$  of *C* can be covered by a collection of connected vertex-disjoint regular subgraphs (not necessarily induced and not necessarily of the same degree of regularity for different subgraphs) of order at least two.

**Proof.** Note that *C* has at most four chords by Lemma 2. In *C*, the degree of any vertex is at most four, since any vertex with at least three chords out of it produces a proper doubly chorded cycle. Therefore,  $\delta(\overline{C})$  is at least |V(C)| - 5. Hence, if  $|V(C)| = n \ge 10$ , then by Dirac's Theorem,  $\overline{C}$  is Hamiltonian and therefore covered by a 2-regular graph.

For the remaining cases (n = 7, 8, 9) we use the above observations to note that we may assume that  $\overline{C}$  is not Hamiltonian (or we are done), and has minimum degree  $\delta(\overline{C}) \ge n - 5$  and maximum degree  $\Delta(\overline{C}) \le n - 3$ .

Suppose then that n = 9. Then  $\delta(\overline{C}) \ge 4$  and  $\Delta(\overline{C}) \le 6$ . Then the graph  $\overline{C}$  is traceable by Ore's Theorem [11]. Let  $P : v_1, v_2, \ldots, v_9$  be such a path. We note that if  $v_1$  is adjacent to  $v_i$ , then  $v_9$  is not adjacent to  $v_{i-1}$ , or else  $\overline{C}$  is Hamiltonian. Further, if  $v_1$  is adjacent to any of  $v_3, v_5$  or  $v_7$ , then an odd cycle and a matching (or single edge) cover  $V(\overline{C})$  and we are again done. Thus, we may assume these adjacencies do not occur. Similarly, we may assume  $v_9$  is not adjacent to  $v_3, v_5$  or  $v_7$ .

As  $v_1$  has three adjacencies on *P* besides  $v_2$ , these adjacencies must be to  $v_4$ ,  $v_6$  and  $v_8$ . Similarly,  $v_9$  must be adjacent to  $v_2$ ,  $v_4$  and  $v_6$ . Now  $v_7$  has degree at least 4 and is not adjacent to either  $v_1$  or  $v_9$ . If  $v_7$  is adjacent to  $v_5$ , then  $v_1$ ,  $v_2$ ,  $v_3$ ,  $v_4$ ,  $v_1$  and  $v_5$ ,  $v_7$ ,  $v_8$ ,  $v_9$ ,  $v_6$ ,  $v_5$  are two cycles covering the graph. A similar argument applies if  $v_7$  is adjacent to  $v_3$ . Thus,  $v_7$  must be adjacent to  $v_2$  and  $v_4$ . A similar argument shows that  $v_5$  must be adjacent to  $v_2$  and  $v_8$ . Finally, a similar argument shows  $v_3$  is adjacent to  $v_6$  and  $v_8$ , or we have the components we seek. But the graph we now have is  $K_{4,5}$  which cannot be  $\overline{C}$  as its complement is disconnected, not a cycle. Hence, a contradiction is reached completing the case when n = 9.

Next suppose n = 8 with  $\delta(\overline{C}) \ge 3$  and  $\Delta(\overline{C}) \le 5$ . If  $\overline{C}$  is not connected, then the graph is  $K_4 \cup K_4$  and these two components suffice. So assume  $\overline{C}$  is connected. If  $\overline{C}$  is traceable, then an argument similar to the n = 9 case shows the regular components exist. Thus, we may assume that  $\overline{C}$  is not traceable.

Let *P* be a longest path in  $\overline{C}$ . Suppose |P| = 7, say  $P : v_1, v_2, \ldots, v_7$ . Then  $v_8$  is not on this path but has at least three adjacencies on this path, and clearly, these must be  $v_2, v_4$  and  $v_6$ . Now  $v_1$  must have two more adjacencies on *P* and these are not  $v_3, v_5$  or  $v_7$  or a longer path would exist. Thus,  $v_1$  is adjacent to  $v_4$  and  $v_6$ . By symmetry,  $v_7$  is adjacent to  $v_2$  and  $v_4$ .

Now consider  $v_3$  and  $v_5$ , each having an additional adjacency on *P*. If they are themselves adjacent, then the cycle  $v_1$ ,  $v_2$ ,  $v_8$ ,  $v_6$ ,  $v_7$ ,  $v_4$ ,  $v_1$  and the edge  $v_3v_5$  form the two regular components. Otherwise, we already know  $v_1v_3$ ,  $v_3v_7$ ,  $v_1v_5$ 

and  $v_7v_5$  do not exist, thus,  $v_3v_6$  and  $v_2v_5$  are edges of *G*. But the graph thus formed cannot be  $\overline{C}$  as it implies *C* is not connected. Thus, the longest path cannot contain seven vertices. If the longest path *P* has six vertices, then the two vertices off this path cannot be adjacent or  $\overline{C}$  would contain a perfect matching. But then, each of the vertices off the path would be adjacent to three of  $v_2$ ,  $v_3$ ,  $v_4$ ,  $v_5$  and thus have consecutive adjacencies on *P*. But then *P* is not the longest path. Similar arguments show |P| cannot be five or less. Thus, this case is completed.

If n = 7 and  $\delta(\overline{C}) \ge 2$  and  $\Delta(\overline{C}) \le 4$  we note that the graph must contain at least 10 edges. This follows since if  $\overline{C}$  had at most 9 edges, then *C* would have at least 12 edges and hence be a cycle with at least 5 chords. But then a proper doubly chorded cycle would exist by Lemma 2. By inspection of the list of graphs of order 7 and size 10 or more contained in [14], either the graph admits the components we seek or cannot be the complement of *C* for one of the several reasons for contradictions given in earlier cases of this proof. Thus, we conclude that such a cover always exists.  $\Box$ 

#### **Lemma 4.** There is no vertex $x \in H$ and cycle $C \in \mathbb{C}$ so that $\deg_C(x) \ge 5$ .

**Proof.** Suppose that  $deg_C(x) \ge 5$ . Then if  $|V(C)| \ge 6$ , it is easy to find a shorter doubly chorded cycle, contradicting our choice of C. If |V(C)| = 5, then a  $K_4$  would be formed using x, again contradicting our choice of C.  $\Box$ 

**Lemma 5.** Suppose some vertex  $x \in H$  has  $\deg_{C}(x) = 4$  for some  $C \in C$ . Then  $|C| \le 5$ .

**Proof.** Suppose  $|V(C)| \ge 9$ . Then it is easy to find a shorter doubly chorded cycle using *x* and omitting a segment of *C* with at least two vertices.

Next suppose |V(C)| = 8, say  $V(C) = \{v_1, v_2, ..., v_8\}$ . Then, without loss of generality,  $N_C(x) = \{v_1, v_3, v_5, v_7\}$  or a shorter doubly chorded cycle can again be found. Now note that if any chord of *C* is contained within  $v_1, v_2, ..., v_5$ , then a shorter doubly chorded cycle exists using these vertices and *x*. Similarly, if a chord exists within the vertices  $v_3, v_4, ..., v_7$  or within  $v_5, v_6, v_7, v_8, v_1$  or  $v_7, v_8, v_1, v_2, v_3$  we can again find a shorter doubly chorded cycle. Thus, the two chords of *C* must be  $v_2v_6$  and  $v_4v_8$ . But now,  $x, v_3, v_4, v_5, v_6, v_2, v_1, x$  is a 7-cycle with chords  $xv_5$  and  $v_2v_3$ , contradicting our choice of *C*.

Next assume |V(C)| = 7. Let  $V(C) = \{v_1, v_2, ..., v_7\}$  and, without loss of generality, assume  $N_C(x) = \{v_1, v_3, v_5, v_7\}$ . As before, a chord in any of the segments  $v_1 - v_5$ ,  $v_3 - v_7$ ,  $v_5 - v_1$ , or  $v_7 - v_3$  produces a shorter doubly chorded cycle. But now, the only possible chord which does not produce such a cycle is  $v_2v_6$ . Hence, this case also cannot happen.

Now suppose |V(C)| = 6, say  $V(C) = \{v_1, v_2, ..., v_6\}$ . Note that no vertex off *C* can be adjacent to four consecutive vertices of *C*, or a shorter doubly chorded cycle would exist. First assume  $N_C(x) = \{v_1, v_3, v_5, v_6\}$ . Note that no chord can be contained within the vertices  $v_3 - v_6$  or within the vertices  $v_6 - v_3$  or a shorter doubly chorded cycle is immediate. Thus, the only possible chords are  $v_1v_5$ ,  $v_1v_4$ ,  $v_2v_4$  and  $v_2v_5$ . If  $v_1v_5$  is a chord, then a  $K_4$  exists, again a contradiction. If  $v_1v_4$  is a chord, then x,  $v_1$ ,  $v_4$ ,  $v_5$ ,  $v_6$ , x is a 5-cycle with chords  $v_1v_6$  and  $xv_5$ , a contradiction. Thus, there are no chords from  $v_1$ . If the chords are  $v_2v_4$  and  $v_2v_5$ , then x,  $v_5$ ,  $v_4$ ,  $v_2$ ,  $v_3$ , x is a 5-cycle with chords  $v_3v_4$  and  $v_2v_5$ , a contradiction.

Next assume that  $N_C(x) = \{v_1, v_2, v_4, v_5\}$ . Note that this is the only other case we must consider. Now there can be no chords within the vertices  $v_1 - v_4$ , or within  $v_2 - v_5$  or  $v_4 - v_1$  or  $v_5 - v_2$  or a shorter doubly chorded cycle exists. Thus, the only possible chord is  $v_3v_6$ , a contradiction completing this case.  $\Box$ 

**Lemma 6.** If *C* contains a cycle of length at least 7, then  $|H| \ge 9$ .

**Proof.** Recall,  $H = G - (C_1 \cup C_2 \cup \cdots \cup C_{k-1})$  is the remainder after a minimal set of k-1 vertex disjoint cycles, C is removed from G. Order the cycles  $C_i \in C$  so that  $|C_1| \ge |C_2| \ge \cdots |C_{k-1}|$ . Suppose  $|C_1| \ge 7$  and  $|H| \le 8$ . We show that a shorter cycle system exists, proving the claim.

By Lemma 3,  $C_1$  can be covered by a collection  $R_i$ , each a  $d_i$ -regular subgraph of the complement of  $C_1$  and for each such subgraph

$$\sum_{e=xy\in R_i} (\deg x + \deg y) = d_i \sum \deg x \ge \sigma_2(G)d_i|R_i|/2.$$

Thus,  $\sum_{x \in V(C_1)} \text{ deg } x \ge \sigma_2(G) \sum |R_i|/2 = \sigma_2(G)|C_1|/2$ . By Lemma 2 and minimality,  $e(C_1) \le |V(C_1)| + 4$ , so

$$e(C_1, G - C_1) \ge |V(C_1)|(\sigma_2(G)/2 - 1) - 4 \ge 3|C_1|(k - 2) + 4.5|C_1| - 4.$$

By Lemma 5, if any vertex of *H* has 4 or more adjacencies on  $C_1$ , then a doubly chorded cycle smaller than  $C_1$  exists, a contradiction. Now

$$e(C_1, G - (C_1 \cup H)) \ge 3|C_1|(k-2) + 4.5|C_1| - 4 - 3|H|.$$

Combining with our assumptions that  $|C_1| \ge 7$  and  $|H| \le 8$  implies that  $e(C_1, G - (C_1 \cup H)) > 3|C_1|(k-2)$ .

Therefore,  $C_1$  sends more than  $3|C_1| + 1$  edges to another cycle in C. Now, by Lemma 1, we obtain two smaller cycles, replacing two cycles of C, contradicting our choice of C.  $\Box$ 

Ν	$z \in X_N$	Cycle on $(C - z) \cup \{y\}$	Chords	Satisfies
$v_1, v_2, v_3$	$v_4$	$y, v_1, v_2, v_5, v_3, y$	$yv_2, v_2v_3$	2( <i>b</i> )
	$v_1$	$y, v_2, v_5, v_4, v_3, y$	$v_2v_3, v_3v_5$	Edge: $v_2v_3$
$v_1, v_2, v_4$	$v_4$	$y, v_1, v_5, v_3, v_2, y$	$v_2v_1, v_2v_5$	2( <i>a</i> )
	$v_3$	$y, v_4, v_5, v_1, v_2, y$	$yv_1, v_2v_5$	
	$v_1$	$y, v_2, v_3, v_5, v_4, y$	$v_2v_5, v_3v_4$	
$v_1, v_3, v_4$	$v_1$	$y, v_3, v_2, v_5, v_4, y$	$v_3v_5, v_3v_4$	2( <i>b</i> )
	$v_2$	$y, v_1, v_5, v_4, v_3, y$	$yv_4, v_3v_5$	Edge: $v_3v_4$
$v_1, v_3, v_5$	$v_4$	$y, v_1, v_2, v_3, v_5, y$	$yv_3, v_2v_5$	2( <i>b</i> )
	$v_2$	$y, v_1, v_5, v_4, v_3, y$	$yv_5, v_3v_5$	Edge: $v_1v_5$
$v_1, v_4, v_5$	$v_2$	$y, v_1, v_5, v_3, v_4, y$	$yv_5, v_4v_5$	2( <i>a</i> )
	$v_3$	$y, v_1, v_2, v_5, v_4, y$	$yv_5, v_1v_5$	
	$v_4$	$y, v_1, v_2, v_3, v_5, y$	$v_1v_5, v_2v_5$	
	$v_1$	$y, v_4, v_3, v_2, v_5, y$	$v_3v_5, v_4v_5$	
$v_2, v_3, v_4$	$v_1$	$y, v_2, v_5, v_4, v_3, y$	$v_2v_3, v_3v_5$	2( <i>b</i> )
	$v_4$	$y, v_2, v_1, v_5, v_3, y$	$v_2v_3, v_2v_5$	Edge: $v_2v_3$

Table 1
Cycles for chords $v_2v_5$ and $v_3v_5$

**Corollary 1.** Without loss of generality,  $|H| \ge 6$ . Furthermore, if C contains at least one 5-cycle then  $|H| \ge 7$ , and if C contains a 4-cycle or two 5-cycles then  $|H| \ge 8$ .

**Proof.** This follows as  $|H| = |V(G)| - \sum_{C_i \in \mathcal{C}} |C_i|$  and  $|V(G)| \ge 6k$  by assumption. Under the assumption that |H| < 9, then the maximum cycle length is 6.  $\Box$ 

Beyond this basic control over |H|, we require some additional lemmas describing when and how we may perform exchanges that preserve |H|, allowing us to assert further control over the properties of *H*. A simple fact, following immediately from the preceding lemmas is the following.

**Lemma 7.** Suppose  $x, y \in V(H)$  are such that  $\deg_C(x) + \deg_C(y) \ge 7$  for some cycle  $C \in C$ . Then, without loss of generality  $\deg_C(x) = 4$  and  $\deg_C(y) \ge 3$ .

A more complicated version is the following. While the ultimate statement is quite technical, it is set up in a way to conveniently use later. On an initial reading one might ignore the conditions 2(a), 2(b) and 2(c) which the lemma asserts, instead focusing on the first condition. A number of immediate consequences are described below, and are also easier on the reader.

**Lemma 8.** Suppose  $C \in C$  is such that there is some vertex  $x \in H$  with four neighbors in C. Suppose  $y \in H$  is incident to  $N \subseteq C$  with |N| = 3. Let  $X_N = \{z : (C - z) \cup \{y\}\}$  is a DCC.

1.  $(N_C(x) \setminus N) \subseteq X_N$ 

2. Also, with a single exception, the following occurs:  $|N_C(x) \cap X_N| \ge 2$  and at least one of

(a)  $|N_C(x) \cap X_N| > 2$ 

- (b)  $|N_C(x) \cap X_N| = 2$  and  $N \setminus (N_C(x) \cap X_N)$  spans an edge of *C* (this edge may be a chord)
- (c)  $C \setminus (N \cup (C \setminus (N_C(x) \cap X_N))) = \{r\}$ , and there exists an  $s \in (C \setminus (N_C(x) \cap X_N))$  with  $r \sim s$  and if  $x \neq y$ ,  $\{x, y\} \cup (C \setminus \{r, s\})$  induces a DCC.

Furthermore, in the exceptional case it is still the case that  $X_N = C \setminus N$ .

**Remark.** One may take x = y in the application of this lemma—in which case N can be any three vertices of  $N_C(x)$ .

**Proof.** Suppose |V(C)| = 5, say  $V(C) = \{v_1, v_2, v_3, v_4, v_5\}$ . Without loss of generality assume  $N_C(x) = \{v_1, v_2, v_3, v_4\}$ . Then  $v_1v_3$  and  $v_2v_4$  cannot be chords of *C* or a  $K_4$  would exist, contradicting our choice of cycles. Thus the two (or more) chords of *C* must come from  $v_1v_4$ ,  $v_2v_5$ , and  $v_3v_5$ . To complete the proof of the lemma, we illustrate for all possible *N*, and  $z \in (C \setminus N)$  the DCC formed on  $(C-z) \cup \{y\}$ . Note that we do not include *N* which induces a triangle in *C*, as then there would be a  $K_4$  which contradicts minimality of *C*. For each *N*, we exhibit sufficient vertices in  $X_N$  to verify that one of 2(a), 2(b), or 2(c) holds. We label which of these it holds, and in case (b) we list the edge, in case (c) we list  $v_1$ ,  $v_2$ .

First we consider the case where the chords are  $v_2v_5$  and  $v_3v_5$ , in Table 1.

Next we consider the case where the chords are  $v_1v_4$  and  $v_3v_5$  in Table 2.

Note that the singular case referenced in the statement is the case where  $N = v_1, v_3, v_4$ .

The case with chords  $v_1v_4$  and  $v_2v_5$  is completely symmetric.

If |C| = 4, the statement is completely clear as  $\{y\} \cup N$  induces a DCC, and hence the singular  $z \in (C \setminus N)$  is the desired N.  $\Box$ 

Two immediate consequences are exchange Lemmas which state instances in which a vertex can be swapped from *H* into a cycle.

Ν	Ζ	Cycle on $(C - z) \cup \{y\}$	Chords	Satisfies
$v_1, v_2, v_3$	$v_2$	$y, v_1, v_5, v_4, v_3, y$	$v_1v_4, v_3v_5$	2(c)
	$v_4$	$y, v_1, v_5, v_3, v_2, y$	$v_1v_2, yv_3$	$r = v_4, s = v_5$
	$v_5$	$y, v_1, v_4, v_3, v_2, y$	$yv_3, v_2v_1$	
$v_1, v_2, v_4$	$v_2$	$y, v_1, v_5, v_3, v_4, y$	$v_1v_4, v_4v_5$	2( <i>b</i> )
	$v_3$	$y, v_2, v_1, v_5, v_4, y$	$yv_1, v_1v_4$	Edge: $v_1v_4$
	$v_5$	$y, v_1, v_2, v_3, v_4, y$	$yv_2, v_1v_4$	
$v_1, v_2, v_5$	$v_4$	$y, v_1, v_2, v_3, v_5, y$	$yv_2, v_1v_5$	2( <i>a</i> )
	$v_3$	$y, v_2, v_1, v_4, v_5, y$	$yv_1, v_1v_5$	
	$v_2$	$y, v_1, v_4, v_3, v_5, y$	$v_1v_5, v_4v_5$	
$v_1, v_3, v_4$	$v_2$	$y, v_1, v_5, v_4, v_3, y$	$v_1v_4, yv_4$	Exception
	$v_5$	$y, v_1, v_2, v_3, v_4, y$	$yv_3, v_1v_4$	
$v_1, v_3, v_5$	$v_2$	$y, v_1, v_5, v_4, v_3, y$	$v_1v_4, v_3v_5$	2( <i>b</i> )
	$v_4$	$y, v_1, v_2, v_3, v_5, y$	$v_1 v_5, y v_3$	Edge: $v_1v_5$
$v_2, v_3, v_4$	$v_1$	$y, v_2, v_3, v_5, v_4, y$	$yv_3, v_3v_4$	2( <i>b</i> )
	$v_2$	$y, v_3, v_5, v_1, v_4, y$	$v_3v_4, v_4v_5$	Edge: $v_3v_4$
	$v_5$	$y, v_2, v_1, v_4, v_3, y$	$yv_4, v_2v_3$	
$v_2, v_3, v_5$	$v_1$	$y, v_2, v_3, v_4, v_5, y$	$yv_3, v_3v_5$	2( <i>a</i> )
	$v_2$	$y, v_3, v_4, v_1, v_5, y$	$v_3v_5, v_4v_5$	
	$v_4$	$y, v_2, v_1, v_5, v_3, y$	$yv_5, v_2v_3$	
$v_2, v_4, v_5$	$v_1$	$y, v_2, v_3, v_4, v_5, y$	$yv_4, v_3v_5$	2( <i>b</i> )
	$v_3$	$y, v_2, v_1, v_5, v_4, y$	$yv_5, v_1v_4$	Edge: $v_4v_5$

Table 2	
Cycles for chords $v_1v_4$ and $v_3v_5$ .	

