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On the interval chromatic number of proper interval graphs

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ABSTRACT

A perfect graph is a graph every subgraph of which has a chromatic number equal to its clique number (Berge, 1963; Lovász, 1972). A (vertex) weighted graph is a graph with a weight function w on its vertices. An interval coloring of a weighted graph maps each vertex v to an interval of size $w(v)$ such that the intervals corresponding to adjacent vertices do not intersect. The size of a coloring is the total size of the union of these intervals. The minimum possible size of an interval coloring of a given weighted graph is its interval chromatic number. The clique number of a weighted graph is the maximum weight of a clique in it. Clearly, the interval chromatic number of a weighted graph is at least its clique number. A graph is superperfect if for every weight function on its vertices, the chromatic number of the weighted graph is equal to its clique number (Hoffman, 1974). It is known that determining the interval chromatic number of a given interval graph is NP-Complete, implying that interval graphs are not included in the family of superperfect graphs (Golumbic, 2004). The question whether these results hold for simple graph families is open since then. We answer this question affirmatively for proper interval graphs for which most investigated problems are polynomial (http://www.graphclasses.org/classes/gc_298.html). Specifically, we show that determining the interval chromatic number of a proper interval graph is NP-Complete in the strong sense. We present a simple 2-approximation algorithm for this special case, whereas the best known approximation algorithm for interval graphs is a $(2+\epsilon)$-approximation (Buchsbaum et al., 2004).

1. Introduction

1.1. Background

The notion of *perfect graphs* was introduced by Berge in the early 1960s [2]. A graph is perfect if every induced subgraph of it satisfies two properties: (a) its chromatic number is equal to its clique number, and (b) its clique cover number is equal to its independence number. Berge presented the famous perfect graph conjecture that was later proven by Lovász [16], according to which these two properties are equivalent. Therefore, we use only the first property and say that a graph is perfect if every induced subgraph of it has a chromatic number equal to its clique number. Several important graph families are included in the family of perfect graphs. A *chord* of a cycle C in a graph is an edge of the graph connecting two vertices that are non-adjacent in C. A graph is *chordal* (or *triangulated*) if every cycle of it with at least 4 vertices has a chord. Chordal graphs are perfect [12,1]. An *interval graph* is a graph whose vertices correspond to open intervals on the real line and two vertices are adjacent if their corresponding intervals intersect. It is easy to see that interval graphs are chordal, therefore

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perfect. A *proper interval* graph is an interval graph where none of the intervals corresponding to the vertices is properly contained in another.

The notions of coloring and perfect graphs are extended as follows. Consider a graph with a weight function w on its vertices. An *interval coloring* c of a weighed graph maps each vertex v to an interval $c(v)$ of size $w(v)$ such that $c(u)$ and $c(v)$ do not intersect whenever u and v are adjacent in the graph. Interval coloring extends the classical notion of vertex coloring. The size of an interval coloring is the size of the union of the intervals. The *interval chromatic number* of a weighted graph is the minimum possible size of an interval coloring of it. The *clique number* of a weighted graph is the maximum total weight of a clique of it. The interval chromatic number of a weighted graph is at least its clique number. Indeed, given a clique K of weight equal to the clique number of the weighted graph, the vertices of K have to be assigned pairwise disjoint intervals. A graph is superperfect if for every weight function w, the chromatic number of the corresponding weighted graph is equal to its clique number [14].

1.2. Related work

The problem of determining the interval chromatic number of an interval graph is known also as the *shipbuilding* problem (see [10]). It is also mentioned in the literature as the *dynamic storage allocation* problem. (see problem (SR2) in [6]). The problem can be described as coloring of *floating* rectangles that are free to move along one axis. Another rectangle coloring variant discussed in the literature is the *berth allocation problem*, in which the rectangles can move along both axes, though not completely freely [18].

Many problems that are NP-Hard in general graphs, such as maximum independent set, maximum clique, chromatic number, can be solved in polynomial time for interval graphs. In particular, for proper interval graphs most investigated problems are polynomial [11]. Larry Stockmeyer showed in 1976 that determining the interval chromatic number of a weighted interval graph is NP-Complete. In fact he showed that the question whether the interval chromatic number of a weighted interval graph is equal to its clique number poses an NP-Complete problem (for a proof see [3]). This implies that some interval graphs are not superperfect, because the problem of determining the clique number of a weighted interval graph is clearly polynomial.