**Lemma 9** (Single Bypass Lemma). Suppose  $x \in V(H)$  and  $C \in C$  satisfy  $\deg_C(x) \ge 4$ . Then for any vertex  $z \in C$ ,  $(C - z) \cup \{x\}$  is a doubly chorded cycle.

**Proof.** This follows immediately from Lemma 8. For  $z \in N_C(x)$ , applying the lemma with  $N = (N_C(x) \setminus \{z\})$  yields the conclusion by the first bulleted conclusion. In the case where |C| = 5, and  $z \notin N_C(x)$  a DCC is clear where both chords are incident to x.  $\Box$ 

**Lemma 10.** Suppose  $x, y \in V(H)$  are nonadjacent vertices with  $\deg_C(x) = 4$  and  $\deg_C(y) = 3$  for some  $C \in C$ . Then there exist vertices  $z_x, z_y \in V(C)$  such that  $z_x$  is adjacent to x and  $z_y$  is adjacent to y and both  $(C - z_x) \cup \{y\}$  and  $(C - z_y) \cup \{x\}$  induce doubly chorded cycles.

**Proof.** This follows almost immediately from Lemma 8. Note that for  $N = N_C(y)$ , that some vertex in  $N_C(x)$  is always one of the admissible *z*—this is  $z_x$ . It is easily seen that any vertex in  $N_C(y)$  can serve as  $z_y$  by Lemma 9.  $\Box$ 

In some instances in the main part of the proof, more complicated exchanges are necessary as well. Our main tool is the following lemma which gives conditions on when two vertices in *H* may be exchanged for two vertices in a cycle.

**Lemma 11** (Double Bypass Lemma). Suppose C is a doubly chorded 5 cycle. Further suppose that there are vertices  $x, y, z, w \notin C$ with  $\deg_C(x) = 4 \deg_C(y) = \deg_C(z) = 2$  and  $\deg_C(w) = 3$ . Suppose that  $N_C(y) \cap N_C(w) = \emptyset$  and  $N_C(z) \subseteq N_C(w)$ . Further suppose that neither  $N_C(x)$  nor  $N_C(w)$  spans a triangle in C, and neither  $N_C(z)$  nor  $N_C(y) \cap N_C(x)$  spans an edge of C. Then there exist two vertices  $u, v \in C$  so that  $u \sim y, v \sim z, u \sim v$  and so that  $(C \setminus \{u, v\}) \cup \{x, w\}$  is a DCC.

**Proof.** Let  $V(C) = \{c_1, c_2, c_3, c_4, c_5\}$  and  $N_C(x) = \{c_1, c_2, c_3, c_4\}$ . Then, without loss of generality, the chords are either  $(c_1c_4, c_3c_5)$  or  $(c_2c_5, c_3c_5)$ . Note that the facts that  $N_C(z) \subseteq N_C(w)$  and  $N_C(z)$  cannot span an edge of C mean that if  $N_C(w)$  consists of three consecutive vertices, then  $N_C(z)$  contains the first vertex and the third vertex in  $N_C(w)$ , and if  $N_C(w)$  consists of two consecutive vertices plus a third, nonconsecutive vertex, then the third vertex must be in  $N_C(z)$ . We present Table 3 displaying u and v for various  $N_C(w)$ . Note that  $N_C(w)$  cannot span a triangle. Also, as many  $N_C(w)$  are the same by symmetry, we only include one representative from each symmetry class.  $\Box$ 

We finally reach the true crux of the proof: In this lemma we build on the exchange lemmas established above, resulting in a strong conclusion about the structure that we can assume on *H*.

**Lemma 12.** Without loss of generality H contains a Hamiltonian path, on  $v_1, \ldots, v_{|H|}$ . Furthermore, this can be chosen so that either  $v_1 \sim v_{|H|}$  or  $v_2 \sim v_{|H|}$ .

**Proof.** Claim 0. We may assume, without loss of generality, that *H* is connected.

**Proof of Claim 0.** If not, then as long as *H* is disconnected, there exist nonadjacent vertices *x* and *y* in different components and with at least 7 edges to one cycle of *C* by Lemma 7, and by repeated application of the Exchange Lemma (Lemma 10), *H* can be made connected.  $\Box$ 

$N_C(w)$	( <i>u</i> , <i>v</i> )	Cycle on $(C \setminus \{u, v\}) \cup \{x, w\}$	Chords
$c_1, c_2, c_3$	$(c_5, c_1)$	$x, c_2, w, c_3, c_4, x$	$xc_3, c_2c_3$
$c_1, c_2, c_4$	$(c_5, c_4)$	$x, c_1, w, c_2, c_3, x$	$xc_2, c_1c_2$
$c_1, c_2, c_5$	$(c_4, c_5)$	$x, c_1, w, c_2, c_3, x$	$xc_2, c_1c_2$
$c_1, c_3, c_5$	$(c_2, c_3)$ if chords $c_1c_4$ and $c_3c_5$	$x, c_1, w, c_5, c_4, x$	$c_1c_4, c_1c_5$
	$(c_4, c_3)$ if chords $c_2c_5$ and $c_3c_5$	$x, c_1, w, c_5, c_2, x$	$c_1c_2, c_1c_5$
$c_1, c_4, c_5$	$(c_2, c_1)$	$x, c_3, c_5, w, c_4, x$	<i>c</i> <sub>3</sub> <i>c</i> <sub>4</sub> , <i>c</i> <sub>4</sub> <i>c</i> <sub>5</sub>
$c_2, c_3, c_5$	$(c_4, c_5)$	$x, c_1, c_2, w, c_3, x$	$xc_2, c_2c_3$

**Table 3** Two vertices (u, v) and cycles for chords  $(c_1c_4, c_3c_5)$  or  $(c_2c_5, c_3c_5)$ .

Suppose, of all connected H with minimal cycle system size, we choose the one with the longest path. Let P be a longest path in H. If such exists, we choose P so that the first vertex is incident to one of the last two vertices—in which case we will be done.

**Claim 1.** We may assume, without loss of generality, that the first and last vertices of *P* are of degree at most 2.

**Proof of Claim 1.** Let  $P = v_1, v_2, \ldots, v_t$ . There are possibly many spanning paths on this vertex set; we choose P so the degree of  $v_1$  and  $v_t$  is at most 2 if possible. Suppose it is not possible that the degree of  $v_1$  is 2. Then every vertex which can start the path (we call these *start vertices*) has degree at least 3. Of all such paths, choose  $v_1$  so that its neighbors are  $v_i$  and  $v_j$  with j as large as possible. Then  $v_{i-1}$  is a potential start to the path. By our assumption  $\deg(v_{i-1}) \ge 3$  and the maximality of P, it must send a chord into the path and by our assumption that  $v_j$  is as large as possible, it must send the chord into the cycle  $v_1, \ldots, v_j$ , which is now easily seen to be a DCC with the other chord being  $v_1v_i$ . Thus if H does not contain a DCC, we may assume that  $\deg(v_1) \le 2$ .

Arguing similarly, taking *P* to be the arrangement of  $v_1, \ldots, v_t$  with  $v_1$  to be the fixed vertex of degree 2, and looking at potential end vertices of the path we observe we may assume that  $v_t$  has degree at most 2 as well.  $\Box$ 

**Claim 2.** We may assume that if  $deg(v_1) = 2$  and  $deg(v_2) = 4$ , and  $v_q$  is the highest indexed neighbor of  $v_2$ , then there is a vertex  $v_r$  of degree 2 with 1 < r < q.

**Proof of Claim 2.** We build on the proof of Claim 1, to observe that there are at least two start vertices of degree 2 unless the first four vertices induce a  $K_4^-$  where  $v_1$  and  $v_4$  are the non adjacent vertices of the  $K_4^-$ . Let  $v'_1, v'_2, \ldots, v'_t$  be the path *P* on vertex set  $v_1, \ldots, v_t$  so that  $v_t = v'_t$ , but  $v'_1$ 's neighbor is  $v'_i$  for *i* as large as possible.

If deg<sub>p</sub>( $v'_1$ ) = 3, then  $v'_1$  has neighbors  $v'_i$  and  $v'_j$  where j < i. Then both  $v'_{i-1}$  and  $v'_{j-1}$  are start vertices. If  $j \neq i-1$ , either of these vertices having degree 3 would create at DCC, so we have two start vertices of degree 2 as claimed. Hence j = i - 1. But then  $v'_2$  is a start vertex, as one can take  $v'_2, v'_3, \ldots, v'_j, v'_1, v'_i, v'_{i+1}, \ldots$ . Here, both  $v'_2$  and  $v'_{i-1}$  must have degree 2 or we have a DCC. Thus we are done unless 2 = i - 1, that is unless i = 3. In this case  $v'_1, v'_2, v'_3, v'_4$  are a  $K_4^-$ . Since  $v_1, \ldots, v_t$  is obtained by taking the same path with  $v'_2$  as  $v_1$ , we have the purported structure in this case.

If  $\deg_P(v'_1) = 2$ , then  $v'_1$  has neighbor  $v'_{i'}$ . If  $\deg_P(v'_{i-1}) = 2$ , we are done. Otherwise,  $v'_{i-1}$  has another neighbor on P,  $v'_j$ . Note that j < i by the maximality of i. Then  $v'_{j+1}$  is also a start vertex as witnessed by the path  $v'_{j+1}, v'_{j+2}, \ldots, v'_{i-1}, v'_j, v'_{j-1}, \ldots, v'_1, v'_i, v'_{i+1}, \ldots$ . Hence  $v'_{j+1}$  cannot have any neighbors other than  $v'_j$  and  $v'_{j+2}$  in  $v'_1, \ldots, v'_i$  without creating a DCC and none in  $v'_{i+1}, v'_{i+2}, \ldots$  by the maximality of i. Thus  $v'_{j+1}$  has degree 2 and  $v'_1$  and  $v'_{j+1}$  are the two degree 2 vertices.

If the initial four vertices give the purported  $K_4^-$  in such a way, note that  $v_2$  cannot have any additional neighbors without creating a DCC, so if deg $(v_2) = 4$  we are not in this case.

Note that  $\{v_1, \ldots, v_i\} = \{v'_1, \ldots, v'_i\}$  and thus the two exhibited vertices of degree 2 are  $v_1$  and  $v_s$  for some s < i. If  $v_t$ , the highest indexed neighbor of  $v_2$  has  $t \le i$ , then  $v'_1, \ldots, v'_i$  contains both of the chords from  $v_2$  and is hence a DCC. Thus s < i < t and we have the conclusion of the claim.  $\Box$ 

Now we turn to proving that the path is Hamiltonian. Our first step is to rule out the existence of small degree vertices off of the path.

**Claim 3.** deg<sub>*H*</sub>(v)  $\geq$  4 for all  $v \in (H \setminus P)$ .

**Proof of Claim 3.** We may assume that the vertices in *P* do not span a cycle, or we are done by maximality of *P*. Thus  $v_1$  is not incident to  $v_t$ . Suppose  $v \notin P$ . Note  $v \nsim v_1$  and  $v \nsim v_t$ . If deg<sub>*H*</sub>(v)  $\leq$  3, then consider the three pairs { $v_1, v_t$ }, { $v_1, v_1$ , { $v_t, v_t$ }. All three of these pairs are of pairwise non-adjacent vertices. Applying our  $\sigma_2$  condition to  $v_1$  and v yields

$$\deg(v_1) + \deg(v) \ge (6k - 1) = 6(k - 1) + 5.$$
<sup>(1)</sup>

If  $\deg_{C}(v_{1}) + \deg_{C}(v) \ge 7$  for any  $C \in C$  the exchange lemma implies that the path can be lengthened; exchanging v for a vertex incident to  $v_{1}$ . That fact, along with (1) imply that  $v_{1}$  and v together have *exactly* 6 incidences into every  $C \in C$ . The same holds for  $v_{t}$  and v.

Now applying the  $\sigma_2$  condition to  $v_1$  and  $v_t$ , and noting that  $\deg_H(v_1) + \deg_H(v_t) \le 4$ , we observe that there is some cycle *C* with  $\deg_C(v_1) + \deg_C(v_t) \ge 7$ . The fact that  $v_1$  along with v and also  $v_2$  along with v have the same number of edges into every cycle imply that  $\deg_C(v_1) = \deg_C(v_t) = 4$  and  $\deg_C(v) = 2$ , and  $|C| \le 5$ . Note that  $v_t$  and v must have a common neighbor in *C*, and the Single Bypass Lemma implies that we may exchange  $v_1$  for some  $z_{v_1} \in C$  with  $z_{v_1} \sim v_t$  and  $z_{v_1} \sim v$ . This contradicts the maximality of the path.  $\Box$ 

## **Claim 4.** Every component of $H \setminus P$ has cardinality at least 4.

**Proof of Claim 4.** By Claim 3,  $\deg_H(v) \ge 4$  for every vertex not on *P*. Suppose *X* is a connected component of  $H \setminus P$ . If |X| = 1, then the single vertex has four neighbors on the path yielding a DCC. If |X| = 2 then the pair of vertices in *X* each have 3 neighbors on the path and this is settled by the  $(3, 3) \hookrightarrow P$  case of Lemma 14. The |X| = 3 case is similarly settled by the  $(2, 2, 2) \hookrightarrow P$  case of Lemma 14. Thus every component outside of  $H \setminus P$ , if any, is of cardinality at least 4.  $\Box$ 

**Claim 5.** We may assume that one of the edges  $v_1 \sim v_t$ ,  $v_2 \sim v_t$  or  $v_1 \sim v_{t-1}$  is present.

Note that Claim 5 completes the Lemma, as it along with the maximality of P rule out any components in  $H \setminus P$ .

We say that *P* is set up if either  $v_1$ ,  $v_2$ ,  $v_3$ ,  $v_4$  or  $v_{t-3}$ ,  $v_{t-2}$ ,  $v_{t-1}$ ,  $v_t$  induces a  $K_4^-$  so that  $v_1$  (or  $v_t$ ) is a vertex of degree 2. We assume that the *P* meets all the assumed qualifications:

(†) *P* is of maximum length, has end vertices of degree 2, and all components of  $H \setminus P$  have cardinality at least 4 and that subject to these, if possible, *P* is set up.

## Proof of Claim 5. Suppose none of the purported edges are present.

Note that deg<sub>H</sub>( $v_1$ ), deg<sub>H</sub>( $v_t$ )  $\leq 2$  by assumption and deg<sub>H</sub>( $v_2$ ), deg<sub>H</sub>( $v_{t-1}$ )  $\leq 4$  as otherwise there would be a doubly chorded cycle. Since  $v_1 \not\sim v_{t-1}$  and  $v_2 \not\sim v_t$ , we may apply our  $\sigma_2$  condition to see that

$$\deg(v_1) + \deg(v_2) + \deg(v_{t-1}) + \deg(v_t) \ge 2(6k-1) = 12(k-1) + 10.$$
(2)

**Case 1:**  $\deg_H(v_1) + \deg_H(v_2) + \deg_H(v_{t-1}) + \deg_H(v_t) < 10.$ 

In this case, combining with (2) and averaging over cycles implies that there are at least 13 edges between  $v_1, v_2, v_{t-1}$ and  $v_t$  and some  $C \in C$ . Note this means that  $\deg_C(v_i) \ge 4$  for some  $v_i$  and hence  $|C| \le 5$ . Without loss of generality, we may assume  $\deg_C(v_1) + \deg_C(v_2) \ge \deg_C(v_{t-1}) + \deg_C(v_t)$ . This means that  $\deg_C(v_1) + \deg_C(v_2) \ge 7$  and hence either  $\deg_C(v_1) = 4$  or  $\deg_C(v_2) = 4$ , while the other vertex must have degree at least 3.

Suppose first that  $\deg_C(v_1) = 4$ . If  $\deg_C(v_2) + \deg_C(v_t) > 5$  or  $\deg_C(v_2) + \deg_C(v_{t-1}) > 5$  we are done. Indeed, this implies that for some  $i \in \{t - 1, t\}, v_2$  and  $v_i$  have a common neighbor  $z \in C$ . By the Single Bypass Lemma, we can exchange  $v_1$  for that z, obtaining the desired path structure. One of these always occurs however, as one of  $\deg_C(v_t)$  or  $\deg_C(v_{t-1})$  is at least

$$\frac{13 - 4 - \deg_{\mathbb{C}}(v_2)}{2} = \frac{9 - \deg_{\mathbb{C}}(v_2)}{2}$$

Now assume  $\deg_{C}(v_1) = 3$  but  $\deg_{C}(v_2) = 4$ . If |C| = 4, and one of  $v_t$  or  $v_{t-1}$  is incident to the vertex in C which is not incident to  $v_1$  may finish by exchanging  $v_1$  for that vertex (which is also incident to  $v_2$ ). Thus, designating the vertices on C to be  $c_1, c_2, c_3, c_4$  we may assume that the neighbors of  $v_1, v_{t-1}$  and  $v_t$  are all  $\{c_1, c_2, c_3\}$ . If either end is set up, then we are done. Indeed, suppose  $v_1, v_2, v_3, v_4$  is set up: Depending on the configuration of the  $K_4^-$ , either  $v_1, c_1, v_2, v_3, v_4, v_1$  is a cycle with chords  $v_1v_2$  and  $v_2v_4$  or  $v_1, c_1, v_2, v_4, v_3, v_1$  is a cycle with chords  $v_1v_2$  and  $v_2v_3$ . Similar cycles exist if the other end is set up. Otherwise, we may exchange  $v_{t-1}$  and  $v_t$  for  $c_1$  and  $c_4$ , with  $c_4, c_1, v_1, v_2$  as the new initial vertices of the path—which now has a set-up end.

In order to see this yields a set-up path matching the conditions (†), we must verify that it can be chosen with end degree 2 while maintaining the  $K_4^-$  at the end. Note that it is clear  $c_4$  has degree 2 already, as if it had another adjacency in H, there would clearly be a doubly chorded cycle or longer path. We only need to verify that we can find an ordering that *ends* with a vertex of degree 2 but that still has the initial  $K_4^-$ . We run the argument of Claim 1, taking the ordering of the path starting with  $c_4$ ,  $c_1$ ,  $v_1$ ,  $v_2$  and so that the vertex ending the path has its neighbor as close to the start as possible. If the argument does not run smoothly preserving the  $K_4^-$ , there would have to be a potential start vertex for the end of the path that sent a chord into  $c_4$ ,  $c_1$ ,  $v_1$ . The vertex  $c_1$  here is the only actual option as if  $v_1$  had another neighbor on P it would form a DCC. But if  $c_1$  were incident to a start vertex we would have a structure that satisfies the conclusion of the claim.

The case |C| = 5 remains. Without loss of generality,  $v_2$  is incident to  $c_1$ ,  $c_2$ ,  $c_3$ ,  $c_4$  and  $v_1$  is incident to  $c_5$ ,  $c_1$  and one of  $c_3$  or  $c_4$ —if  $v_1$  is incident to  $c_4$  then the chords of C are  $c_5c_2$  and  $c_5c_3$ . In any case, note that there is a DCC including  $v_1$  and  $C \setminus c_i$  for any  $c_i$  not incident to  $v_1$ . That implies that the neighbors of  $v_t$  and  $v_{t-1}$  must match those of  $v_1$  as otherwise an exchange can be done— $v_1$  for a common neighbor of both  $v_2$  and one of  $v_t$  or  $v_{t-1}$ . But then a  $K_4$  is formed from  $v_t$ ,  $v_{t-1}$  along with  $c_1$  and  $c_5$ , contradicting the minimality of C.

**Case 2:**  $\deg_H(v_1) + \deg_H(v_2) + \deg_H(v_{t-1}) + \deg_H(v_t) \in \{10, 11\}.$ 

In this case, either  $v_1$ ,  $v_2$ ,  $v_{t-1}$ , and  $v_t$  cumulatively send 13 edges to *some* cycle  $C \in C$  or they cumulatively send 12 edges into *every* cycle  $C \in C$  except for perhaps one where they send 11 edges. The prior case, where they send at least 13 into one cycle is handled in Case 1. Thus we focus on the second possibility. We note that  $v_1 \not\sim v_t$  and applying the  $\sigma_2$  condition there we find some cycle *C* so that  $v_1$  and  $v_t$  cumulatively send at least 7 edges. Hence we may assume  $\deg_C(v_1) = 4$  and  $|C| \le 5$ . The vertices together send at least 11 edges to *C*, and this is what we handle in this case.

First assume that  $\deg_C(v_t) = 4$  as well. Then without loss of generality  $\deg_C(v_2) \ge 2$ . Thus  $v_2$  and  $v_t$  have a common neighbor in *C*, and exchanging  $v_1$  for the common neighbor finishes this case. Thus we may assume that  $\deg_C(v_t) = 3$ , whence  $\deg_C(v_2) + \deg_C(v_{t-1}) \ge 4$ . If  $\deg_C(v_2) > 2$  then we may exchange  $v_1$  for a common neighbor of  $v_2$  and  $v_t$ . If |C| = 4, then this covers the case where  $\deg_C(v_2) = 2$  as well.

If |C| = 4 and  $\deg_C(v_2) = 1$ , then  $\deg_C(v_{t-1}) \ge 3$ . We fail an immediate exchange only if the neighbors of  $v_{t-1}$  in C are the same as those of  $v_t$  and different than the neighbor of  $v_2$ . In this case, either one end is set up or we can set up one of the ends as in Case 2.

If |C| = 5 and  $\deg_C(v_2) = 1$  then again  $\deg_C(v_{t-1}) \ge 3$ . Let  $N = N_C(v_t)$  and we apply Lemma 8 with this choice, where  $x = v_1$  and  $y = v_t$  in this application. Let  $X_N$  be as in the statement of Lemma 8.

If  $v_{t-1}$  and  $v_1$  have a common neighbor, say z, in  $X_N$ , then by replacing C with the DCC  $(C - z) \cup \{v_t\}$  gives a Hamiltonian H along with a cycle system of the same size. This is exactly what happens when 2(a) or 2(b) occurs. If N satisfies 2(a), the facts that  $|X_N \cap N_C(v_1)| \ge 3$ , deg<sub>C</sub> $(v_{t-1}) \ge 3$ , and |C| = 5 immediately yields a common neighbor of  $v_1$  and  $v_{t-1}$  in  $X_N$ .

If *N* satisfies 2(*b*) or 2(*c*), it is theoretically possible that  $v_1$  and  $v_{t-1}$  have no common neighbor. In this case since  $|X_N \cap N_C(x)| \ge 2$  this is enough to determine  $N_C(v_{t-1})$ . It must be the case that  $N_C(v_{t-1}) = N \setminus (N_C(x) \cap X_N)$ . In case 2(*b*), this spans an edge of *C* (this edge may possibly be a chord of *C*). This means that  $v_t$  and  $v_{t-1}$  span a  $K_4$ , which contradicts the minimality of *C*.

If *N* satisfies 2(c), note that if  $v_2$  is incident to any neighbor of either  $v_t$  or  $v_{t-1}$  on *C* then we may exchange  $v_1$  for that vertex and obtain an *H* as the lemma hypothesizes. Since  $N_C(v_{t-1}) = N \setminus (N_C(x) \cap X_N)$ , this means that the neighbor of  $v_2$  is the vertex 'r' of condition 2(c). In this case, the conclusion of the Lemma is exactly that we may exchange  $v_1$  and  $v_2$  for the {r, s} in *C* and we obtain a Hamiltonian *H*.

There is a single remaining case, the exceptional case, when  $N_C(v_t) = \{c_1, c_3, c_4\}$ . As already discussed,  $v_{t-1}$  cannot be incident to any vertex in  $X_N \cap N_C(v_1) = \{c_2\}$  and  $v_2$  cannot be incident to any vertex in  $N_C(v_{t-1}) \cup N_C(v_t)$ . Note that it also cannot be the case that both  $\{c_3, c_4\}$  or  $\{c_1, c_4\}$  are in  $N_C(v_{t-1})$  as otherwise  $v_t$  and  $v_{t-1}$  will span a  $K_4$ . This restricts the case to  $N_C(v_{t-1}) = \{c_1, c_3, c_5\}$  and  $N_C(v_2) = \{c_2\}$ .

Here, if either end of *P* is set up, we have exhibited two DCCs (similar to the previous case, depending on the end set up and the configuration of the  $K_4^-$ ). If neither end is setup, replacing  $c_1c_2c_3c_4c_5$  with the DCC  $v_tc_3c_4c_5v_{t-1}v_t$  leads to a set up path starting with  $c_1, c_2, v_1, v_2, ...$  This completes the case where deg<sub>C</sub>( $v_2$ ) = 1, and deg<sub>C</sub>( $v_{t-1}$ )  $\geq$  3.

Now suppose  $\deg_C(v_2) = 0$ , then  $\deg_C(v_{t-1}) = 4$ . In this case we exchange  $v_t$  for a common neighbor of  $v_1$  and  $v_{t-1}$  that this possibly follows from the degrees and Lemma 8.

The final possibility is if |C| = 5 and  $\deg_C(v_2) = 2$ . Then  $\deg_C(v_{t-1}) \ge 2$ . If  $v_2$  shares a neighbor with  $v_{t-1}$  or  $v_t$  we are done by the Single Bypass Lemma. Thus  $v_{t-1}$ 's neighbors are two of the neighbors of  $v_{t-1}$ .

In this case we use the Double Bypass Lemma to exchange  $v_1$  and  $v_t$  for two vertices in *C*—one of which is incident to  $v_2$  and one of which is incident to  $v_{t-1}$ . This yields a cycle of the same length as *P* and completes Case 2.

**Case 3:**  $\deg_H(v_1) + \deg_H(v_2) + \deg_H(v_{t-1}) + \deg_H(v_t) = 12.$ 

This is the pessimal case: in this case, we must have  $\deg_H(v_2) = \deg_H(v_{t-1}) = 4$ . Note that since all components outside of *H* have cardinality at least 3,  $v_2$  and  $v_{t-1}$  cannot be incident to them so all neighbors of  $v_2$  and  $v_{t-1}$  in *H* are on *P*. Let  $v_s$  be the highest indexed neighbor for  $v_2$  and  $v_t$  be the lowest indexed neighbor of  $v_{t-1}$ . It is easy now to observe that if t < s, there is a DCC, so  $s \le t$ . (This DCC depends somewhat on how the neighbors of  $v_2$  and  $v_t$  interlace, but all are formed according to the following rule: starting at  $v_2$  follow *P* until the highest indexed neighbor of  $v_{t-1}$  with index strictly less than *s*. Follow this edge to  $v_{t-1}$ , then proceed back to the smallest indexed neighbor of  $v_2$  which is not already in use then follow the edge back to  $v_2$ .)

By Claim 2, applied to both ends of the path, there are two additional vertices of degree 2, say  $v_q$  and  $v_r$  with q < s and r > t. Note that  $\{v_1, v_q\}, \{v_r, v_t\}$  and  $\{v_2, v_{t-1}\}$  are all disjoint pairs of vertices and applying our  $\sigma_2$  condition we see that these vertices have total degree at least

$$3 \cdot (6k - 1) \ge 18(k - 1) + 15.$$

Furthermore, their degrees in *H* are 16. Therefore, either these 6 vertices send a total of 19 edges to some  $C \in C$  or they send at least 17 edges to every cycle in  $C \in C$ .

Suppose the latter holds. Applying the  $\sigma_2$  condition to just  $v_1$  and  $v_t$  there is some  $C \in C$  where  $\deg_C(v_1) + \deg_C(v_2) \ge 7$ . To this cycle,

$$\deg_{C}(v_{1}) + \deg_{C}(v_{2}) + \deg_{C}(v_{t-1}) + \deg_{C}(v_{t}) < 11$$

or we are in the situation dealt with in Case 2. Thus  $\deg_C(v_q) + \deg_C(v_r) \ge 7$ . Without loss of generality  $\deg_C(v_1) = 4$ . Then  $\deg_C(v_q) + \deg_C(v_t) \ge 6$ , so  $v_q$  and  $v_t$  have a shared neighbor. Furthermore taking the portion of the path between  $v_q$  and  $v_t$  along with the shared neighbor is a DCC, with two chords incident to  $v_{t-1}$  lie along this path.  $v_1$  incident to the rest of *C* is also a DCC by the Single Bypass Lemma.

Otherwise, suppose the former holds and consider the cycle C so that these 6 vertices send at least 19 edges to C. If

$$\deg_{C}(v_{1}) + \deg_{C}(v_{2}) + \deg_{C}(v_{t-1}) + \deg_{C}(v_{t}) \ge 13,$$

the Case 1 argument applies so  $\deg_{C}(v_q) + \deg_{C}(v_r) \ge 7$ .

Note that if some  $v \in \{v_1, v_2, v_{t-1}, v_t\}$  has deg<sub>C</sub>(v) = 4, then a very similar argument to the case we just dealt with—if  $v \in \{v_1, v_2\}$  one of  $v_{t-1}$  or  $v_t$  will have a common neighbor with  $v_q$ .

On the other hand,  $\deg_C(v_q) + \deg_C(v_r) \le 8$  so at least one of  $\{v_1, v_2\}$  or  $\{v_{t-1}, v_t\}$  – say,  $\{v_1, v_2\}$  – has total degree 6 into the cycle. If this is the case,  $v_q$  and one of  $v_{t-1}$  or  $v_t$  have a common neighbor, x. As before x along with P between  $v_1$  and  $v_{t-1}$  (or  $v_t$ ) creates a DCC. Furthermore,  $\{v_1, v_2\} \cup (C \setminus x)$  yields a DCC as the minimum degree of  $v_1$  and  $v_2$  into  $(C \setminus x)$  is 2 and  $(2, 2) \hookrightarrow \{P_3, P_4\}$  yields a DCC by Lemma 14. This is all possibilities, completing Case 3.  $\Box$ 

## This completes the proof of the lemma. $\Box$

The final lemma will complete the proof of Theorem 3 by using the structure of *H* that we have established to *k* disjoint doubly chorded cycles and hence contradicting the fact that we assumed we had a counterexample.

#### **Lemma 13.** There exists a cycle $C \in C$ so that $H \cup C$ contains two vertex disjoint DCCs.

**Proof.** By Lemma 12, *H* can be assumed to be either a Hamiltonian cycle or a Hamiltonian path with  $v_2 \sim v_t$ .

In the first case, there can be at most one chord, so *H* contains at most |H| + 1 edges. In the second case, there can be at most one chord in the cycle, and  $v_1$  can be incident to at most 3 vertices on the cycle. But if  $v_1$  has 3 neighbors in the cycle and there also exists a chord in the cycle, a DCC is easily found—so there are always at most |H| + 2 edges within *H*. Note that our observations also imply  $|H| \ge 7$ . Indeed, it is easy to see that there are two non-adjacent vertices of degree 2 in *H*. By our degree condition and averaging, we see that there is a vertex of degree 4 into some cycle—this shows that *C* contains a 5-cycle and hence  $|H| \ge 7$  by the Corollary to Lemma 6.

Now we consider the total degrees of vertices in *H*. First note that if two vertices in *H* have degree less than 3*k*, they must be adjacent by the  $\sigma_2$  condition. Since *H* does not contain a  $K_4$ , there are at most 3 such vertices.

Suppose first there are 3 such vertices, and their degrees are  $3k-t_1$ ,  $3k-t_2$  and  $3k-t_3$ , with  $t_1 \ge t_2 \ge t_3$ . All other vertices have degree at least  $3k + (t_3 - 1)$  by the fact that H contains no  $K_4$ . All but at most two have degree at least  $3k + (t_1 - 1)$  as H has maximum (interior) degree 4. Furthermore, all but at most one has degree at least  $3k + (t_2 - 1)$ . We can see this as follows: if two vertices had degree less than  $3k + (t_2 - 1)$ , then H would contain a  $K_{3,2}$  with an edge connecting the vertices in the partite set of size 2. This cannot happen: H is connected, and if it contains two vertices of interior degree 4 then all other vertices of H have interior degree 2. But since |H| > 5 and H is connected, the  $K_{3,2}$  we have cannot occur as part of H-no other edge can be connected to it without violating one of our conditions.

Thus the total degree in H is at least

$$(3k - t_1) + (3k - t_2) + (3k - t_3) + [3k + (t_2 - 1)] + [3k + (t_3 - 1)] + (|H| - 5)(3k + (t_1 - 1)) = 3k|H| - 2 + (|H| - 5)(t_1 - 1) - t_1$$

$$\geq 3k|H| - 3 = 3(k - 1)|H| + 3|H| - 3$$

The same count works if there are fewer such vertices (essentially, this follows by setting  $t_3 = 0$  or both  $t_2 = 0$  and  $t_3 = 0$ ). Indeed, in these cases a slightly stronger count applies.

Now, if |H| = 7, the degree sum within the cycle is at most 2(|H| + 2) = 2|H| + 4, and hence there are at least

$$3(k-1)|H| + 3|H| - 3 - [2(|H|+2)] = 3(k-1)|H| + |H| - 7 = 3(k-1)|H|$$

edges to the cycles, and hence either 22 edges to some cycle or exactly 21 edges to every cycle.

Suppose there is a Hamiltonian path in *H* starting at a vertex sending at least 4 edges to some cycle *C*, and there are at least 21 edges between *C* and *H*. Then  $|C| \le 5$  and there is some vertex in *C* with at least  $\lceil \frac{(21-4)}{|C|} \rceil \ge 4$  other vertices in *H*. This gives two doubly chorded cycles—one involving the vertex in the cycle and the remainder of *H* and the other involving the start vertex and the remainder of the cycle.

Note that  $v_1$  starts a Hamiltonian path, as does any successor or predecessor of neighbor of  $v_1$ . Thus  $v_t$  and  $v_3$  also start a Hamiltonian path. If they are connected, note that  $v_{t-1}$  and  $v_4$  also start a Hamiltonian path. This, plus the fact that there is at most one chord in the cycle, is enough to conclude that there are two nonadjacent vertices starting a Hamiltonian path with degree at most 2 in H. By the  $\sigma_2$  condition, these vertices send at least 7 edges to some cycle, and hence there *is* a vertex starting a Hamiltonian path with 4 edges to a single cycle. If H sent at least 21 edges to this path we are done, so we may assume it sends fewer. Thus there is some different cycle, where H has at least 22 adjacencies. But then there is some vertex in H sending at least 4 vertices to the cycle. This vertex cannot start a Hamiltonian path and – in particular – this cycle is a different cycle than the one just considered. Thus there are two vertices  $h_1$ ,  $h_2 \in H$  and two cycles  $C_1$ ,  $C_2 \in C$  so that  $h_1$  has four neighbors in  $C_1$  and  $h_2$  has four neighbors in  $C_2$ . This means that  $|C_1| \leq 5$  and  $|C_2| \leq 5$ , so  $|H| \geq 8$  by the Corollary to Lemma 6.

If  $|H| \ge 8$ , then the degree from H to the cycles is at least 3(k-1)|H| + 1, and hence there are at least 3|H| + 1 edges from H to some particular  $C \in C$ . Thus there exists some vertex  $v \in H$  with at least 4 neighbors in C, and so  $|C| \ge 4$ . Likewise there is some vertex  $c \in C$  with at least  $\lceil \frac{(3|H|+1)-4}{|C|} \rceil \ge 5$  neighbors in H other than v. If v starts a Hamiltonian path this is clearly enough by the above. Otherwise, v starts a path of length |H| - 1, not including  $v_1$ . But then this vertex in the  $c \in C$  has 4 neighbors other than v on this path, and this gives two DCCs as desired.  $\Box$ 

This immediately implies Theorem 3.  $\Box$ 



Fig. 1. Two exceptions to Lemma 14.



Fig. 2. The inside degree configuration.

## 3. Proof of Lemma 1

Our next goal is to finally prove Lemma 1, which is fairly involved.

Recall that we have two cycles *C* and *C'* with  $\ell = |C| \ge |C'|$ ,  $\ell \ge 7$  and  $e(C, C') \ge 3|C| + 1$ , and our objective is to find two doubly chorded cycles whose union is smaller than |C| + |C'|.

The key to the proof is simply observing that many collections of edges between two paths yield (in many cases proper) doubly chorded cycles. We will require a library of cases in order to complete the proof of Lemma 1. Since these cases are plentiful, it is helpful to introduce some notation.

We use the notation  $(d_1, d_2, \ldots, d_i) \hookrightarrow P_k$  to indicate a path with consecutive vertices of degree  $(d_1, d_2, \ldots, d_i)$  into a  $P_k$ . We use the symbol  $\star$  to indicate a *non-empty* sequence of arbitrary degrees. We use the notation  $P_k \hookrightarrow_d P_j$  to indicate an arbitrary *d* edges from a  $P_k$  to  $P_j$ . If no length is indicated then *P* represents a path of arbitrary length. In the following lemma, we indicate various degree sequences into short paths which always admit doubly chorded cycles and, in many cases proper doubly chorded cycles.