A 6-approximation algorithm for the problem of determining the interval chromatic number of a weighted interval graph was presented in [15]. In [7] and [8] this ratio was improved to 5 and 4, respectively. The best approximation ratio for this problem is $(2+\epsilon)$ [4].

To develop our positive results we use tools introduced in [17], that investigates a related problem.

It was noted by Alan Hoffman that comparability graphs are superperfect. An infinite family of non-comparability superperfect graphs is demonstrated in [9]. In [5] a forbidden subgraph characterization of comparability split graphs is provided. Since these forbidden subgraphs are not superperfect and comparability graphs are superperfect, this implies a characterization of superperfect graphs within the family of split graphs.

1.3. Our contribution

In this work we consider proper interval graphs. In Section 2 we introduce definitions and notation with some preliminary results. In Section 3 we show that determining whether the interval chromatic number of a given weighted proper interval graph is equal to its clique number is NP-Complete in the strong sense. This implies that the family of proper interval graphs is not included in the family of superperfect graphs, and an infinite family of non-superperfect proper interval graphs can be constructed from our proofs. In Section 4 we initiate the study of approximation algorithms for the interval coloring problem of proper interval graphs by presenting a simple 2-approximation algorithm for it. We conclude with a summary and open questions in Section 5.

2. Preliminaries

Interval Notation: We denote an open (resp. closed) interval I on the real line as $(s(I), t(I))$ (resp. $[s(I), t(I)]$) where $s(I) < t(I)$. For two intervals I, I', we denote by $I < I'$ the fact that $t(I) \leq s(I')$. For a number x and an interval (s, t) we define $x \cdot (s,t) \stackrel{def}{=} (x \cdot s, x \cdot t)$, and $x + (s,t) \stackrel{def}{=} (x+s, x+t)$. We extend these definitions to sets of intervals and functions of intervals as expected. Namely, given a set $\mathcal{I}$ of intervals and a function $f : D \rightarrow \mathcal{I}$ over some domain D and a number x we denote $x \cdot \mathcal{I} \stackrel{def}{=} \{x \cdot I : I \in \mathcal{I}\}$, $x + \mathcal{I} \stackrel{def}{=} \{x + I : I \in \mathcal{I}\}$, $(x \cdot f)(d) = x \cdot f(d)$ and $(x + f)(d) = x + f(d)$ for every $d \in D$. The size $size(I)$ of an interval is $t(I) - s(I)$. The size $size(\mathcal{I})$ of a set of pairwise disjoint intervals $\mathcal{I}$ is $size(\cup \mathcal{I}) \stackrel{def}{=} \sum_{I \in \mathcal{I}} size(I)$.

Superperfect Graphs: A *clique* of a graph $G = (V(G), E(G))$ is a subset $K \subseteq V(G)$ of vertices that are pairwise adjacent in G. A clique is *maximal* if it is not contained in any other clique, and it is *maximum* if it contains the biggest number of vertices among all cliques. A *clique cover* of G is a set $\mathcal{K}$ of cliques of G that covers all the vertices, i.e. $\bigcup \mathcal{K} = V(G)$. The *clique number* of G is the size of its maximum cliques and is denoted by $\omega(G)$. A stable (or independent) set of a graph is a subset of its vertices that are pairwise non-adjacent. A vertex is called *simplicial* if all its neighbors constitute a clique. In other words, a vertex is simplicial if it is contained in exactly one maximal clique. A *coloring* of a graph G is a labeling of its vertices, such that a set of vertices labeled with the same label constitutes a stable set. The *chromatic number* of a graph is the minimum possible number of labels used by a coloring, and is denoted by $\chi(G)$. A graph G is *perfect* if $\chi(G[S]) = \omega(G[S])$ for every $S \subseteq V(G)$, where $G[S]$ denotes the subgraph of G induced by S, and $w(S) \stackrel{def}{=} \sum_{v \in S} w(v)$.

$(G; w)$ denotes the weighted graph consisting of the graph G and the weight function $w : V(G) \rightarrow \mathbb{N}$ on its vertices. The *clique number* of $(G; w)$ is $\omega(G; w) \stackrel{def}{=} \max \{w(K) : K \text{ is a clique of } G\}$. We term a clique K that achieves this maximum as a *critical clique*, i.e. a clique K is critical if $w(K) = \omega(G; w)$. Clearly, a critical clique is maximal. An *interval coloring* c of $(G; w)$ is a labeling of the vertices of G with intervals, such that the intervals $c(u)$ and $c(v)$ corresponding to two adjacent vertices u and v do not intersect. The size *size*(c) of interval coloring c of $(G; w)$ is *size*$(\cup_{v \in V(G)} c(v))$. We denote by $\chi(G; w)$ the *interval chromatic number* of $(G; w)$ which is the minimum size of an interval coloring c of $(G; w)$. A graph G is *superperfect* if $\chi(G; w) = \omega(G; w)$ for every weight function w on its vertices.

Interval Graphs: A graph G is an interval graph if every $v \in V(G)$ corresponds to an interval I_v such that $\{u, v\} \in E(G)$ if and only if $I_u \cap I_v \neq \emptyset$. The set of intervals $\{I_v : v \in V(G)\}$ is termed an *interval representation* of G. It is known that an interval representation of a given interval graph can be found in polynomial time. Therefore, we assume that an interval graph is given together with an interval representation. We also assume without loss of generality that G is connected, implying that the union of the intervals I_v is an interval I. It is well known that intervals have the Helly property, i.e. a clique of an interval graph G is represented by set of intervals that have a common intersection. We say that this intersection $I_K \stackrel{def}{=} \cap_{v \in K} I_v$ *represents* the clique K. The intervals $I_K, I_{K'}$ representing two maximal cliques K, K' do not intersect, because otherwise their intersection implies a clique $K \cup K'$, contradicting the maximality of K and K'. Therefore, the intervals representing maximal cliques are totally ordered. Such a total order induces a total order on the maximal cliques of an interval graph.

Example 2.1. Fig. 1(b) depicts a possible interval representation of the graph G_0 in Fig. 1(a). The maximal cliques of the graph are $K_1 = \{1, 2, 3\}$, $K_2 = \{2, 3, 4\}$, $K_3 = \{3, 4, 5\}$, $K_4 = \{4, 5, 6\}$, and $K_5 = \{5, 6, 7\}$. The intersection of the intervals $I_1 = (0, 1)$, $I_2 = (0, 2)$, $I_3 = (0, 3)$ that represent the vertices of K_1 is $(0, 1)$. In general, for $i \in [1, 5]$, the maximal clique K_i is represented by the interval $(i - 1, i)$. The union of all intervals is $I = (0, 7)$.

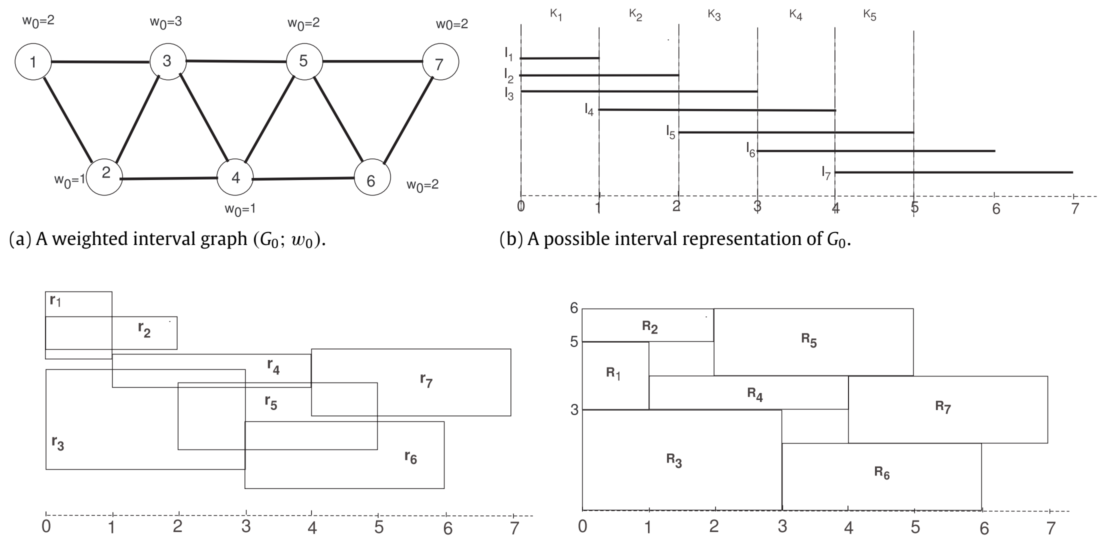
It is convenient to describe a weighted interval graph $(G; w)$ as a set of axis-parallel rectangles on the plane. Every vertex v is represented by a *floating rectangle* $r_v = (I_v, w(v))$ where I_v is the projection of r_v on the horizontal axis and $w(v)$ is its *height*, i.e. the length of its projection on the vertical axis. An interval coloring c determines the vertical position of the floating rectangle r_v, in other words it assigns to it a (non floating) *rectangle* $R_v = (I_v, c(v))$ where I_v and $c(v)$ are its projections on the horizontal and vertical axes respectively and *size*$(c(v)) = w(v)$. Note that an interval coloring is valid if and only if the rectangles are pairwise disjoint. The size of the coloring is the size of the projection of all the rectangles on the vertical axis. Without loss of generality we assume that this projection is the interval $(0, W_c)$ for some integer W_c.