**Lemma 14.** 1. The following ensures a doubly chorded cycle:  $\{(2, 2), (2, \star, 2)\} \hookrightarrow P_3, (2, 2) \hookrightarrow P_4, \{(3, 2), (3, \star, 2)\} \hookrightarrow P, P_2 \hookrightarrow_5 P, (3, 1, 1) \hookrightarrow P, \{(2, 1, 2), (2, 1, \star, 2)\} \hookrightarrow P, P_3 \hookrightarrow_6 P, (2, 2, 1, 1) \hookrightarrow P \text{ and } \{P_4, P_5\} \hookrightarrow_6 P_5.$ 

- 2.  $P_3 \hookrightarrow_5 P_3$  yields a doubly chorded cycle except for the degree sequence (1, 3, 1) and a proper doubly chorded cycle except in the case (2, 1, 2), as pictured in Fig. 1.
- 3. The following ensures a proper doubly chorded cycle:  $(3, \star, 2) \hookrightarrow \{P_3, P_4\}, P_2 \hookrightarrow_5 P_4, P_2 \hookrightarrow_6 P, \{(3, 1, 1), (3, \star, 1, 1)\} \hookrightarrow P_3, (3, 1, 1) \hookrightarrow P_4, (2, 1, 2) \hookrightarrow P_4, P_3 \hookrightarrow_6 P_5, \{P_3\} \hookrightarrow_6 \{P_3, P_4\}, \{P_3, P_4, P_5\} \hookrightarrow_7 P, P_6 \hookrightarrow_8 P, P_7 \hookrightarrow_9 P, P \hookrightarrow_{10} P', (1, 2, 2, 1) \hookrightarrow P_5 and (2, 2, 1, 1) \hookrightarrow P_5.$
- 4.  $P_3 \hookrightarrow_5 P_4$  yields a proper doubly chorded cycle except for (1, 3, 1).
- 5.  $P_4 \hookrightarrow_5 P_4$  yields a doubly chorded cycle except for the degree sequences (1, 3, 1, 0), (1, 3, 0, 1), (1, 2, 1, 1), or an inside (2, 0, 2, 1) (see Fig. 2).
- 6.  $P_4 \hookrightarrow_6 P_4$  yields a doubly chorded cycle, which is necessarily proper in all cases except for the degree sequence (2, 1, 1, 2).

We defer the proof of Lemma 14, which involves fairly extensive case analysis, to the end of the section. To restrict cases later, it is useful to bound the maximum degree between C and C'.

**Lemma 15.** The maximum degree of any vertex in C or C' to the other is at most 5.

**Proof.** Let  $m \ge 6$  denote the maximum degree of a vertex in one of *C*, *C'* to the other, and let *v* have that degree. We do not know which of *C*, *C'* that *v* lies on, so we say that *v* lies on *C*<sup>\*</sup>, with *m* edges to *C*<sup>\*\*</sup>. Then there are at least  $3\ell + 1 - m \ge 2\ell + 1$  edges between *C*<sup>\*\*</sup> and *C*<sup>\*</sup> \ *v*. By averaging, some adjacent pair of vertices in *C*<sup>\*\*</sup> sends at least 5 edges to *C*<sup>\*</sup> \ *v*. If  $m \ge 7$ , we are done, as this pair creates a doubly chorded cycle with *C*<sup>\*</sup> \ *v*, by Lemma 14 part (1), and *v* must be adjacent to at least 5 other vertices of *C*<sup>\*\*</sup>, so it is easy to ensure a proper doubly chorded cycle. If m = 6 we are only in trouble if our adjacent pair of vertices and both of their neighbors on the cycle are adjacent to *v*. There are at most 3 pairs of this type, which is achieved only if 6 neighbors of *v* are consecutive. These pairs may have at most 5 edges to *C*<sup>\*</sup> as otherwise there is a proper doubly chorded cycle containing them and  $C^* \setminus v$  (this is the third part of Lemma 14). Each of the other pairs have at most 4 edges. Thus we have an upper bound on the number of edges between *C*<sup>\*\*</sup> and *C*<sup>\*</sup> \ *v* of  $\frac{1}{2}(5 \times 3 + 4 \times (\ell - 3)) = 2\ell + \frac{3}{2}$ . On the other hand, we have a lower bound of  $3\ell + 1 - 6 = 3\ell - 5$ . Since  $\ell \ge 7$ , and 15.5 < 16, this is a contradiction proving the lemma; unless  $|C^{**}| = 6$ . In this case if there is an edge of *C*<sup>\*\*</sup> with at least 6 edges to *C*<sup>\*</sup> \ *v*, the same argument works. Thus, each edge of *C*<sup>\*\*</sup> must send at most 5 edges to *C*<sup>\*</sup> \ *v*. But, there are at least  $3 \times 7 + 1 - 6 = 16$  edges to *C*<sup>\*\*</sup>, and so one of the edges must have at least 6 edges to *C*<sup>\*</sup> \ *v*.



Fig. 3. Path partitions of the two cycles with edges sent to each part.

We frequently use the following simple fact.

**Lemma 16.** Let  $C \in C$  denote a cycle of length  $\ell$  with maximum degree at most 5 and with  $\deg(C) = \sum_{v \in C} \deg(v)$ . Then C can be partitioned into two paths P and Q (with degree of any path defined in a manner similar to that of  $\deg(C)$ ) with  $|P| = \lceil \frac{\ell}{2} \rceil$  and  $|Q| = \lfloor \frac{\ell}{2} \rfloor$ , and  $|\deg(P) - \deg(Q)| \le 5$ .

**Proof.** This is a standard intermediate value theorem proof. Let  $m = \lceil \frac{\ell}{2} \rceil$ , and order the vertices of *C*, say  $c_1, c_2, \ldots, c_\ell$  (with indices mod  $\ell$ ). Let  $P^i = \{c_i, \ldots, c_{i+m-1}\}$  and  $Q^i = C \setminus P^i$ . Note that  $\deg(P^i) + \deg(Q^i) = \deg(C)$ . Consider  $x_i = \deg(P^i) - \deg(Q^i)$ . In the  $\ell$  even case,  $x_1 = -x_{m+1}$ . Since  $|x_i - x_{i+1}| \le 10$  by the maximum degree condition, some  $x_i$  has  $|x_i| \le 5$  and this is our desired partitioning. For  $\ell$  odd, we note that  $|x_1 + x_{m+1}| = 2\deg(C) \le 10$ . In particular, if  $|x_1| > 5$ , then  $x_{m+1}$  has an opposite sign and again we find our desired partition. Otherwise,  $|x_1| \le 5$  and already  $P^1$ ,  $Q^1$  is our desired partition.  $\Box$ 

At this point we partition *C* into two paths with degrees (r, s) to *C'*, respectively, and *C'* into two paths with degrees (t, u) to the paths of *C*, respectively (see Fig. 3). We choose the balanced partition guaranteed by Lemma 16, except in the cases when the cycle has length  $\ell = 7$ , 8. If  $\ell = 7$  or  $\ell = 8$ , we choose the decomposition on the cycle that is most balanced in terms of  $|\deg(P) - \deg(Q)|$  (which may not be the one guaranteed by the pairs) and we assume that *P* is as short as possible. We have the caveat that, if there exists a  $P_2 \cup P_{\ell-2}$  decomposition so that  $|\deg(P_2) - \deg(P_{\ell-2})| \le 5$ , we take this even if there is a more balanced partitioning. (Note that we only do this when  $\ell = 7$ , for (9, 13) and  $\ell = 8$  for (10, 15).)

**Claim 1.** Assume  $\ell = 7$ . In the case of (9, 13) we may assume that |P| = 2. In the case of (10, 12), if |P| = 4, then we may assume that the degree of one of the end points of P is at most 1.

**Proof.** We begin by noting that if (9, 13) is the most balanced setup and the order of the first path is greater than 2, then only degrees 4 and 5 (or 0) may be present (or a more balanced partition is easily obtained). Indeed, suppose there is a positive degree less than 4. Start at this vertex and add the degrees of adjacent vertices, clockwise around the cycle until the degree sum is larger than 9. Then the degree sum must be at least 13, or it would be a more balanced partition. If it is exactly 13, moving the vertex of degree less than 4 to the other side yields a more balanced partition. Otherwise, using Lemma 15, the sum is exactly 14 and this implies that the initial degree is 1 and the last degree is 5. That is, we have  $(1, d_1, \ldots, d_i, 5)$  with  $\sum_{t=1}^{i} d_t = 8$ . We may further assume that i = 2: if i > 2, then repeating in a counter-clockwise direction around the cycle yields an identical situation where i = 2—here we strongly use that  $\ell = 7$ . Now,  $d_1 + d_2 = 8$ . If  $d_1 \in \{4, 5\}$ , then there is the desired 9 segment of order two or a more balanced partition. Hence  $d_1 = 3$ , and  $d_2 = 5$ . Now consider the three vertices between the vertex of degree 1 and the vertex of degree 5 when transversing the cycle in the other direction. These vertices have degree sum 5, but also it is easy to check that none of them may have non-zero degree without violating the balanced condition. Indeed, consider the vertex closest to the degree 1 vertex with non-zero degree d<sub>1</sub> = 3. If this has degree 4 or 5, then there is a segment of degree 10 or 11 starting with the vertex  $d_2 = 5$ .

Thus the remaining possibility is that all positive degrees are at least 4. Furthermore, each degree 5 vertex must be surrounded by degree four vertices, not a priori adjacent. However, this says that the degrees of the vertices in the cycle are 5, 4, 5, 4, 4, possibly with some degree zero vertices in between. However, since the order of the cycle is at most 7, a degree 5 and a degree 4 vertex must be adjacent, given |P| = 2.

Next we consider the case |P| = 4. Clearly, then, |Q| = 3, because the maximum degree is 5. Furthermore, the endpoints of *P* have positive degree, as otherwise they could be added to *Q*. The only degree sequence options for *Q* are (4, 4, 4), (5, 3, 4) and (5, 2, 5); these are the only (sorted) options because an adjacent (5, 4) would give the preferred (9, 13)

C pair $(9, 13)$ vs. all $(t, u)$ pairs for C'.						
(9, 13)						
9	( <u>4</u> , 5)	(≤3, ≥ <u>6</u> )				
13	(5, <u>8</u> )	(≥ <u>6</u> , ≤7)				
10	$(\geq \underline{5}, \leq 5) \\ (\leq 4, \geq \underline{8})$	(4,6)	(≤3, ≥ <u>7</u> )			
12		(5,7)	(≥ <u>6</u> , ≤6)			
11	(≥ <u>5</u> , ≤6)	(4, <u>7</u> )	(≤ 3, ≥ <u>8</u> )			
11	(≤4, ≥ <u>7</u> )	( <u>5</u> , 6)	(≥ <u>6</u> , ≤5)			

partition and adjacent (5, 5) would give a *P* of order 2. In the first case, it is clear that the only neighbors of the 4 are either another 4 or 1, otherwise a *P* of order 3 would exist, a (5, 4) would exist or a more balanced (11, 11) partition would exist. However, if a degree 4 vertex is adjacent, one may shift over and repeat the argument and as not every vertex can have degree 4, one eventually gets a 1 as desired. The other cases are similar. In the (5, 3, 4) case it is clear that the only positive number allowable incident to the 5 is 1 as others yield a more preferred partition. In the (5, 2, 5) case only 1 or 2 is permissible next to the 5's. If 2 is present in both fives, there are two remaining spots which must add degree 6. The options for these are (5, 1) (which would yield the *P* beginning with 1), (4, 2) or (3, 3) (both of which yield more preferred partitions).  $\Box$ 

## **Claim 2.** Assume $\ell = 8$ . In the case of a P, Q decomposition of type (10, 15) we may assume that |P| = 2.

**Proof.** The proof is similar to the (9, 13) case, but easier. We claim that if (10, 15) is the most balanced decomposition then every vertex with positive degree must be of degree 5. Indeed, if there were a vertex of degree less than 5, the strategy above yields a more balanced decomposition. But then there are five vertices of degree 5 on a cycle of length 8 so two must be adjacent. This gives the desired (10, 15) decomposition.  $\Box$ 

Finally, we are ready to prove Lemma 1.

Table 4

**Proof of Lemma 1.** Recall, we have two cycles with  $\ell = |C| \ge |C'|$  and  $\ell \ge 7$ . We partition *C* and *C'* into paths with degrees (r, s) and (t, u), balanced as above. We next note that each of the pairs for the two cycles can further be partitioned into two parts based on the number of edges sent to each subpath of the other cycle. We consider these pairs as  $(a_1, b_1)$  and  $(a_2, b_2)$  as shown in Fig. 3. Thus, for *C* we have  $r = a_1 + b_1$  and  $s = a_2 + b_2$ . While for cycle *C'* we have  $t = a_1 + a_2$  and  $u = b_1 + b_2$ .

At this point, there are many cases based on  $\ell$ , and the partitions  $(a_1, b_1)$  and  $(a_2, b_2)$ . Our primary tool to handle these cases is Lemma 14. Recall that in all cases, we wish to show that the graph induced by  $C \cup C'$  contains a pair of doubly chorded cycles with at least one of them proper. We will proceed based on  $\ell$ .

#### **Case 1:** Suppose $\ell = 7$ .

Possible pairs are reflected in the interior parts of the tables. The tables show all the cases based upon the path partitions for *C* and *C'*. Note that the first column lists the (r, s) pair for *C* vs. all possible (t, u) pairs for *C'*. The interior pairs are the  $(a_1, a_2)$  and  $(b_1, b_2)$  splits based upon the corresponding (r, s) and (t, u) values for each cycle.

The remainder of the proof is to verify that in each case, one of the splits  $(a_1, b_2)$  and  $(b_1, a_2)$  allows us to find two disjoint doubly chorded cycles that, in total, use fewer vertices than C and C' together.

In most cases, the results of Lemma 14 make this transparent. In the cases where (9, 13) is against other pairs the fact that  $P_2 \hookrightarrow_6 P$  gives a proper doubly chorded cycle, and  $\{P_3, P_4, P_5\} \hookrightarrow_7 P$  gives proper doubly chorded cycles makes finding a proper doubly chorded cycle easy. In the tables, we indicate these cases by underlining the choices that most easily produce the desired result, and bolding the segment which guarantees a proper doubly chorded cycle. We will not include all the details here (see Tables 4–8).

There remains a few more difficult cases where we must argue a bit more.

In the (10, 12) vs. (10, 12) case where we have (4, 6) vs. (6, 6), note that we have two cases, (4, <u>6</u>) vs. (<u>6</u>, 6) and (4, <u>6</u>) vs. (<u>6</u>, 6)—which case occurs depends on which of the segments involved is shorter, but since the total length is 7, at least one of the segments of the first cycle must have length at most 3. The fact that { $P_2$ ,  $P_3$ }  $\hookrightarrow$  <sub>6</sub>{ $P_3$ ,  $P_4$ ,  $P_5$ } yields a proper DCC is enough to show that the system is proper. Similar reasoning reveals that in the (11, 11) vs (11, 11) case where we have (6, 5) vs. (5, 6) both '6's guarantee a DCC, and since one of the segments must have length 3 one guarantees a proper DCC.

In the (9, 13) vs. (10, 12) case, where we have (4, 6) vs. (5, 7), if the 10 segment has order 2, so that 4 edges guarantees a doubly chorded cycle, (<u>4</u>, 6) and (5, <u>7</u>) gives the desired cycles. Otherwise, we need that the 6 edges from the 10 segment to the 13 segment of the other cycle gives a proper doubly chorded cycle. If the 10 segment has order 3, then  $P_3 \hookrightarrow_6 P_5$ guarantees a proper doubly chorded cycle by Lemma 14. In the case where the 10 segment has order 4, we have that one end has degree at most 1 into the  $P_5$ , by Claim 1. If it has degree zero, then we have  $P_3 \hookrightarrow_6 P_5$ , yielding the proper cycle. If it has degree 1, excluding that vertex, we have  $P_3 \hookrightarrow_5 P_5$ , and we are done unless there is no doubly chorded cycle (as we have already excluded a vertex). Note that (3, 1, 1)  $\hookrightarrow P$  yields a doubly chorded cycle by Lemma 14, as does (2, 1, 2)  $\hookrightarrow P$ , (3, 2)  $\hookrightarrow P$  and (3,  $\star$ , 2)  $\hookrightarrow P$ . If the maximum degree is 3, we are left with (1, 3, 1)  $\hookrightarrow P$ , which after adding back the vertex of degree one to the path, gives a (3, 1, 1)  $\hookrightarrow P$  and hence the desired proper doubly chorded cycle. In the cases with maximum degree 2, except for (2, 1, 2), the maximum degree two cases not covered, after adding back the degree 1 vertex,