Example 2.2. For the graph G_0, let w_0 be the following weight function: $w_0(1) = 2$, $w_0(2) = 1$, $w_0(3) = 3$, $w_0(4) = 1$, $w_0(5) = 2$, $w_0(6) = 2$, $w_0(7) = 2$. Fig. 1(c) shows a floating rectangle representation of $(G_0; w_0)$ whose projection on the horizontal axis is the interval representation in Fig. 1(b). We note that $w_0(K_1) = w_0(K_3) = w_0(K_5) = \omega(G_0; w_0) = 6$, i.e. K_1, K_3 and K_5 are critical cliques. The rectangles depicted in Fig. 1(d) correspond to the following coloring c_0 of (G_0, w_0): $c_0(1) = (3, 5)$, $c_0(2) = (5, 6)$, $c_0(3) = (0, 3)$, $c_0(4) = (3, 4)$, $c_0(5) = (4, 6)$, $c_0(6) = (0, 2)$, and $c_0(7) = (2, 4)$. The size of c_0 is *size*$((0, 6)) = 6$, therefore it is optimal and $\chi(G_0, w_0) = 6$.

The interval chromatic number of $(G; w)$ is the smallest number W_c such that the rectangles R_v can fit within the rectangle $(I, (0, W_c))$ without intersecting each other. The *dual* of a subinterval (s, t) of $(0, W_c)$ is the subinterval $\overline{(s, t)} \stackrel{def}{=} (W_c - t, W_c - s)$. The dual $\bar{c}$ of a coloring c is the coloring that assigns to every vertex v the interval $\overline{c(v)}$. In terms of rectangles, $\bar{c}$ is obtained by rotating the rectangles of c around the horizontal axis $W_c/2$.

Observation 2.1. (i) *size*$(c) =$ *size*$(\bar{c})$,
(ii) $\bar{\bar{c}} = c$, *and*
(iii) *c is a valid interval coloring of $(G; w)$ if and only if $x \cdot c$ is a valid interval coloring of $(G; x \cdot w)$ for any $x \geq 0$.*

Proper Interval Graphs: A proper interval graph is an interval graph having an interval representation such that no interval is properly included in another. This implies that $s(I_u) \leq s(I_v)$ if and only if $t(I_u) \leq t(I_v)$. Whenever we use terms such as first path and next path, we implicitly refer to the total order of the paths implied by the total order of their start points. For a subset S of vertices, or a subset I_S of intervals, we denote by $\min(S)$, $\max(S)$, $\min(I_S)$, $\max(I_S)$ the minima and maxima in this total order. Note that for a clique K of a proper interval graph we have $I_K = (s(\max(K)), t(\min(K)))$. In this work we allow intervals to properly include another as long as they have one endpoint in common. It is easy to see that any such interval representation can be transformed to a representation that intervals do not properly include each other. Indeed, given a representation containing inclusions with common endpoints we can eliminate them using the following procedure and get an equivalent representation: (a) replace the common start (resp. termination) point with an interval, and (b) modify the start (resp. termination) points within this interval according to the order of the termination (resp. start) points. This modification does not change the intersection relationships among the paths. Therefore, the set of intervals obtained in



(a) A weighted interval graph $(G_0; w_0)$.

(b) A possible interval representation of G_0.

(c) Floating rectangles corresponding to $(G_0; w_0)$.

(d) An interval coloring of (G_0, w_0) of size 6.

Fig. 1. Examples 2.1 and 2.2.

this way is a representation of the same graph. Note that the representation in Fig. 1(b) is a set of proper intervals, that constitutes a representation of the (proper interval) graph G_0 in Fig. 1(a).

We will use the following basic lemma to develop our results.