$\begin{array}{rrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrr$	10, 12)					
$ \begin{array}{rrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrr$	1	(≤4, ≥ <u>7</u> )	(5,6)	(≥ <u>6</u> , ≤5)		
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	1	( <u>≥6</u> , <u>≤</u> 5)	(5,6)	(≤4, ≥ <u>7</u> )		
Table 6         C pair (11, 11) vs. all (t, u) pairs for C'.         (11, 11)         11 $(\leq 4, \geq 7)$ (6, 5)         Table 7         C pair (10, 15) vs. all (t, u) pairs for C'.         (10, 15)       (1, 1)       (1, 1)         10 $(\leq 3, \geq 7)$ (4, 6)         15 $(\geq 7, \leq 8)$ (6, 4)         11 $(\geq 5, \leq 6)$ $(\leq 4, \geq 7)$ 14 $(\leq 5, \geq 9)$ $(\geq 6, \leq 8)$ 12 $(\geq 5, \leq 7)$ $(\leq 4, \geq 8)$ 13 $(\leq 5, \geq 8)$ $(\geq 6, \leq 7)$	0	(≤3, ≥ <u>7</u> )	(4, 6)	(5, 5)	(≥ <u>6</u> , :	≤4)
Table 6         C pair (11, 11) vs. all $(t, u)$ pairs for C'.         (11, 11)         11 $(\leq 4, \geq 7)$ $(6, 5)$ Table 7         C pair (10, 15) vs. all $(t, u)$ pairs for C'.         (10, 15)       ( $\leq 3, \geq 7$ )       (4, 6)         15 $(\geq 7, \leq 8)$ ( $6, 4$ )         11 $(\geq 5, \leq 6)$ $(\leq 4, \geq 7)$ 14 $(\leq 5, \geq 9)$ $(\geq 6, \leq 8)$ 12 $(\geq 5, \leq 7)$ $(\leq 4, \geq 8)$ 13 $(\leq 5, \geq 8)$ $(\geq 6, \leq 7)$	.2	( <u>≥1</u> , ≤ 5)	(6,6)	(5,7)	( <u>≤</u> 4, :	<u>≥8</u> )
$\frac{C \text{ pair } (11, 11) \text{ vs. all } (t, u) \text{ pairs for } C'.}{(11, 11)}$ $\frac{11}{11} (\leq 4, \geq 7) (6, 5)$ $\frac{11}{11} (\geq 7, \leq 4) (5, 6)$ $\frac{7 \text{ able 7}}{(10, 15) \text{ vs. all } (t, u) \text{ pairs for } C'.}$ $\frac{(10, 15)}{10} (\leq 3, \geq 7) (4, 6)$ $15 (\geq 7, \leq 8) (6, 4)$ $11 (\geq 5, \leq 6) (\leq 4, \geq 7)$ $14 (\leq 5, \geq 9) (\geq 6, \leq 8)$ $12 (\geq 5, \leq 7) (\leq 4, \geq 8)$ $13 (\leq 5, \geq 8) (\geq 6, \leq 7)$ $\frac{(12, 13)}{(4, \geq 7)} (5, 6) (\geq 6, \leq 5) 12 (\leq 5, \geq 7) (\geq 8)$	Table	e 6				
$\begin{array}{c} \underbrace{(11, 11)}{11} & (\leq 4, \geq \underline{7}) & (6, 5) \\ 11 & (\geq \underline{7}, \leq 4) & (5, 6) \end{array}$ <b>Table 7</b> <i>C</i> pair (10, 15) vs. all (t, u) pairs for C'.   (10, 15)   10   15   15   (\geq \underline{7}, \leq 8) & (6, 4) \\ 11 & (\geq \underline{5}, \leq 6) & (\leq 4, \geq \underline{7}) \\ 14 & (\leq 5, \geq \underline{9}) & (\geq \underline{6}, \leq 8) \\ 12 & (\leq \underline{5}, \leq 7) & (\leq 4, \geq \underline{8}) \\ 13 & (\leq 5, \geq \underline{8}) & (\geq \underline{6}, \leq 7) \end{array}	C pai	r (11, 11) vs. all (	(t, u) pairs for C	<i>'</i> .		
$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	(11	, 11)				
$ \frac{11}{(\geq \underline{7}, \leq 4)} (5, 6) $ Table 7 C pair (10, 15) vs. all (t, u) pairs for C'. $ \frac{(10, 15)}{10} (\leq 3, \geq \underline{7}) (4, 6) 15 (\geq \underline{7}, \leq 8) (6, 4) 11 (\geq \underline{5}, \leq 6) (\leq 4, \geq \underline{7}) 14 (\leq 5, \geq \underline{9}) (\geq \underline{6}, \leq 8) 12 (\geq \underline{5}, \leq 7) (\leq 4, \geq \underline{8}) 13 (\leq 5, \geq \underline{8}) (\geq \underline{6}, \leq 7) $ and (12, 13) vs. all (t, u) pairs for C'. $ \frac{(12, 13)}{(\underline{4}, \geq \underline{7})} (5, 6) (\geq 6, \leq 5) 12 (\leq 5, \geq \overline{7}) (\geq \underline{6}) $	11		$(\leq 4, \geq \underline{7})$		(6, 5)	
Table 7         C pair (10, 15) vs. all $(t, u)$ pairs for C'.         (10, 15)         (10, 15)         10       ( $\leq 3, \geq 7$ )       (4, 6)         15       ( $\geq 7, \leq 8$ )       (6, 4)         11       ( $\geq 5, \leq 6$ )       ( $\leq 4, \geq 7$ )         12       ( $\geq 5, \leq 7$ )       ( $\leq 4, \geq 8$ )         12       ( $\geq 5, \leq 7$ )       ( $\leq 4, \geq 8$ )       ( $\geq 6, \leq 7$ )       ( $\geq 6, \leq 7$ )         md (12, 13) vs. all $(t, u)$ pairs for C'.         (12, 13)         ( $\geq 6, \leq 5$ )       12       ( $\leq 5, \geq 7$ )       ( $\geq 2$ ( $\geq 2, \leq 5$ )       12       ( $\leq 5, \geq 7$ )       ( $\geq 2$	11		( <u>≥7</u> , <u>≤</u> 4)		(5,6)	
Table 7         C pair (10, 15) vs. all $(t, u)$ pairs for $C'$ .         (10, 15)         (10, 15)         10 $(\leq 3, \geq \mathbf{Z})$ (4, 6)         15 $(\geq \mathbf{Z}, \leq \mathbf{S})$ (6, 4)         11 $(\geq 5, \leq 6)$ $(\leq 4, \geq \mathbf{Z})$ 14 $(\leq 5, \geq 9)$ $(\geq 6, \leq \mathbf{S})$ 12 $(\geq 5, \leq 7)$ $(\leq 4, \geq 8)$ 13 $(\leq 5, \geq 8)$ $(\geq 6, \leq 7)$ (12, 13)         (12, 13)						
$\frac{(10, 15) \text{ vs. all } (t, u) \text{ pairs for } C}{(10, 15)}$ $\frac{(10, 15)}{10} (\leq 3, \geq 7) (4, 6)$ $15 (\geq 7, \leq 8) (6, 4)$ $11 (\geq 5, \leq 6) (\leq 4, \geq 7)$ $14 (\leq 5, \geq 9) (\geq 6, \leq 8)$ $12 (\geq 5, \leq 7) (\leq 4, \geq 8)$ $13 (\leq 5, \geq 8) (\geq 6, \leq 7)$ $\text{nd } (12, 13) \text{ vs. all } (t, u) \text{ pairs for } C'.$ $(12, 13)$ $(\leq 5, \geq 7) (\leq 6, \leq 5) (12) (\leq 5, \geq 7) (\geq 6)$						
$\frac{(10, 15)}{10} (\leq 3, \geq \underline{7}) (4, 6) \\ 15 (\geq \underline{7}, \leq 8) (6, 4) \\ 11 (\leq \underline{5}, \leq 6) (\leq 4, \geq \underline{7}) \\ 14 (\leq 5, \geq \underline{9}) (\geq \underline{6}, \leq 8) \\ 12 (\geq \underline{5}, \leq 7) (\leq 4, \geq \underline{8}) \\ 13 (\leq 5, \geq \underline{8}) (\geq \underline{6}, \leq 7) \\ \hline \\ nd (12, 13) \text{ vs. all } (t, u) \text{ pairs for } C'. \\ \hline \\ \hline \\ (12, 13) \\ \hline \\ (4, \geq \underline{7}) (5, 6) (\geq 6, \leq 5) 12 (\leq 5, \geq \underline{7}) (\geq \underline{6}) \\ \hline \\ \end{cases}$	Table	e7 n (10, 15) vo. ell (	t u) nains fan (	-/		
$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	C pai	e 7 r (10, 15) vs. all (	(t, u) pairs for C			
$ \frac{13}{11} (25, \le 6) (0, 4) $ $ \frac{11}{14} (25, \ge 9) (26, \le 8) $ $ \frac{12}{13} (25, \ge 7) (24, \ge 8) $ $ \frac{13}{13} (25, \ge 8) (26, \le 7) $ $ \frac{12}{13} (25, \ge 8) (26, \le 7) $ $ \frac{12}{13} (25, \ge 8) (26, \le 7) $ $ \frac{12}{13} (25, \ge 7) (25, \ge 7) (25, \ge 7) $	<b>Table</b> C pai (10	e 7 r (10, 15) vs. all ( , 15)	(t, u) pairs for C			
$ \frac{11}{14} (\underline{\geq 5}, \leq 6) (\underline{< 4}, \underline{> 2}) \\ 14 (\underline{< 5}, \underline{> 9}) (\underline{\geq 6}, \underline{< 8}) \\ 12 (\underline{> 5}, \underline{< 7}) (\underline{< 4}, \underline{> 8}) \\ 13 (\underline{< 5}, \underline{> 8}) (\underline{\geq 6}, \underline{< 7}) \\ \frac{14}{13} (\underline{< 5}, \underline{> 8}) (\underline{< 6}, \underline{< 7}) \\ (\underline{< 12, 13}) \\ (\underline{< 4, \underline{> 7}}) (\underline{5, 6}) (\underline{\geq 6}, \underline{< 5}) 12 (\underline{< 5}, \underline{> 7}) (\underline{> 7}) $	<b>Table</b> C pai (10 10	<b>2 7</b> r (10, 15) vs. all ( 1, 15)	(t, u) pairs for C $(\leq 3, \geq \underline{7})$	(4,)	6)	
$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	<b>Table</b> C pai (10 10 15	<b>? 7</b> r (10, 15) vs. all ( , 15)	(t, u) pairs for C $(\leq 3, \geq \underline{7})$ $(\geq \underline{7}, \leq 8)$ (= 5, +5)	(4, 1)	6) 4)	
$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	Table C pai (10 10 15 11	<b>? 7</b> r (10, 15) vs. all ( 1, 15)	(t, u) pairs for C $(\leq 3, \geq \underline{7})$ $(\geq \underline{7}, \leq 8)$ $(\geq \underline{5}, \leq 6)$ $(\leq 5 \geq 9)$		$ \begin{array}{c} 6) \\ 4) \\ 4, \geq 7) \\ 6 < 8) \end{array} $	
$\frac{1}{(12, 13) \text{ vs. all } (t, u) \text{ pairs for } C'.}{(12, 13)}$	Table C pai (10 10 15 11 14	2 <b>7</b> r (10, 15) vs. all ( , 15)	$(\underline{<3, \geq \underline{7}})$ $(\underline{\geq \underline{7}, \leq \underline{8}})$ $(\underline{\geq \underline{5}, \leq \underline{6}})$ $(\underline{\geq 5, \leq \underline{9}})$ $(\underline{>5, <7})$		$ \begin{array}{c} 6) \\ 4) \\ 4, \geq \underline{7}) \\ \underline{5}, \leq 8) \\ 4, > 8) \end{array} $	
$\frac{d (12, 13) \text{ vs. all } (t, u) \text{ pairs for } C'.}{(12, 13)}$ $(4, \geq 7)  (5, 6)  (\geq 6, \leq 5)  12  (\leq 5, \geq 7)  (\geq 6, \leq 5)$	Table           C pai           (10)           10           15           11           14           12           13	<b>2 7</b> r (10, 15) vs. all ( 1, 15)	$(\leq 3, \geq \underline{7})$ $(\geq \underline{7}, \leq 8)$ $(\geq \underline{5}, \leq 6)$ $(\leq \underline{5}, \leq 9)$ $(\geq \underline{5}, \leq 7)$ $(\leq \underline{5}, \geq \underline{8})$	(4, (6, (24) (24) (24) (24)	$ \begin{array}{c} 6) \\ 4) \\ 4, \geq \underline{7}) \\ \underline{5}, \leq 8) \\ 4, \geq \underline{8}) \\ \underline{5}, \leq 7) \end{array} $	
$\begin{array}{c} \text{ ad } (12, 13) \text{ vs. all } (t, u) \text{ pairs for } C'. \\ (12, 13) \\ \hline \\ $	Table           C pai           (10)           10           15           11           14           12           13	2 <b>7</b> r (10, 15) vs. all ( 1, 15)	$(\leq 3, \geq \underline{7})$ $(\geq \underline{7}, \leq 8)$ $(\geq \underline{5}, \leq 6)$ $(\leq 5, \geq \underline{9})$ $(\geq \underline{5}, \leq 7)$ $(\leq 5, \geq \underline{8})$	(4, (6, ) (6, ) (24) (24) (24) (24)	$ \begin{array}{c} 6) \\ 4) \\ 4, \geq \mathbf{Z}) \\ 5, \leq 8) \\ 4, \geq \mathbf{S}) \\ 5, \leq 7) \\ \hline \end{array} $	
$(12, 13)$ $(4, \ge 7)  (5, 6)  (\ge 6, \le 5)  12  (\le 5, \ge 7)  (\ge 6, \le 5)$	Table           C pai           (10)           10           15           11           14           12           13	2 7 r (10, 15) vs. all ( , 15)	$(\leq 3, \geq 7)$ $(\geq 7, \leq 8)$ $(\geq 5, \leq 6)$ $(\leq 5, \leq 9)$ $(\geq 5, \leq 7)$ $(\leq 5, \geq 8)$	(4, ) (6, ) (54 (24 (24) (24) (24)	$ \begin{array}{c} 6) \\ 4) \\ 4, \geq \mathbf{Z}) \\ 5, \leq 8) \\ 4, \geq 8) \\ 4, \geq 8) \\ 5, \leq 7) \end{array} $	
$(4, \ge 7)$ (5, 6) ( $\ge 6, \le 5$ ) 12 ( $\le 5, \ge 7$ ) ( $\ge$	Table C pai (10) 10 15 11 14 12 13 	2 <b>7</b> r (10, 15) vs. all ( , 15)	$(\leq 3, \geq 7)$ $(\geq 7, \leq 8)$ $(\geq 5, \leq 6)$ $(\leq 5, \leq 9)$ $(\geq 5, \leq 7)$ $(\leq 5, \geq 8)$ s for C'.	(4, ) (6, ) (54 (54) (54) (54) (54)	$ \begin{array}{c} 6) \\ 4) \\ 4, \geq 7) \\ 5, \leq 8) \\ 4, \geq 8) \\ 4, \geq 8) \\ 5, \leq 7) \end{array} $	
	Table           C pai           (10)           10           15           11           14           12           13	2 <b>7</b> r (10, 15) vs. all ( , 15)	$(\pm, u)$ pairs for C $(\pm 3, \geq 7)$ $(\pm 7, \leq 8)$ $(\pm 5, \leq 6)$ $(\pm 5, \leq 9)$ $(\pm 5, \leq 7)$ $(\pm 5, \geq 8)$ s for C'.	(4, ) (6, ) (≤∠ (≥€ (≥€ (≥€) (12, 13)	$ \begin{array}{c} 6) \\ 4) \\ 4, \geq 7) \\ 5, \leq 8) \\ 4, \geq 8) \\ 5, \leq 7) \end{array} $	
	$ \begin{array}{c}     Table \\     C pai \\     \hline     (10) \\     10 \\     15 \\     11 \\     14 \\     12 \\     13 \\     \hline     (12, 13) \\     \hline     (12, 13) \\     \hline     (28) \\   \end{array} $	2 <b>7</b> r (10, 15) vs. all ( , 15) ) vs. all ( <i>t</i> , <i>u</i> ) pair (5, 6) (6, 8)	$(\pm, u) \text{ pairs for } C$ $(\leq 3, \geq 7)$ $(\geq 7, \leq 8)$ $(\geq 5, \leq 6)$ $(\leq 5, \geq 9)$ $(\geq 5, \leq 7)$ $(\leq 5, \geq 8)$ s for C'. $(\geq 6, \leq 5)$ $(\leq 5, \geq 9)$		$6)$ $4, \ge \mathbf{Z})$ $5, \le 8)$ $4, \ge 8)$ $5, \le 7)$ $(\le 5, \ge)$ $(\ge \mathbf{Z}, \le 6)$	<u>7</u> ) (≥i ;) (≤i

are  $(2, 2, 1, 1) \hookrightarrow P_5$  and  $(1, 2, 2, 1) \hookrightarrow P_5$ , both of which are shown by Lemma 14 to give a proper doubly chorded cycle, finishing the case.

By far the most difficult is the (10, 12) vs. (11, 11) case where we have (5, 6) vs. (5, 6). We know by 16 that the 10 segment has order 2, 3 or 4. If the 10 segment has order 2, since then  $P_2 \hookrightarrow_5 \{P_3, P_4\}$  yields a doubly chorded cycle, and  $P_3 \hookrightarrow_6 P_5$  yields a proper doubly chorded cycle by Lemma 14, we find our desired pair of doubly chorded cycles.

If the 10 segment has order 3, by Lemma 14,  $P_3 \hookrightarrow_5 \{P_3, P_4\}$  yields a doubly chorded cycle, except in the case where the  $P_3$  has degree sequence (1, 3, 1), and  $P_4 \hookrightarrow_6 \{P_3, P_4\}$  yields a proper doubly chorded cycle except in the case  $(2, 1, 1, 2) \hookrightarrow P_4$ . Since the 10 segment of *C* cannot have degrees (1, 3, 1) into both segments of *C'*, we find our desired pair of doubly chorded cycles except in a singular case—that in which both *P* and *Q* are divided into segments of length 3 and 4. If the segments of length 3 and 4 on opposite sides.

The bad case here is where both length 3 segments are connected with degree sequence (2, 1, 2) and both length 4 segments are connected with degree sequence (2, 1, 1, 2). In this case the  $P_3$  segment of the (10, 12) path has degrees (3, 4, 3). If either of the ends of the  $P_4$  on the (10, 12) cycle had degree 3 or 4 we would be done. This gives either a more balanced partition of the top or a  $P_3$  with a (4, 3, 3) degree configuration that does not support this bad case. If either of the ends of the  $P_4$  on the (10, 12) cycle had degree 4, we would also be done. Since this degree 5 vertex sends 2 edges into the  $P_4$  on the (11, 11) side, it must send 3 into the  $P_3$  on the (11, 11) side. Combining it with its neighbor on the  $P_3$  gives  $(3, 2) \hookrightarrow P_3$ , giving a DCC and the remainder gives a  $P_5 \hookrightarrow_7 P_4$  which guarantees a proper DCC. Thus both ends of the  $P_4$  on the (10, 12) have degree 2 (both into the  $P_4$  on the other side, and in sum). Then combining one of those vertices with the adjacent two vertices in the  $P_3$  give a degree sequence of  $(2, 1, 3) \hookrightarrow P_4$ , giving a proper DCC. The middle two vertices of the  $P_4$  have a total degree sum of  $12 - 2 \cdot 2 = 8$  and only send two edges to the  $P_4$  which gives a  $P_2 \hookrightarrow_6 P_3$  and hence also a proper DCC. This completes the case of this (10, 12) vs. (11, 11) split where the 10 segment has order 3.

The case where the 10 segment of *C* has order 4 also requires some additional argument. Let us denote by *P* the segment of order 4 in *C* and by  $Q_1$  and  $Q_2$  the two segments in our partition of *C'*. If we can find a DCC between *P* and either  $Q_1$  or  $Q_2$ then we are done, as the remaining  $P_3 \hookrightarrow_6 \{Q_1, Q_2\}$  will provide the desired proper doubly chorded cycle. Unfortunately we are not guaranteed the existence of a DCC between *P* and either  $Q_1$  or  $Q_2$ . Instead, we will argue as follows: in the event that no DCC lies between *P* and either  $Q_1$  or  $Q_2$ , we will use Lemma 14 to assert structural information about the edge set. We

 C pairs (11, 1)

 (11, 14)

 11

14

12

13

 $(\geq 6, \leq 7)$ 

 $(\leq 5, \geq 8)$ 

will then use this information to find a new partition  $Q'_1$  and  $Q'_2$  of C' so that there exist DCCs both between P and  $Q'_1$  and between P and  $Q'_2$ . Then as the remaining  $P_3$  of C must send at least 6 edges to one of  $Q'_1$  and  $Q'_2$ , we will be done.

We proceed as follows. Consider the two partitions  $Q_1$  and  $Q_2$ , both of which send 5 edges into *P*. If both  $|Q_1| = |Q_2| = 3$  and neither yields a DCC with *P*, then by Lemma 14 part (4), the degree sequence of both is (1, 3, 1) so the degree sequence of the cycle *C'* into *P* is (1, 3, 1, 1, 3, 1). With such a choice, we repartition according to (1, **3**, **1**, **1**, 3, 1) (taking the bolded vertices to be  $Q'_1$  and the non-bolded to be  $Q'_2$ ). Both  $Q'_1$  and  $Q'_2$  have degrees (3, 1, 1) into *P* and thus yield DCCs.

Thus we assume  $|Q_1| = 4$  and  $|Q_2| = 3$ . If neither yields a DCC with *P*, by Lemma 14 part (4), the degree sequence of  $Q_2$  is (1, 3, 1) and the degree sequence of  $Q_1$  is one of (1, 3, 1, 0), (1, 3, 0, 1), or (1, 2, 1, 1) or an 'inside' (2, 0, 2, 1). (Technically, this is up to symmetry, but we orient the cycle so that this is the case and since (1, 3, 1) is symmetric we can do this without loss of generality.) Below we list all possible cases with our choice for  $Q'_1$  in bold and  $Q'_2$  in non-bold.

#### (1, **3**, 1, 0, 1, 3, 1) (1, **3**, 0, 1, 1, 3, 1) (1,2,1,1,1,3,1) (2, 0, 2, 1,1, 3, 1)

This yields  $Q'_1$  and  $Q'_2$  both of which are guaranteed to yield DCCs with *P*, completing the argument. One final word of explanation is warranted: Quite crucially, we note that if the original degree sequence of  $Q_1$  involved an 'inside' (2, 0, 2, 1)' the (**2**, **0**, **2**, **1**) of  $Q'_1$  is *not* 'inside' and this explains why, in this case,  $Q'_1$  yields a DCC with *P* while the original  $Q_1$  did not.

In the (10, 12) vs. (10, 12) case where we have (5, 5) vs. (5, 7), we are done if the 5 edges between the two 10 parts create a doubly chorded cycle, as the 7 edges between the other parts make a proper doubly chorded cycle. If it does not, then the arguments are similar to (5, 6) vs. (5, 6).

This completes the proof when  $\ell = 7$ . Fortunately, for larger  $\ell$  the number of edges increases so the proofs are mostly easier.

#### **Case 2:** Suppose $\ell = 8$ .

Recall that we have a (10, 15) decomposition where the 10 edge path has order 2 by Claim 2. We must verify the cases, but they are easy using Lemma 14. Recall that  $P_2 \hookrightarrow_5 P$  and  $\{P_4, P_5\} \hookrightarrow_6 P_5$  yield doubly chorded cycles, and  $P_2 \hookrightarrow_6 P$ ,  $P_3 \hookrightarrow_6 P_5$ ,  $\{P_3, P_4\} \hookrightarrow_6 \{P_3, P_4\}$  and  $\{P_3, P_4, P_5\} \hookrightarrow_7 P$ , yield proper doubly chorded cycles. These cover nearly all of the possible partitions; again we give tables showing how the partitions are covered.

In the (11, 14) vs. (11, 14) case where we have (5, 6) vs. (6, 8), the argument is somewhat more complicated as well. If either of the segments supporting 6 is of length 3 (note both segments are of length at least 3) then we may use the fact that  $P_3 \hookrightarrow_6 \{P_3, P_4, P_5\}$  yields a proper doubly chorded cycle and the fact that  $\{P_4, P_5\} \hookrightarrow_6 \{P_3, P_4, P_5\}$  yields a DCC to get a proper system.

A more complicated scenario occurs when both segments of both cycles have length 4, as  $P_4 \hookrightarrow_6 P_4$  guarantees only a DCC, but not a proper one in the case of  $(2, 1, 1, 2) \hookrightarrow P_4$ . Here, suppose the first cycle is  $x_1, \ldots, x_8$  and the second cycle is  $y_1, \ldots, y_8$ .

We assume  $x_1, \ldots, x_4$  has a total of 11 edges incident to it and  $x_5, \ldots, x_8$  has 14. Likewise, assume  $y_1, \ldots, y_4$  has 11 edges incident to it, and  $y_5, \ldots, y_8$  has 14. The problem case is when there are 6 edges between  $x_1, \ldots, x_4$  and  $y_5, \ldots, y_8$  and likewise between  $y_1, \ldots, y_4$  and  $x_5, \ldots, x_8$ . If neither of the DCC's from the  $P_4 \hookrightarrow_6 P_4$  is proper, the degrees of  $x_1, \ldots, x_4$  into  $y_5, \ldots, y_8$  must be (2, 1, 1, 2) and likewise the degrees of  $x_5, \ldots, x_8$  must be (2, 1, 1, 2).

Now, suppose that either  $x_1$ , or  $x_4$  has degree at most 1 into  $y_1$ , ...,  $y_4$ . Without loss of generality, say  $x_4$  has this property. Now consider the new partition ( $x_4$ ,  $x_5$ ,  $x_6$ ,  $x_7$ ) and ( $x_8$ ,  $x_1$ ,  $x_2$ ,  $x_3$ ).

 $(x_4, x_5, x_6, x_7)$  has a total of 2 + (8 - 3) = 7 edges into  $y_5, y_6, y_7, y_8$ . This follows as there are a total of 2 edges from  $x_4$  into this set (by our degree case) and a total of 8 edges from  $x_5, x_6, x_7, x_8$  into this set. Since  $x_8$  has degree 2 into  $y_1, y_2, y_3, y_4$  it has degree at most 3 into  $x_5, x_6, x_7, x_8$  giving the count.

On the other hand  $(x_8, x_1, x_2, x_3)$  has 5 - 1 + 2 = 6 edges into  $(y_1, y_2, y_3, y_4)$ . This is as we already know that  $x_8$  has degree 2 in this set.  $x_1, x_2, x_3, x_4$  has degree 5 into this set, and we assume that  $x_4$  has degree 1 into this set. Combining gives the desired bound. Thus we have  $P_4 \hookrightarrow_6 P_4$  giving a DCC, and  $P_4 \hookrightarrow_7 P_4$  giving a proper DCC.

Thus the degree of both  $x_4$ , and  $x_1$  is at least 2 into  $y_1, y_2, y_3, y_4$ . Consider the same pairs:  $(x_4, x_5, x_6, x_7)$  still has 7 neighbors in  $y_5, y_6, x_7, y, y_8$  and gives a proper DCC.  $x_8, x_1$  into  $y_1, y_2, y_3, y_4$  is  $(2, 2) \hookrightarrow P_4$  which gives a DCC. This completes the case.

#### **Case 3:** Suppose $\ell = 9$ .

Consider that our (r, s) decomposition is into paths of order 4 and 5 by Lemma 16, and we have at least  $3\ell + 1 = 28$  edges. Thus, our balanced partite sets are (12, 16), (13, 15) and (14, 14). Recall that  $\{P_4, P_5\} \hookrightarrow_6 P_5$  yield doubly chorded cycles, and  $\{P_4, P_5\} \hookrightarrow_7 P$  yield proper doubly chorded cycles by Lemma 1. Thus, it is easy to verify that we always get a pair that guarantees our proper doubly chorded cycle.

#### **Case 4:** Suppose $\ell \geq 10$ .

Again, for  $\ell = 10$ , we get parts of order 5, and the same verification as above works. After this point, the verification gets easier. As above, we consider the balanced partition of both cycles, as guaranteed by Lemma 16. This gives a (P, Q) decomposition of the *C* with  $|P| = \lceil \frac{\ell}{2} \rceil$  and a  $Q = \lfloor \frac{\ell}{2} \rfloor$ , along with a (P', Q') decomposition of *C'*. The total degree of *C* into *C'* is at least  $3\ell + 1$ . Consider the one of *P* and *Q* with the lower degree; it sends at least half of its edges to one of *P'* or *Q'*. Now consider the 'parallel pair'. By the minimality of *P*, this must have degree at least as high as the initially considered pair. For  $\ell = 11$ , these are both at least 8. Since  $P_5 \hookrightarrow_7 P$  and  $P_6 \hookrightarrow_8 P$  yield DCCs, this completes the proof. For  $\ell = 12$ , these are both at least 8 as well, and that  $P_6 \hookrightarrow_8 P$  yields a DCC as well.

For  $\ell = 13$ , the minimum degree of the first pair is 9 as the minimum degree in the balanced partition is 18. If the degree is at least 10 the fact that  $P \hookrightarrow_{10} P$  yields a proper DCC. However, if the degree in the first pair is 9, the degree from whichever of *P* or *Q* which has minimum degree must be 9 into *both P'* and *Q'*. This also implies that the other of *P* or *Q* must have degree at least 9 into both *P'* and *Q'*. By pairing up the larger cardinality of *P* and *Q* with the smaller cardinality of *P'* and *Q'* and vice versa, we may use the fact that  $P_6 \hookrightarrow_8 P$  yields a proper DCC twice to find the desired DCCs. Finally, if  $\ell > 14$ , the proof is trivial: the minimum pair degree is at least 10 and the fact that  $P \hookrightarrow_{10} P$  yields a proper DCC finishes the proof.

Finally, we turn to give the proof of Lemma 14.

**Proof of Lemma 14.** Throughout the proof, for a degree sequence  $(d_1, \ldots, d_i) \hookrightarrow P_k$  we let  $v_j$  denote the *j*th vertex, that is, the vertex of degree  $d_j$ , and  $x_1 < x_2 < \cdots < x_{d_1}$  denote the neighbors of  $v_1$ , and likewise  $y_1 < y_2 < \cdots y_{d_2}$ , and  $z_1 < z_2 < \cdots < z_{d_3}$ ,  $w_1 < w_2 < \cdots w_{d_4}$  denote the neighbors of the vertices with degrees  $d_2$  and  $d_3$ ,  $d_4$  in increasing order with respect to some ordering of the  $P_k$ . For sequences with  $\star$ , the  $v_j$ ,  $x_\ell$ ,  $y_{\ell'}$ , etc. only refer to the neighbors of vertices with specified degrees. Note that  $v_j$  is denoted for positive degree  $d_j$ .

**Claim** ( $\alpha$ ): {(2, 2), (2,  $\star$ , 2)}  $\hookrightarrow$  *P* yields a DCC unless  $x_1 < y_1 < y_2 < x_2$  (up to reordering  $v_1$  and  $v_2$ ).

**Proof.** Let *Q* be a path with degree sequence (2, 2) or (2,  $\star$ , 2) and vertices  $v_1 < v_2$ , each of degree 2. If  $y_1 \le x_1 < y_2 \le x_2$  (say), then  $v_1, x_2, P^-, y_1, v_2, Q^-, v_1$  is a cycle with chords  $v_2y_2$  and  $v_1x_1$ . If  $x_1 = y_1 < x_2 < y_2$ , consider  $v_1, x_1, P, y_2, v_2, Q^-, v_1$ , which is a DCC even when  $x_1 < x_2 \le y_1 < y_2$ .  $\Box$ 

This immediately yields that  $(2, 2) \hookrightarrow P_3$  and  $(2, \star, 2) \hookrightarrow P_3$  yield a DCC, and also that  $(2, 2) \hookrightarrow P_4$  does as well. If  $(2, 2) \hookrightarrow P_4$ , then either there is a clear DCC (as in the proof above) or by Claim ( $\alpha$ ) because  $P = P_4$  we need only consider the case  $x_1 < y_1 < y_2 < x_2$ , and  $v_1, x_1, y_1, v_2, y_2, x_2, v_1$  is a DCC with chords  $y_1y_2$  and  $v_1v_2$ .

 $(2, 0, 2, 1) \hookrightarrow P_4$  yields a DCC except for an inside (2, 0, 2, 1) by Claim  $(\alpha)$ .

**Claim** ( $\beta$ ): {(3, 2), (3,  $\star$ , 2)}  $\hookrightarrow$  *P* yields a DCC.

**Proof.** Let *Q* be a path with degree sequence as above and vertices  $v_1 < v_2$ . After removing the edge  $v_1x_1$ , we have either a DCC or  $x_2 < y_1 < y_2 < x_3$  or  $y_1 < x_2 < x_3 < y_2$  by Claim ( $\alpha$ ). Then, in either case, adding  $x_1$  yields a DCC. Indeed, in the first case  $v_1, x_1, P, y_2, v_2, Q^-, v_1$  has chords  $v_1x_2$  and  $v_2y_1$  and in the second case it has chords  $v_1x_2$  and  $v_1x_3$ .  $\Box$ 

From here it is easy to see that  $(3, \star, 2) \hookrightarrow \{P_3, P_4\}$  yields a proper DCC. Indeed, for  $(3, \star, 2) \hookrightarrow P_3$ , it is clear that the resulting cycle is proper if  $x_1 = y_1 < x_2 = y_2 < x_3$ , as  $x_3$  is not required for a DCC. Thus, the remaining case is  $x_1 = y_1 < x_2 < x_3 = y_2$ . This has cycle  $v_1, x_1, v_2, y_2, x_2, v_1$  which has chords  $x_1x_2$  and  $v_1x_3$ . (Note that this is only proper because  $\star$  is posited to be a *non-empty* sequence.) Taken into  $P_4$ , the same argument applies in the first case (replacing  $x_2 = y_2$  with  $x_2 \leq y_2$ ). In the second case, note that either  $x_1x_2 \in E(P_4)$  or  $x_2x_3 \in E(P_4)$ , so the cycle above works. Next, assume  $y_1 < x_1 < x_2 < x_3$ . This gives an obvious cycle in which case  $y_1$  or  $x_3$  can be avoided. The remaining cases are of the form  $x_1 < x_2 = y_1 < y_2 < x_3$  (or the mirror with  $x_2 = y_2$ ). But these also have obvious cycles avoiding  $x_3$ .

Since it is easy to see that (4)  $\hookrightarrow$  *P* yields a DCC, this and Claim ( $\beta$ ) give that  $P_2 \hookrightarrow_5 P$  yields a DCC and both  $P_2 \hookrightarrow_5 P_4$ and  $P_2 \hookrightarrow_6 P$  yield proper DCCs. Indeed, the only case of  $P_2 \hookrightarrow_6 P$  that is not automatic is that (3, 3)  $\hookrightarrow$  *P*. Removing the edges  $v_1x_1$  and  $v_2y_1$  yields either  $x_2 < y_2 < y_3 < x_3$  or  $y_2 < x_2 < x_3 < y_3$ , by Claim ( $\alpha$ ). Adding back  $x_1$  in the first case or  $y_1$  in the second case yields a proper DCC avoiding  $x_3$  or  $y_3$  respectively.

**Claim**  $(\gamma)$ : {(3, 1, 1), (3,  $\star$ , 1, 1)}  $\hookrightarrow$   $P_3$ , and (3, 1, 1)  $\hookrightarrow$   $P_4$  yield proper DCCs.

**Proof.** Let *Q* be a path with degree sequence as above and vertices  $v_1 < v_2 < v_3$ . First consider the case {(3, 1, 1), (3,  $\star$ , 1, 1)}  $\hookrightarrow$  *P*<sub>3</sub>. Note that the *P*<sub>3</sub> is precisely  $x_1 < x_2 < x_3$  and if  $y_1 \in \{x_1, x_3\}$ , then if  $y_1 = x_1$  then  $v_1, x_3, x_2, x_1, v_2, Q^-, v_1$  is a proper DCC avoiding  $v_3$  and has chords  $v_1x_1$  and  $v_1x_2$ ; while if  $y_1 = x_3$  then  $v_1, Q^-, v_2, x_3, x_2, x_1, v_1$  is a proper DCC avoiding  $v_3$  and with chords  $v_1x_2$  and  $v_1x_3$ . Thus  $y_1 = x_2$ . With this, it is easy to check that regardless of whether  $z_1$  is  $x_1, x_2$  or  $x_3$  then there is a proper DCC. In the case (3, 1, 1)  $\hookrightarrow$  *P*<sub>4</sub> the argument is similar. Let *g* denote the vertex not adjacent to  $v_1$  on the *P*<sub>4</sub>. Without loss of generality either  $g < x_1 < x_2 < x_3$  or  $x_1 < g < x_2 < x_3$ . In the first case, note that (as before)  $y_1 = x_2$  or we are done, and similarly, regardless of  $z_1$ , there is a proper DCC. In the second case,  $y_1 \in \{g, x_2\}$  or we are done as before. However, regardless of the placement of  $z_1$ , similar proper DCCs exist. The more difficult case is  $y_1 = g$  and  $z_1 = x_3$ . Then  $v_1, x_2, g, v_2, v_3, x_3, v_1$  is a proper DCC avoiding  $x_1$  and with chords  $v_1v_2$  and  $x_2x_3$ .

**Claim** ( $\delta$ ): (3, 1, 1)  $\hookrightarrow$  *P*, (3, 1,  $\star$ , 1)  $\hookrightarrow$  *P*<sub>4</sub>, and (3,  $\star$ , 1, 1)  $\hookrightarrow$  *P*<sub>4</sub> each yield a DCC.

**Proof.** If  $y_1 \le x_1$  or  $y_1 \ge x_3$ , this is clear. If  $y_1 = x_2$ , it is as in Claim ( $\gamma$ ). Thus we may assume that, without loss of generality  $x_1 < y_1 < x_2$ . Now consider  $z_1$ . Again,  $x_1 < z_1 < x_3$  or we are done. If  $x_2 \le z_1$ , then a doubly chorded cycle  $v_1, x_1, P, z_1, v_3, v_2, v_1$  is clear. Similarly if  $z_1 \le y_1$ , then  $v_1, x_3, P^-, z_1, v_3, v_2, v_1$  is a DCC. The remaining case is  $x_1 < y_1 < z_1 < x_2 < x_3$ . But then  $v_1, x_1, P, x_3, v_1$  has chords  $v_1x_2$  and  $v_1v_2$ .

For the second part, let Q be a path with degree sequence  $(3, 1, \star, 1)$  and vertices  $v_1 < v_2 < v_3$ . As in Claim  $(\gamma)$  we have either  $g < x_1 < x_2 < x_3$  or  $x_1 < g < x_2 < x_3$ . In the first case, if  $y_1$  or  $z_1$  is one of  $g, x_1, x_3$  there exist DCCs. Thus,  $y_1 = z_1 = x_2$  and  $v_1, Q, v_3, x_2, x_1, v_1$  is a cycle with chords  $v_1 x_2$  and  $v_2 x_2$ . In the second case, we may assume  $y_1, z_1 \in \{g, x_2\}$ . Suppose

 $y_1 = g$ . If  $z_1 = g$ , then  $v_1$ , Q,  $v_3$ , g,  $x_2$ ,  $x_3$ ,  $v_1$  is a cycle with chords  $v_1x_2$  and  $v_2g$ . If  $z_1 = x_2$ , then  $v_1$ , Q,  $v_3$ ,  $x_2$ , g,  $x_1$ ,  $v_1$  is a cycle with chords  $v_1x_2$  and  $v_2g$ . Suppose  $y_1 = x_2$ . Then there exist DCCs regardless of  $z_1$  by the same argument as the case  $y_1 = g$ .

The proof of the third part is analogous to that of the second part.  $\Box$ 

**Claim** ( $\epsilon$ ): (2, 1, 2),  $\hookrightarrow$  *P*<sub>4</sub> yields a proper DCC.

**Proof.** Let  $g_1$  and  $g_2$  denote the vertices not adjacent to  $v_1$  on  $P_4$ . Without loss of generality, then, the cases are  $x_1 < x_2 < g_1 < g_2$ , and  $x_1 < g_1 < x_2 < g_2$ , and  $x_1 < g_1 < x_2 < g_2$ , and  $x_1 < g_1 < x_2 < g_2$ , and  $x_1 < x_1 < x_2 < g_2$ . In the first two cases note that we are done unless  $z_2 = g_2$ , (as then we have  $(2, \star, 2) \hookrightarrow P_3$  of Claim ( $\alpha$ ) and the cycle is now proper as  $g_2$  is not included). Suppose  $x_1 < x_2 < g_1 < z_2$ . In the case that  $z_1 \in \{x_2, g_1\}$ , then it is easy to find proper DCCs regardless of the placement of  $y_1$ , avoiding either  $x_1$  or  $z_2$ . The more difficult case is when  $x_1 = z_1 < x_2 < g_1 < z_2$ . If  $y_1 \in \{x_1, x_2\}$  it is easy to build proper DCCs avoiding  $\{g_1, z_2\}$ . If  $y_1 = z_2$ , then  $v_1, x_2, x_1, v_3, z_2, v_2, v_1$  avoids  $g_1$  and has chords  $v_1x_1$  and  $v_3v_2$ . If  $y_1 = g_1$ , then  $v_1, x_1, v_3, v_2, g_1, x_2, v_1$  avoids  $z_2$  and has chords  $x_1x_2$  and  $v_1v_2$ . Next suppose  $x_1 < g_1 < x_2 < z_2$ . If  $z_1 \in \{g_1, x_2\}$ , we can find proper DCCs avoiding either  $x_1$  or  $g_2$ , depending on  $y_1$ . Thus, we may assume  $z_1 = x_1$ . If  $y_1 \in \{x_1, g_1, x_2\}$ , it is easy to find a proper DCC avoiding  $g_2$ . In the case where  $y_1 = g_2 = z_2$ , then  $v_1, x_1, v_3, v_2, z_2, x_2, v_1$  has chords  $v_3z_2$  and  $v_1v_2$  and avoids  $g_1$ . Next suppose  $x_1 < g_1 < g_2 < x_2$ . If  $x_1 = z_1$  and  $z_2 = g_2$ , or  $z_1 = g_1$  and  $z_2 = x_2$ , then this is similar to the cases already examined. If  $x_1 = z_1$  and  $x_2 = z_2$ , then if  $y_1 = x_1$  (or symmetrically  $y_1 = x_2$ ) the cycle  $v_1, x_1, v_2, v_3, z_2, v_1$  has chords  $v_1v_2$  and  $v_3x_1$  and avoids  $g_2$  and hence is a proper doubly chorded cycle. If  $z_1 = g_1$  and  $z_2 = g_2$  or  $z_1 = x_1$  and  $z_2 = g_1$  (or symmetrically  $z_1 = g_2$  and  $z_2 = g_2$  or  $z_1 = x_1$  and  $z_2 = g_1$  (or symmetrically  $z_1 = g_2$  is similar.  $\Box$ 

**Claim**  $(\zeta)$ : {(2, 1, 2), (2, 1,  $\star$ , 2)}  $\hookrightarrow$  *P* yields a DCC.

**Proof.** Note that as we just require a DCC, and not a proper one, we may assume that we have  $x_1 < z_1 < z_2 < x_2$  (or  $z_1 < x_1 < x_2 < z_2$ ) or we are done by Claim ( $\alpha$ ). Regardless of  $y_1$ , we can find a doubly chorded cycle.  $\Box$ 

By Claims  $(\beta)$ ,  $(\gamma)$  and  $(\zeta)$  (resp.  $(\epsilon)$ ), and since  $(2, 2) \hookrightarrow P_3$  (resp.  $P_4$ ) yields a DCC by Claim  $(\alpha)$  and  $(3, \star, 2) \hookrightarrow P_3$  (resp.  $P_4$ ) yields a proper DCC by Claim  $(\beta)$ , the assertion of Lemma 14 (2) (resp. (4)) holds.