Lemma 2.1. *Let $\mathcal{K}$ be a clique cover of a connected proper interval graph G consisting of maximal cliques. There exists a subset $\mathcal{K}' = \{K'_1, K'_2, \ldots\}$ of $\mathcal{K}$, such that every vertex of G is either an element of exactly one clique from $\mathcal{K}'$, or element of two consecutive cliques $K'_i, K'_{i+1} \in \mathcal{K}'$.*

Proof. Let $\mathcal{I}$ be a proper interval representation of G. We recall that for two distinct cliques $K, K' \in \mathcal{K}, I_K \cap I_{K'} = \emptyset$. Let $\mathcal{K} = \{K_1, \ldots, K_k\}$ be a clique cover of G consisting of maximal cliques, where the indices are such that $I_{K_1} < \cdots < I_{K_k}$. We determine the cliques of $\mathcal{K}'$ using Algorithm 1. We start with $K'_1 = K_1$, then we choose as K'_2 the last clique of $\mathcal{K}$ that contains the interval $\max(K'_1)$, and so on. By the way the cliques are chosen, it is clear that $\mathcal{K}' \subseteq \mathcal{K}$. Moreover, no interval is contained in more than two cliques. Indeed, if $I_v \in K'_i \cap K'_{i+2}$, then $\max(K'_i) \in K'_{i+2}$, contradicting the fact that K'_{i+1} is chosen by the algorithm to $\mathcal{K}'$. It remains to show that $\mathcal{K}'$ is a clique cover. Assume by contradiction that an interval I_v is not covered by $\mathcal{K}'$. Let i be the unique number such that $I_{K'_i} < I_v < I_{K'_{i+1}}$. Let $u = \max(K'_i)$. By the way the algorithm chooses the cliques we have $u \in K'_{i+1}$. Then $I_u \supseteq I_{K'_i} \cup I_{K'_{i+1}}$, implying that $I_u \supseteq I_v$ and they do not have a common endpoint. A contradiction to the fact that $\mathcal{I}$ is a proper set of intervals.

Algorithm 1

Require: $\mathcal{K} = \{K_1, \ldots, K_k\}$ is a clique cover
Require: $I_{K_1} < \cdots < I_{K_k}$
$k' \leftarrow i \leftarrow 1$
$K'_{k'} \leftarrow K_i$
while $i < k$ **do**
  $i \leftarrow \max\{j : \max(K_i) \in K_j\}$
  $k' \leftarrow k' + 1$
  $K'_{k'} \leftarrow K_i$
end while

□

Related Problems We now describe two NP-Complete problems that we use in this work (see [6]).

PARTITION
Input: A set X of n positive numbers.
Question: Can X be partitioned into two sets X_1, X_2 such that $\sum_{x \in X_1} x = \sum_{x \in X_2} x$?

PARTITION is NP-Complete in the weak sense, i.e. the sub-problem in which the numbers of X are bounded from above by some polynomial in n can be solved in polynomial time.

3-PARTITION **Input:** A set X of $3 \cdot m$ numbers, such that $\sum_{x \in X} = m \cdot B$ and $B/4 < x < B/2$ for every $x \in X$. **Question:** Can X be partitioned into m sets $X_1, \ldots, X_m$, such that $\lvert X_i \rvert = 3$ and $\sum_{x \in X_i} x = B$ for every $i \in [1, m]$?

3-PARTITION is NP-Complete in the strong sense, i.e. it remains NP-Complete even if the numbers are bounded by a polynomial in n.

3. Hardness results

In this section we show that determining the interval chromatic number of a weighted proper interval graph is NP-Complete in the strong sense, implying that the family of proper interval graphs is not included in the family of superperfect graphs. We first give a simple proof that shows that the problem in NP-Complete in the weak sense, and then extend the result to show that it is NP-Complete in the strong sense.

We start by presenting a construction that will serve us as a basic building block in the proofs in this section.

Proposition 3.1. *The weighted proper interval graph* (G_0, w_0) *depicted in Fig. 1 (a) has exactly two colorings of size* 6 *. These are (a) the coloring* c_0 *shown in Fig. 1 (d), and (b) its dual* $\bar{c_0}$*.*

Proof. Let c be a coloring of size $\omega(G_0; w_0) = 6$. We recall that K_1, K_3 and K_5 are critical cliques of $(G_0; w_0)$. As K_1 is critical, $c(1), c(2), c(3)$ are three consecutive subintervals of $(0, 6)$. As the claim is symmetric on c and its dual $\bar{c}$, we can assume without loss of generality that $c(1) < c(2)$. We consider the three possible cases.