**Claim**  $(\eta)$ :  $(2, 1, 1, 2) \hookrightarrow P_4$  yields a DCC.

**Proof.** This actually follows from Claim ( $\epsilon$ )'s argument.  $\Box$ 

**Claim** ( $\theta$ ):  $P_3 \hookrightarrow_6 P$  yields a DCC.

**Proof.** Since  $(4) \hookrightarrow P$  yields a DCC, we have max degree 3. Since  $\{(3, 2), (3, \star, 2)\} \hookrightarrow P$  yields a DCC by Claim ( $\beta$ ), the only remaining case is  $(2, 2, 2) \hookrightarrow P$ , which follows from Claim ( $\zeta$ ).  $\Box$ 

**Claim** ( $\iota$ ):  $P_3 \hookrightarrow_6 P_5$  yields a proper DCC.

**Proof.** If (3, 2) or (4) arises in the degree sequence of the  $P_3$  we are done as we do not use all of the  $P_3$ . Thus, our options are  $(3, 1, 2) \hookrightarrow P_5$ , or  $(3, 0, 3) \hookrightarrow P_5$ , or  $(2, 2, 2) \hookrightarrow P_5$ . Consider  $(3, 1, 2) \hookrightarrow P_5$ , which is quite similar to Claim  $(\beta)$ . Let  $g_1, g_2$  denote the vertices not adjacent to  $v_1$ . Then without loss of generality, the possibilities are  $x_1 < x_2 < x_3 < g_1 < g_2$ ,  $x_1 < x_2 < g_1 < x_3 < g_2, x_1 < g_1 < x_2 < g_2 < x_3$  and  $x_1 < x_2 < g_1 < g_2 < x_3$  and  $x_1 < g_1 < x_2 < x_3 < g_2$  and  $g_1 < x_1 < x_2 < x_3 < g_2$ . In the first two cases it is clear that  $z_2 = g_2$  (as otherwise we have  $(3, \star, 2) \hookrightarrow P_4$  of Claim  $(\beta)$ ). It is easy to find proper DCC's then, regardless of  $z_1$ . In the case where  $x_1 < g_1 < x_2 < x_3 < g_2$  are straightforward. In the cases for  $x_1 < x_2 < g_1 < g_2 < x_3$ , it is easy to find proper DCC's then, regardless of  $z_1$ . In the case where  $x_1 < g_1 < x_2 < x_3 < g_2$  are straightforward. In the cases for  $x_1 < x_2 < g_1 < g_2 < x_3$ ,  $x_1 < g_1 < x_2 < x_3 < g_2$  and  $g_1 < x_1 < x_2 < x_3 < g_2$  are straightforward. In the case  $(3, 0, 3) \hookrightarrow P_5$ , the argument is similar. It remains to consider  $(2, 2, 2) \hookrightarrow P_5$ . The case  $(2, 2) \hookrightarrow P_5$  already yields the desired proper DCC unless either  $x_1 < y_1 < g < y_2 < x_2$ , where g denotes the gap (or similarly  $x_1 < g < y_1 < y_2 < x_2$ ), or  $y_1 < x_1 < g < x_2 < y_2$  (or similarly  $y_1 < g < x_1 < x_2 < y_2$  by Claim  $(\alpha)$ . In the first case  $z_1 = x_1$  and  $z_2 = x_2$  or we are done by Claim  $(\alpha)$ ). Then  $v_1, x_1, y_1, v_2, v_3, z_2, v_1$  is the proper DCC avoiding  $\{g, y_2\}$  and with chords  $v_1v_2$  and  $v_3x_1$ . In the second case the argument is similar.  $\Box$ 

**Claim** ( $\kappa$ ): (2, 2, 1, 1)  $\hookrightarrow$  *P* yields a DCC.

**Proof.** We may assume that  $x_1 < y_1 < y_2 < x_2$  or  $y_1 < x_1 < x_2 < y_2$  or the  $(2, 2) \hookrightarrow P$  already gives the desired cycle by Claim ( $\alpha$ ). In the first case, note that if  $z_1 \le y_1$  or  $y_2 \le z_1$ , DCCs are easy to find. A similar argument applies for  $w_1$ . Thus  $y_1 < z_1 < y_2$  and  $y_1 < w_1 < y_2$ . If  $z_1 \le w_1$ , then  $v_1, x_1, P, w_1, v_4, v_3, v_2, v_1$  is a DCC with chords  $v_2y_1$  and  $v_3z_1$ . Otherwise, if  $z_1 > w_1$ , we use  $v_1, x_2, P^-, w_1, v_4, v_3, v_2, v_1$ . Thus, we are in the case  $y_1 < x_1 < x_2 < y_2$ . Here we have  $y_1 < z_1 < y_2$  and  $y_1 < w_1 < y_2$ . Note that, possibly by reversing P, we may assume that  $z_1 \le w_1$ . We now simply list the cases and cycles and chords in the table that follows.  $\Box$ 

By Claims $(\alpha)$ , $(\beta)$	, $(\delta)$ , $(\zeta)$ and $(\sigma)$	, the assertion of Lemma 14	part (5) holds.
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Case	Cycles	Chords
$y_1 < z_1 < x_1 < x_2 < y_2, z_1 < w_1 < x_2$	$v_4, w_1, P, x_1, v_1, x_2, P, y_2, v_2, y_1, P, z_1, v_3, v_4$ (Note: $w_1, P^-, x_1$ if $x_1 < w_1$ )	$v_1v_2, v_2v_3$
$ \begin{array}{l} y_1 < x_1 = z_1 < w_1 < x_2 < y_2 \\ y_1 < z_1 = w_1 \le x_1 < x_2 < y_2 \\ y_1 < z_1 \le x_1 < x_2 \le w_1 < y_2 \end{array} $	$v_4, w_1, P, x_2, v_1, x_1, P^-, y_1, v_2, v_3, v_4$ $v_4, w_1, P, x_1, v_1, x_2, P, y_2, v_2, v_3, v_4$ $v_4, w_1, P^-, x_2, v_1, x_1, P^-, y_1, v_2, v_3, v_4$	$v_1v_2, v_3x_1 v_3z_1, v_1v_2 v_3z_1, v_1v_2$
$y_1 < x_1 < z_1 \le x_2 < y_2, z_1 < w_1 < y_2$	$v_4, w_1, P, x_2, v_1, v_2, y_1, P, z_1, v_3, v_4$ (Note: $v_4, w_1, P^-, x_1, v_1, v_2, v_3, v_4$ if $x_2 < w_1$ )	$v_1 x_1, v_2 v_3$ $v_1 x_2, v_3 z_1$
$y_1 < x_1 < z_1 = w_1 \le x_2 < y_2 y_1 < x_1 < x_2 < z_1 \le w_1 < y_2$	$v_4, w_1, P, x_2, v_1, x_1, P^-, y_1, v_2, v_3, v_4$ $v_4, w_1, P^-, x_2, v_1, x_1, P^-, y_1, v_2, v_3, v_4$	$v_1v_2, v_3z_1 \\ v_1v_2, v_3z_1$

**Claim** ( $\lambda$ ): { $P_3$ }  $\hookrightarrow_6$  { $P_3$ ,  $P_4$ } yields a proper DCC while  $P_4 \hookrightarrow P_4$  yields a DCC which is proper except in the (2, 1, 1, 2)  $\hookrightarrow P_4$  case.

**Proof.** Consider the case  $P_k \hookrightarrow_6 P_j$  where  $k \leq j$ . Note that  $(4) \hookrightarrow P_4$  yields a DCC so the max degree (of either) is 3. Since  $(3, 2) \hookrightarrow P_3$  yields a DCC by Claim  $(\beta)$ , and  $(3, \star, 2) \hookrightarrow \{P_3, P_4\}$  and  $(3, 2) \hookrightarrow P_4$  yield proper DCCs by Claim  $(\beta)$ , if the maximum degree of  $P_k$  is 3, then all other vertices have degree 1 and we have only to consider  $P_4 \hookrightarrow_6 P_4$ . But  $(3, 1, 1) \hookrightarrow P_4$  yields a proper DCC by Claim  $(\gamma)$ . Thus, the maximum degree is 2. Note that  $(2, 2) \hookrightarrow \{P_3, P_4\}$  yields a DCC by Claim  $(\alpha)$ , so if  $P_k$  has two adjacent vertices of degree 2 we are done. Thus, we are done unless k = j = 4. Then our options are (2, 1, 2, 1) or (2, 1, 1, 2). Since we have already shown in Claims  $(\zeta)$  and  $(\eta)$  that  $(2, 1, 2) \hookrightarrow P_4$  yields a DCC and  $(2, 1, 1, 2) \hookrightarrow P_4$  yields a DCC we are done.  $\Box$ 

**Claim** ( $\mu$ ): { $P_3$ ,  $P_4$ ,  $P_5$ }  $\hookrightarrow_7 P$  yields a proper DCC.

**Proof.** Note that it suffices to show that this yields a DCC. If there is such a cycle, there is necessarily a proper one by Lemma 2. We have already shown that  $P_3 \hookrightarrow_6 P$  yields a DCC by Claim ( $\theta$ ), so this case is trivial. In the cases of  $P_4 \hookrightarrow_7 P$  or  $P_5 \hookrightarrow_7 P$ , note that (4)  $\hookrightarrow P$  yields a DCC and {(3, 2), (3,  $\star$ , 2), (3, 1, 1)}  $\hookrightarrow P$  all yield DCCs by Claims ( $\beta$ ) and ( $\delta$ ), and hence the maximum degree is 2. But {(2, 1, 2), (2, 1,  $\star$ , 2), (2, 2, 1, 1)}  $\hookrightarrow P$  yield DCCs by Claims ( $\zeta$ ) and ( $\kappa$ ) and one of these must occur as  $P_5 \hookrightarrow_7 P$  must have at least 2 vertices of degree 2 (and if there are vertices of degree 0, then there are additional vertices of degree 2).  $\Box$ 

**Claim** (v):  $P_6 \hookrightarrow_8 P$  yields a proper DCC.

**Proof.** Again we have that  $\{(4), (3, 2), (3, \star, 2), (3, 1, 1)\} \hookrightarrow P$  yield DCCs, so we are done unless the  $P_6$  has maximum degree 2. Note that we then cannot avoid  $\{(2, 1, 2), (2, 1, \star, 2), (2, 2, 1, 1)\}$  and hence are guaranteed a DCC (and hence a proper one by Lemma 2).  $\Box$ 

**Claim**  $(\xi)$ :  $P_7 \hookrightarrow_9 P$  yields a proper DCC.

**Proof.** Analogous to Claim ( $\nu$ ).  $\Box$ 

**Claim** (*o*):  $P \hookrightarrow_{10} P'$  yields a proper DCC.

**Proof.** As in the Claim ( $\mu$ ) it suffices to show the existence of one DCC. In this case, we order the vertices of *P* and *P'*. Then the chords give a permutation  $\sigma \in S_{10}$  (the symmetric group) induced by the endpoints of the edges between the paths, breaking ties arbitrarily. The Erdős–Szekeres Theorem [5] guarantees an increasing or decreasing sequence of length 4 which easily gives the desired DCC.  $\Box$ 

**Claim**  $(\pi)$ :  $(1, 2, 2, 1) \hookrightarrow P_5$  yields a proper DCC.

**Proof.** If the  $(2, 2) \hookrightarrow P_5$  does not already yield a DCC, then we have, two cases to consider. First, if  $y_1 < z_1 < g < z_2 < y_2$ , where g is a gap, it is trivial to find a proper DCC unless  $w_1 = g$ , so that we have  $y_1 < z_1 < w_1 < z_2 < y_2$ . But then  $v_2, y_1, z_1, v_3, v_4, w_1, z_2, y_2, v_2$  avoids  $v_1$  and has chords  $z_1w_1$  and  $v_2v_3$ . Next, if  $y_1 < z_1 < z_2 < g < y_2$ , then it is easy to find a proper DCC by considering  $w_1$ .  $\Box$ 

**Claim** ( $\rho$ ): (2, 2, 1, 1)  $\hookrightarrow$  *P*<sub>5</sub> yields a proper DCC.

**Proof.** If  $(2, 2) \hookrightarrow P_5$  does not already yield a DCC, then we have  $x_1 < y_1 < g < y_2 < x_2$ , or  $y_1 < x_1 < g < x_2 < y_2$ where *g* is a gap. The first case is analogous to Claim ( $\pi$ ), and  $v_4$  can be avoided. In the second case, if  $z_1 \in \{y_1, g, y_2\}$ , DCCs are easy to find; the most difficult being when  $z_1 = g$  and a proper DCC is  $v_1, x_1, y_1, v_2, v_3, z_1, x_2, v_1$  with chords  $v_1v_2$  and  $x_1z_1$ . Thus we may assume that  $z_1 = x_1$  (or symmetrically  $z_1 = x_2$ ). Likewise it is easy to see that we either have that  $w_1 = x_1$  or  $w_1 = x_2$ , or we may omit one of  $y_1, y_2$  and find a proper DCC. The more difficult case is  $w_1 = g$ . Then  $v_1, x_1, v_3, v_4, g, x_2, y_2, v_2, v_1$  is a cycle with chords  $v_1x_2$  and  $v_2v_3$  avoiding  $y_1$ . In either case it is quite easy to find a proper DCC avoiding both  $y_1$  and  $y_2$ .  $\Box$ 

#### **Claim** ( $\sigma$ ): (2, 1, 1, 1) $\hookrightarrow$ { $P_3$ , $P_4$ } yields a DCC.

**Proof.** We begin by noting that  $(2, 1, 1, 1) \hookrightarrow P_3$  yields a DCC. Indeed, if there were no DCC, then the  $P_3$  would have degree sequence (1, 3, 1). We may assume, without loss of generality, that  $x_1$  is the first vertex of the  $P_3$ . Then  $w_1$ , the neighbor of  $v_4$ , must be one of the bottom two vertices of the  $P_3$ . Hence, the cycle containing the paths along with  $v_1x_1$  and  $v_4w_1$  avoids only (possibly) the edge incident to the bottom vertex of the  $P_3$  and is thus a DCC. Now we consider the case  $(2, 1, 1, 1) \hookrightarrow P_4$ . We repeatedly use the fact that if we can find a cycle containing 7 of the 8 vertices, omitting only an end vertex of degree 1, then the cycle is doubly chorded. This fact follows by counting edges as such a cycle has 5 path edges and 4 cross edges, for a total of 9 edges induced on a 7 vertex cycle. We denote the ordered vertices on  $P_4$  by  $u_1, u_2, u_3, u_4$ . First we consider the case where  $x_1 = u_1$ . The vertex  $u_4$  has degree at least 1, as otherwise we are in the case  $(2, 1, 1, 1) \hookrightarrow P_3$ . If  $u_4 = z_1$  or  $u_4 = w_1$ , either the resulting graph is Hamiltonian or contains a DCC on 7 vertices. Thus either  $u_4 = x_2$  or  $u_4 = y_1$  (or both). If  $u_4 = x_2$ , then by symmetry,  $w_1 = u_3$  and we have a cycle on 7 vertices. Therefore  $u_4$  must have degree 2 and hence  $u_4 = y_1$ . But then  $v_1, u_4, v_2, v_3, v_4, u_3, u_2, u_1, v_1$  is Hamiltonian. Thus, we may assume that  $u_4 \neq x_2$  but instead that  $u_4 = y_1$ .

Since then the degree of  $u_4$  is 1, we may assume that  $w_1 \neq u_3$  (as otherwise there would be a cycle avoiding only  $u_4$ ). Thus  $w_1 = u_1$  or  $w_1 = u_2$ . First suppose  $w_1 = u_1$ . If  $x_2 = u_2$ , then  $v_1, u_1, v_4, v_3, v_2, u_4, u_3, u_2, v_1$  is Hamiltonian. Suppose  $x_2 = u_3$ . Likewise if  $z_1 = u_2$ , then  $v_1, v_2, u_4, u_3, u_2, v_3, v_4, u_1, v_1$  is Hamiltonian. Therefore,  $u_2$  has degree 0 across and by edge counting the cycle  $v_1, u_1, v_4, v_3, v_2, u_4, u_3, v_1$  which contains every vertex except  $u_2$  and induces every cross edge, is a DCC. If  $x_2 = u_4$ , then  $v_1, v_2, v_3, v_4, u_1, u_2, u_3, u_4, v_1$  is Hamiltonian. Next suppose  $w_1 = u_2$ . Consider the second neighbor of  $v_1$ . Suppose  $x_2 = u_2$ . Then we may assume  $z_1 = u_3$ , or we can find a DCC easily. Then  $v_1, v_2, u_4, u_3, v_3, v_4, u_2, v_1$  is a cycle with chords  $v_2v_3$  and  $u_2u_3$ . Suppose  $x_2 = u_3$ . Then  $v_1, u_1, u_2, v_4, v_3, v_2, u_4, u_3, v_1$  is Hamiltonian. Suppose  $x_2 = u_4$ . If  $z_1 \in \{u_2, u_3, u_4\}$ , then  $v_1, v_2, v_3, v_4, u_2, u_3, u_4, v_1$  is a cycle with chords  $v_2u_4$  and  $v_3z_1$ . If  $z_1 = u_1$ , then  $v_1, v_2, v_3, v_4, u_2, u_3, u_4, v_1$  is a cycle with chords  $v_2u_4$  and  $v_3z_1$ . If  $z_1 = u_1$ , then  $v_1, v_2, u_4, u_3, u_2, v_4, v_3, u_1, v_1$  is Hamiltonian. This completes the case where  $x_1 = u_1$  (and by symmetry the case where  $x_2 = u_4$ ).

The remaining case is when  $x_1 = u_2$  and  $x_2 = u_3$ . Again, both  $u_1$  and  $u_4$  must have positive degree (as otherwise we reduce to the case where  $(2, 1, 1, 1) \hookrightarrow P_3$ ). If one of  $u_1, u_4$  has degree 3, we reduce to the case where  $(2, 1, 1, 1) \hookrightarrow P_3$ . If one of  $u_1, u_4$  has degree 2, then we also have the degree sequence (2, 1, 1, 1) within the  $P_4$ , and (up to symmetry) this gives  $y_1 = z_1 = u_1$  and  $w_1 = u_4$ . Then  $v_1, u_2, u_1, v_2, v_3, v_4, u_4, u_3, v_1$  is Hamiltonian. Thus,  $u_1$  and  $u_4$  both have degree 1. If  $w_1 = u_4$ , then this gives a cycle avoiding just  $u_1$  (which has degree 1). By symmetry, the case  $w_1 = u_1$  is the same. Therefore,  $y_1 = u_1$  and  $z_1 = u_4$ . Then  $v_1, u_2, u_1, v_2, v_3, u_4, u_3, v_1$  is a cycle with chords  $v_1v_2$  and  $u_2u_3$ . This completes the case.  $\Box$ 

## **Claim** $(\tau)$ : { $P_4$ , $P_5$ } $\hookrightarrow_6 P_5$ yields a DCC.

**Proof.** As in Claim ( $\lambda$ ) cases, we may assume that the maximum degree in the  $P_4 \hookrightarrow_6 P_5$  or  $P_5 \hookrightarrow_6 P_5$  is 2. The cases not covered by a combination of Claims ( $\zeta$ ), ( $\pi$ ), ( $\rho$ ) are (2, 0, 2, 0, 2)  $\hookrightarrow$   $P_5$  and (2, 2, 0, 1, 1)  $\hookrightarrow$   $P_5$  (or its reverse sequence or some permutation of the middle three terms of these two sequences), and {(1, 2, 2, 0, 1), (1, 2, 0, 2, 1), (1, 0, 2, 2, 1)}  $\hookrightarrow$   $P_5$ , and the case where the minimum degree in the  $P_5$ 's is 1.

For  $(2, 0, 2, 0, 2) \hookrightarrow P_5$ , the fact that  $(2, 0, 2) \hookrightarrow P_5$  yields no DCC implies that (with *g* the vertex nonadjacent to either vertex of degree two) we either have  $x_1 < y_1 < g < y_2 < x_2, x_1 < y_1 < y_2 < g < x_2$  or  $x_1 < y_1 < y_2 < x_2 < g$  or similar inequalities with the roles of *x* and *y* reversed. If the  $x_i$  are in the exterior vertices, then the  $y_i$  and  $z_i$  are forced to be interior. But the  $y_i$  being interior force the  $z_i$  to be exterior, since, for example in the  $x_1 < y_1 < g < y_2 < x_2$  case, if  $z_1 = y_1$  and  $z_2 = g$ , then  $v_1, v_2, v_3, v_4, v_5, y_1, g, y_2, x_2, v_1$  is proper (as  $x_1$  is omitted) and has chords  $v_3y_1$  and  $v_3y_2$ . Similar cycles can be found in the other cases for the adjacencies of  $v_5$ . But the  $x_i$  and  $z_i$  both exterior implies a DCC exists as shown earlier. If  $x_i$  is interior, then  $y_i$  are exterior, but again there are DCCs once  $z_i$  is included in either the interior or exterior.

For  $(2, 2, 0, 1, 1) \hookrightarrow P_5$ , this almost follows the proof of Claim ( $\rho$ ). As in Claim ( $\rho$ ) if  $(2, 2) \hookrightarrow P_5$  does not already produce a DCC, then  $x_1 < y_1 < g < y_2 < x_2$  or  $y_1 < x_1 < g < x_2 < y_2$ . In the first case, if  $z_1 = x_2$ , then  $v_1, v_2, v_3, v_4, x_2, y_2, g, y_1, x_1, v_1$  has chords  $v_2y_1$  and  $v_2y_2$ . If  $z_1 = y_2$  then a similar cycle with the same chords is obvious. If  $z_1 = g$ , then  $v_1, x_1, y_1, g, v_4, v_3, v_2, y_2, x_2, v_1$  has chords  $v_1v_2$  and  $v_2y_1$ . If  $z_1 = y_1$ , then  $v_1, x_1, y_1, v_4, v_3, v_2, y_2, x_2, v_1$  has chords  $v_1v_2$  and  $v_2y_1$ . If  $z_1 = y_1$ , then  $v_1, x_1, y_1, v_4, v_3, v_2, y_2, x_2, v_1$  has chords  $v_1v_2$  and  $v_2y_1$ . If  $z_1 = y_1$ , then  $v_1, x_1, y_1, v_4, v_3, v_2, y_2, x_2, v_1$  has chords  $v_1v_2$  and  $v_2y_2$ . In the second case, if  $z_1 = y_2$ , then  $v_1, x_1, y_1, v_2, v_3, v_4, y_2, x_2, v_1$  has chords  $v_1v_2$  and  $v_2y_2$ . If  $z_1 = y_1$ , then  $v_1, v_2, v_3, v_4, y_1, x_1, g, x_2, v_1$  has chords  $v_1v_2$  and  $v_2y_2$ . If  $z_1 = x_1$  or  $x_2$ , we can produce DCCs by considering  $w_1$ .

The cases of the reverse sequence or where the middle terms are permuted are also all similar, and amount to merely interchanging the degrees along the path. In each case a DCC is easily found. The cases (1, 2, 2, 0, 1), (1, 2, 0, 2, 1) and (1, 0, 2, 2, 1) are all similar to Claim  $(\pi)$ .

Finally, we consider the case where the minimum degree in the  $P_5$ 's is 1. Note that there is only one vertex of degree 2 in both  $P_5$ 's. The proofs of these cases are slightly different than those we have considered before. Let  $p_1, \ldots, p_5$  denote the ordered vertices of the first  $P_5$  and  $q_1, \ldots, q_5$  denote the ordered vertices of the other path. We assume (possibly reordering the  $p_i, q_i$  or swapping the roles of the two  $P_5$ 's) that  $p_1$ 's neighbor is  $q_j$  for j as small as possible. Note that the two paths and the edges between them comprise 10 vertices and 14 edges. If a Hamiltonian cycle exists, then there are 4 chords. If a cycle avoiding only one vertex exists, then if that vertex is an end vertex of one of the paths, (so that it has degree at most 3 in G) there are at least 2 chords for the cycle. This is also true if an internal vertex of degree at most 3 in G is avoided. Case 1: Suppose  $p_1q_1 \in E$ .

Then consider neighbors of  $p_5$ . If  $p_5q_5$  is an edge, the graph is Hamiltonian, and there is a DCC. If  $p_5q_4$  is an edge, the cycle C:  $p_1, q_1, q_2, q_3, q_4, p_5, p_4, p_3, p_2, p_1$  avoids  $q_5$  and no other vertex and so is a DCC.

## Subcase 1.1: Suppose $p_5q_3 \in E$ .

Then consider the  $p_j$  (max j) adjacent to  $q_5$ . By symmetry,  $j \in \{1, 2, 3\}$  or we would be in an earlier case. Now consider the cycles  $C_1 : p_j, \ldots, p_5, q_3, q_4, q_5, p_j$  and  $C_2 : p_1, p_2, \ldots, p_5, q_3, q_2, q_1, p_1$  and  $C_3 : p_1, q_1, \ldots, q_5, p_j, \ldots, p_1$ . These three cycles can be thought of as breaking the paths and edges into path segments  $S_1 = [p_j, p_5], S_2 = [q_3, q_5]$  and  $S_3 = p_j, p_{j-1}, \ldots, p_1, q_1, q_2, q_3$ . Note that  $p_j$  and  $q_3$  each belong to two of these segments. Now there are three more edges between the paths. We would have a DCC unless these three edges each join a distinct pair of the segments. Note that under this restriction,  $p_i$  and  $q_3$  cannot be incident to any of these three edges.

If this is the case, consider  $p_4$ . If  $p_4q_4 \in E$ , then  $p_1, \ldots, p_j, q_5, q_4, p_4, p_5, q_3, q_2, q_1, p_1$  is Hamiltonian if j = 3, or misses only  $p_3$  if j = 2. If  $p_3$  sends only one edge to the other path we are done. Otherwise, there are two edges from  $p_3$ . If  $p_3$  sends an edge to  $S_2$ ,  $C_1$  is a DCC. Thus,  $p_3q_1 \in E$  and  $p_3q_2 \in E$ . Then  $C_2$  is a DCC. If j = 1, then each of  $p_2$  and  $p_3$  are incident to (at least) one of the remaining edges. If both edges go into  $S_3$ , then  $C_2$  is a DCC. Thus, one of these edges goes into  $S_2$  (one of  $q_4$ or  $q_5$ ), then  $C_1$  is a DCC.

Next suppose  $p_4q_3 \in E$ . Now any edge  $p_rq_2$ ,  $r \in \{1, 2, 3, 4, 5\}$  implies that  $C_2$  has chords  $q_2p_r$  and  $p_4q_3$ .

Now suppose that  $p_4q_2 \in E$ . Then  $p_1$ ,  $q_1$ ,  $q_2$ ,  $p_4$ ,  $p_5$ ,  $q_3$ ,  $q_4$ ,  $q_5$ ,  $p_j$ , ...,  $p_1$  is Hamiltonian if j = 3. This cycle avoids only  $p_3$  if j = 2 and is a DCC if  $deg(p_3) = 3$ . Thus, suppose that  $p_3$  is incident to two edges to the other path. One of these must go to  $q_4$ . Now  $p_5$ ,  $p_4$ ,  $q_2$ ,  $q_1$ ,  $p_1$ ,  $p_2$ ,  $p_3$ ,  $q_4$ ,  $q_3$ ,  $p_5$  avoids only  $q_5$ . If j = 1, then note that  $p_2$  and  $p_3$  must each send at least one edge to the other path. If either sends an edge to  $\{q_1, q_2, q_3\}$ , then  $C_2$  is a DCC using that edge and  $p_4q_2$ . Thus, both edges must go to  $q_4$  and now  $C_1$  is a DCC. This completes the  $p_4q_2$  subcase.

Next suppose that  $p_4q_1 \in E$ . Assume j = 3, i.e., that  $q_5p_3$  is an edge. Then  $p_2$  must send at least one edge to the other path. If this edge goes to any of  $\{q_1, q_2, q_3\}$ , then  $C_2$  is clearly a DCC. Thus,  $p_2$  has an adjacency in  $\{q_4, q_5\}$ . Then  $p_1, q_1, p_4, p_5, q_3, q_4, q_5, p_3, p_2, p_1$  has both  $p_2q_4$  (or  $p_2q_5$ ) and  $p_3p_4$  as chords. Thus we next assume j = 2. Then  $p_3$  has at least one adjacency to the other path. If  $p_3q_r \in E$ ,  $r \in \{1, 2, 3\}$ , then  $C_2$  is a DCC with chords  $p_3q_r$  and  $p_4q_1$ . If r = 4, then  $p_2, q_5, q_4, p_3, p_4, p_5, q_3, q_2, q_1, p_1, p_2$  is Hamiltonian. Now assume j = 1, i.e.,  $q_5p_1 \in E$ . If  $p_2$  has an adjacency in  $\{q_1, q_2, q_3\}$  then  $C_2$  is a DCC. Thus,  $p_2q_4 \in E$  and the cycle  $p_1, q_1, q_2, q_3, p_5, \ldots, p_2, q_4, q_5, p_1$  is Hamiltonian. By symmetry these are all the necessary cases when  $p_1q_1 \in E$ . This completes the cases when  $p_4q_1 \in E$  and Subcase 1.1.

#### Subcase 1.2: Suppose $p_5q_2 \in E$ (so by symmetry, $q_5$ is adjacent to $p_2$ or $p_1$ ).

If  $q_5p_1 \in E$ , then  $p_1, \ldots, p_5, q_2, \ldots, q_5, p_1$  avoids only  $q_1$ , so it is a DCC. Now assume instead that  $q_5p_2 \in E$ . Note that each of  $p_3$  and  $p_4$  needs to send at least one edge to the other path. If two such edges are incident in  $\{q_1, q_2\}$ , then  $p_1, \ldots, p_5, q_2, q_1, p_1$  is a DCC. If two such edges are incident in  $\{q_2, q_3, q_4\}$  (note  $q_5$  not possible in this case) then  $p_2, \ldots, p_5, q_2, \ldots, q_5, p_2$  is a DCC. This implies that each of  $p_3$  and  $p_4$  sends exactly one edge to the other path into  $\{q_1\}$  or  $\{q_3, q_4\}$ . If  $p_3q_1 \in E$ , then a Hamiltonian cycle is easy to find. Thus  $p_4q_1 \in E$ . But then a cycle avoiding only  $p_3$  (which must have degree 3, for otherwise  $p_3q_3$  and  $p_3q_4$  are edges and a Hamiltonian cycle is easily found) and we are again done.

## Subcase 1.3: Suppose $p_5q_1 \in E$ .

Then by symmetry,  $q_5p_1 \in E$  and  $p_1, \ldots, p_5, q_1, \ldots, q_5, p_1$  is Hamiltonian. This completes Case 1.

Case 2: Suppose  $p_1q_2 \in E$  (and by symmetry,  $p_1q_1$  and  $p_5q_5$  are not edges).

Thus, the neighbors of  $q_1$  and  $q_5$  lie in  $\{p_2, p_3, p_4\}$ .

#### Subcase 2.1: Suppose $q_1p_2$ and $q_5p_4$ are in *E*.

Suppose  $p_5q_4 \in E$ . Then  $p_1, p_2, \ldots, p_5, q_4, q_3, q_2, p_1$  is a DCC unless at least one of the remaining two edges goes to  $q_1$  or  $q_5$ . But, by our symmetry assumptions, that edge is incident to  $q_1$ . Then  $q_1$  must be adjacent to  $p_3$  or  $p_4$ . If  $p_3q_1 \in E$ , then  $p_1, q_2, q_3, q_4, p_5, p_4, p_3, q_1, p_2, p_1$  avoids only  $q_5$  and hence is a DCC. If  $p_4q_1 \in E$ , then  $p_3q_3 \in E$  by the assumption on the degree sequence and hence,  $p_2, p_3, p_4, q_5, q_4, q_3, q_2, q_1, p_2$  is a DCC.

Next suppose that  $p_5q_3 \in E$ . Assume that  $p_4q_4 \in E$ . Then for any edge  $p_3q_r$ ,  $r \in \{1, 2, 3, 4, 5\}$ ,  $p_2$ ,  $p_3$ ,  $p_4$ ,  $q_5$ ,  $q_4$ , ...,  $q_r$ ,  $p_2$  has chords  $p_3q_r$  and  $p_4q_4$ . If  $p_3q_4 \in E$ , then  $p_5$ ,  $p_4$ ,  $q_5$ ,  $q_4$ ,  $p_3$ ,  $p_2$ ,  $q_1$ ,  $q_2$ ,  $q_3$ ,  $p_5$  avoids only  $p_1$ . If instead,  $p_2q_4 \in E$ , then any edge  $p_3q_r$ ,  $r \in \{1, 2, 3, 4, 5\}$  gives  $p_2$ ,  $p_3$ ,  $p_4$ ,  $q_5$ , ...,  $q_1$ ,  $p_2$  as a DCC. Now suppose that  $p_1q_4 \in E$ . Then  $p_1$ ,  $q_4$ ,  $q_5$ ,  $p_4$ ,  $p_5$ ,  $q_3$ ,  $q_2$ ,  $q_1$ ,  $p_2$ ,  $p_1$  avoids only  $p_3$  which has degree 3, and thus is a DCC.

Next suppose that  $p_5q_2 \in E$ . Then each of  $q_3$  and  $q_4$  sends an edge to  $\{p_1, p_2, p_3, p_4\}$ , or we would be in an earlier case, and so  $p_1, p_2, p_3, p_4, q_5, q_4, q_3, q_2, p_1$  is a DCC. This completes Subcase 2.1.

#### Subcase 2.2: Suppose $q_1p_2$ and $q_5p_3$ are in *E*.

Suppose  $p_5q_3 \in E$ . Now consider the adjacency of  $q_4$  (which cannot include  $p_5$ ). Suppose  $q_4p_4 \in E$ . Then  $p_1, q_2, q_3, p_5, p_4, q_4, q_5, p_3, p_2, p_1$  avoids only  $q_1$ . Next suppose that  $q_4p_3 \in E$ . Now  $p_4$  has an adjacency on the other path. Say  $p_4q_3 \in E$ . Then  $p_3, p_4, p_5, q_3, q_4, q_5, p_3$  has  $p_3q_4$  and  $p_4q_3$  as chords. If  $p_4q_2 \in E$ , then  $p_1, q_2, p_4, p_5, q_3, q_4, q_5, p_3, p_2, p_1$  has  $q_2q_3$  and  $p_3q_4$  as chords. If  $p_4q_1 \in E$ , then  $p_1, q_2, q_1, p_4, p_5, q_3, q_4, q_5, p_3, p_2, p_1$  has  $q_2q_3$  and  $p_3q_4$  as chords. If  $p_4q_1 \in E$ , then  $p_1, q_2, q_1, p_4, p_5, q_3, q_4, q_5, p_3, p_2, p_1$  is Hamiltonian. Next suppose that  $q_4p_2 \in E$ . Then  $p_1, q_2, q_3, p_5, p_4, p_3, q_5, q_4, p_2, p_1$  avoids only  $q_1$ . Finally, suppose  $q_4p_1 \in E$ . Then  $p_1, q_4, q_5, p_3, p_4, p_5, q_3, q_2, q_1, p_2, p_1$  is Hamiltonian.

Finally, if  $p_5q_2 \in E$ , a similar argument applies. This completes Subcase 2.2.

Subcase 2.3: Suppose that  $q_1p_2 \in E$  and  $q_5p_2 \in E$ .

By symmetry,  $p_5$  is adjacent to  $q_2$ . Now  $p_3$  and  $p_4$  each have at least one edge to the other path, and these edges end in  $\{q_3, q_4\}$ . But then,  $p_2, \ldots, p_5, q_2, \ldots, q_5, p_2$  is a DCC. This completes Subcase 2.3.

Similar arguments hold for  $p_1q_2$  when either  $q_1p_3$  or  $q_1p_4$  are assumed along with a possible adjacency for  $q_5$ . By our symmetry assumptions, this completes Case 2.

## Case 3: Suppose $p_1q_3 \in E$ .

By symmetry and our earlier cases,  $p_5q_3$ ,  $q_1p_3$  and  $q_5p_3$  are also edges. Now  $p_2$  and  $p_4$  must have their adjacencies in  $\{q_2, q_4\}$ . If  $p_2q_2$  and  $p_4q_4$  are edges, then  $p_1, p_2, q_2, q_1, p_3, q_5, q_4, p_4, p_5, q_3, p_1$  is Hamiltonian. While if  $p_2q_4$  and  $p_4q_2$  are the edges, then reverse one of the paths to obtain the previous cycle. This completes Case 3.  $\Box$ 

By our symmetry and path reversal assumptions, this completes the proof.  $\Box$ 

#### Acknowledgments

The authors would like to thank the anonymous referees for their extremely careful reading, and valuable comments which helped produce a better manuscript.

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