- $c(1) < c(2) < c(3)$: In this case we have $c(1) = (0, 2), c(2) = (2, 3)$ and $c(3) = (3, 6)$. We now consider the vertices of $K_3 \setminus K_1 = \{4, 5\}$. The interval available for these vertices is $(0, 3)$. We verify that among the two options of coloring 4 and 5 the only valid possibility is $c(4) = (0, 1), c(5) = (1, 3)$, as otherwise $c(2) \cap c(4) = (2, 3) \neq \emptyset$. We proceed with the vertices of $K_5 \setminus K_3 \setminus K_1$, i.e. 6 and 7. The intervals available for them are $(0, 1)$ and $(3, 6)$, however they must be colored with two intervals of size 2 each, which is clearly impossible.
- $c(1) < c(3) < c(2)$: In this case we have $c(1) = (0, 2), c(3) = (2, 5)$ and $c(2) = (5, 6)$. We now consider the vertices of $K_2 \setminus K_1 = \{4, 5\}$. The intervals available for these vertices are $(0, 2)$ and $(5, 6)$. Then $c(4) = (5, 6)$ and $c(5) = (0, 2)$. But now $c(4) \cap c(2) = (5, 6) \neq \emptyset$ and 2, 4 are adjacent in G_0. Therefore this case is impossible too.

We conclude that $c(3) < c(1) < c(2)$, i.e., $c(3) = (0, 3), c(1) = (3, 5)$ and $c(2) = (5, 6)$. The interval available for the vertices 4, 5 is $(3, 6)$. From the two possibilities to color them only one option is valid, namely $c(4) = (3, 4)$ and $c(5) = (4, 6)$. The interval available for the remaining vertices is $(0, 4)$, which leaves only one valid option, i.e. $c(6) = (0, 2)$ and $c(7) = (2, 4)$. We conclude that $c = c_0$. □

Lemma 3.1. *The problem of determining whether* $\chi(G; c) = \omega(G; w)$ *for any weighted proper interval graph* $(G; w)$ *is* NP*-Complete.*

Proof. The proof is by reduction from PARTITION. Given an instance $X = \{x_1, \ldots, x_n\}$ of PARTITION with $B = \sum_{x \in X} x$, we build the following weighted proper interval graph $(G; w)$. G is obtained by adding to G_0 a complete graph K_n with vertex set $V_X = \{v_i : i \in [1, n]\}$ and adding an edge between $7 \in V(G_0)$ and every vertex in V_X. Note that $V_X \cup 7$ is a clique of G. A possible proper interval representation of G is obtained by adding n identical intervals $(6, 7)$ to the interval representation of G_0 depicted in Fig. 1(b). We proceed with the definition of w. We set $w(v) = w_0(v) \cdot B/4$ for every vertex $v \in V(G_0)$ and $w(v_i) = x_i$ for every vertex $v_i \in V_X$. It is easy to verify that $\omega(G, w) = 3B/2$. It remains to show that $\chi(G, w) = 3B/2$ if and only if the PARTITION instance X is a YES instance of PARTITION.

If X is a YES instance, then there is a partition of X into two sets X_1, X_2 such that $\sum_{x \in X_1} x = \sum_{x \in X_2} x = B/2$. $c = \frac{B}{4} c_0$ is a coloring of $(G_0, \frac{B}{4} w_0)$ of size $\frac{B}{4} 6 = 3B/2$. To complete c to a coloring of G, we note that $c(7) = \frac{B}{4}(2, 4) = (\frac{B}{2}, B)$, and we assign consecutive subintervals of $(0, \frac{B}{2})$ (resp. $(B, \frac{3B}{2})$) to the vertices corresponding to the elements of X_1 (resp. X_2). Conversely, assume that there is a coloring c of $(G; w)$ of size $3B/2$. Then c induces a coloring of size at most $3B/2$ of $(G_0; \frac{B}{4} w_0)$ that in turn induces a coloring of size at most 6 of $(G_0; w_0)$. By Proposition 3.1, this coloring is either c_0 or its dual $\overline{c_0}$. However, $c_0(7) = \overline{c_0}(7) = (2, 4)$. Therefore, $c(7) = \frac{B}{4}(2, 4) = (\frac{B}{2}, B)$. Then, the intervals available for the vertices of V_X are $(0, \frac{B}{2})$ and $(B, \frac{3B}{2})$. The set $V_{X1} \subseteq V_X$ of vertices colored with sub-intervals of $(0, \frac{B}{2})$, and the set $V_{X2} \subseteq V_X$ of vertices colored with sub-intervals of $(B, \frac{3B}{2})$ induce a partition of X into two sets X_1 and X_2 as required. □

The following two theorems are corollaries of the above lemma.

Theorem 3.1. *It is* NP*-Complete to determine the interval chromatic number* $\chi(G; w)$ *even if* G *is a proper interval graph.*

Theorem 3.2. *The family of proper interval graphs is not included in the family of superperfect graphs unless* P = NP.

Proof. Assume by way of contradiction that every proper interval graph G is superperfect. Then, $\chi(G; w) = \omega(G; w)$ for every w. As G is an interval graph, $\omega(G; w)$ can be calculated in polynomial time. Then $\chi(G; w)$ can be calculated in polynomial time. Together with Theorem 3.1, this implies P = NP. □

We note that Partition is Np-Complete in the weak sense, i.e. it admits an algorithm that runs in time polynomial to the sizes of the numbers. However this running time is exponential in the size of the input which consists of the binary representations of the numbers. In the rest of this section we extend Theorem 3.1 to show that the problem is Np-Complete in the strong sense.

In the sequel we use the weighted graph $(G_0; w_0)$ as a building block to construct an arbitrarily big weighted proper interval graph by combining these building blocks in a tree structure. In Theorem 3.3, we use this proper interval graph to make a reduction (this time from 3-Partition) in a way similar to Theorem 3.1. Our construction is an inductive one that starts from a representation of G_0, and at each inductive step we combine a representation of G_0 with two copies of the representation obtained in the previous step so that each one of these copies intersects only with segments I_6 and I_7 of G_0. In this process we shift intervals (actually their endpoints) so that the resulting representation is a proper interval representation. For this purpose we introduce the following definitions. Let G be an interval graph, and $\mathcal{I} = \{I_v : v \in V(G)\}$ an interval representation of it. Let $ST(\mathcal{I}) \stackrel{def}{=} \{s(I_v), t(I_v) : v \in G\}$ be the set of endpoints of $\mathcal{I}$. Any real function $f : ST(\mathcal{I}) \rightarrow \mathbb{R}$ induces a function on $\mathcal{I}$ such that $f(I) \stackrel{def}{=} (f(s(I)), f(t(I)))$, also and $f(\mathcal{I}) \stackrel{def}{=} \{f(I) : I \in \mathcal{I}\}$. Such a function is *increasing* or *order preserving* if for every pair of endpoints $x, y \in ST(\mathcal{I})$, $f(x) < f(y)$ whenever $x < y$. For a positive constant C_1 and any constant C_2, clearly f is order preserving if and only if $C_1 \cdot f$ is order preserving if and only if $C_2 + f$ is order preserving.

Observation 3.1. (i) *If f is order preserving then $f(\mathcal{I})$ is also a representation of G.*

(ii) *An order preserving partial function f over $ST(\mathcal{I})$ can be extended to an order preserving (non-partial) function by assigning to every endpoint on which f is not defined, an arbitrary value between the values of the neighboring endpoints.*

Whenever f is an order preserving partial function, we use f to denote also an arbitrary order preserving extension of it. The following lemma describes the inductive step of our construction.

Lemma 3.2. *Let $(G; w)$ a connected weighted proper interval graph on $n \geq 7$ vertices, such that:*

(i) $\chi(G; w) = \omega(G; w)$,

(ii) *there is a clique cover $\mathcal{K} = \{K_1, \ldots, K_k\}$ of G consisting of critical cliques, and*

(iii) *there is a set $S \subsetneq K_k$ of simplicial vertices of G such that in every interval coloring c of $(G; w)$ of size $\omega(G; w)$, the set $(0, \omega(G; w)) \setminus c(S)$ of colors not used by S consists of b disjoint intervals each of which having size x.*

There is a polynomial-time computable connected weighted proper interval graph (G', w') on at most $4n$ vertices, such that:

(i) $\chi(G'; w') = \omega(G'; w') = 3 \cdot \omega(G; w)$,

(ii) *there is a clique cover $\mathcal{K}' = \{K'_1, \ldots, K'_{k'}\}$ of G' consisting of critical cliques, and*

(iii) *there is a set $S' \subseteq K'_{k'}$ of simplicial vertices of G' such that in every interval coloring c' of $(G'; w')$ of size $\omega(G'; w')$, the set $(0, \omega(G'; w')) \setminus c'(S')$ of colors not used by S' consists of $2b$ disjoint intervals each of which having size x.*

Proof. Let $\mathcal{I}$ be set of intervals representing G. We construct a set of intervals $\mathcal{I}'$ representing G'. Whenever there is no risk of confusion we refer to a vertex v and the interval I_v representing it, interchangeably. By Lemma 2.1, we assume without loss of generality that every vertex is in at most two consecutive cliques K_i, K_{i+1}. Also, without loss of generality, all the intervals of K_1 start at the same point and all the intervals of K_k terminate at the same point. Moreover, $K_i \cap K_{i+1} \neq \emptyset$ because G is connected.

Let f be the partial function mapping the intervals representing $\{\max(K_i) : i \in [1, k]\}$, so that $f(\max(K_i)) = (5i, 5i + 11/2)$.

Claim 3.1. *f is order preserving.*

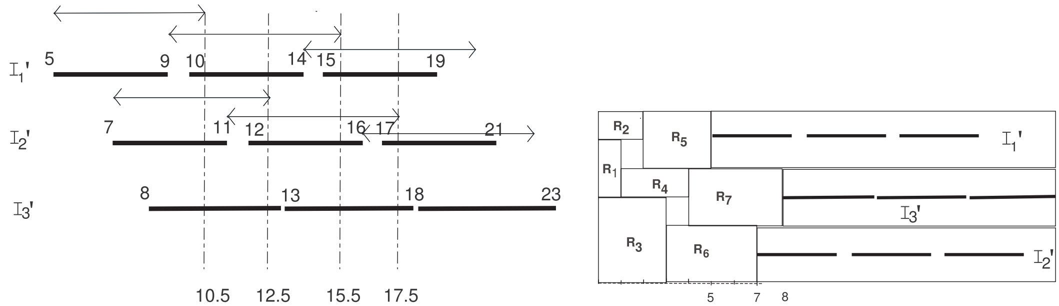
Proof. By definition of f, $f(\max(K_i))$ and $f(\max(K_j))$ intersect if and only if $|i - j| \leq 1$. It suffices to show that the same holds for the intervals $\max(K_i)$ and $\max(K_j)$, i.e. $\max(K_i)$ and $\max(K_j)$ intersect if and only if $|i - j| \leq 1$. Indeed, $\max(K_i) \cap \max(K_{i+1}) \neq \emptyset$ because $\max(K_i) \in K_{i+1}$ (otherwise G is not connected) and therefore $\max(K_i)$ intersects every element of K_{i+1} and in particular $\max(K_{i+1})$. On the other hand for any $k \geq 0$, $\max(K_i) \cap \max(K_{i+2+k}) = \emptyset$, because otherwise $\max(K_i)$ intersects every element of K_{i+2}. Then, $\max(K_i) \in K_{i+2}$, contradicting Lemma 2.1. □

Let $\overline{K}_1 \stackrel{def}{=} K_1$, and for every $i \in [2, k]$, let $\overline{K}_i \stackrel{def}{=} K_i \setminus K_{i-1}$. Clearly, $\{\overline{K}_1, \ldots, \overline{K}_k\}$ is a partition of $V(G)$. Let g be the partial function extending f, such that $g(t(\min(\overline{K}_i))) = 5i + 4$ for every $i \in [1, k]$.

Claim 3.2. *g is order preserving.*

Proof. We verify first that g is an extension of f. We have to show that for every $i, j \in [1, k]$, $\min(\overline{K}_i) \neq \max(K_j)$. Assume by contradiction that for some $i, j \in [1, k]$ we have $\min(\overline{K}_i) = \max(K_j)$. Then, by Lemma 2.1 $j = i - 1$. However, this contradicts the definition of $\overline{K}_i$.

We now show that g is order preserving. $5i + 4 \in (5i, 5i + 11/2)$, i.e. $g(t(\min(\overline{K}_i))) \in g(\max(K_i))$. We have to show that $t(\min(\overline{K}_i)) \in \max(K_i)$ which is equivalent to $\min(\overline{K}_i) \cap \max(K_i) \neq \emptyset$ as G is a proper interval graph. This condition clearly holds, because both $\min(\overline{K}_i)$ and $\max(K_i)$ are elements of the clique K_i. □



(a) The sets $\mathcal{I}'_1, \mathcal{I}'_2, \mathcal{I}'_3$ of intervals for $k = 3$.

(b) The coloring of $\mathcal{I}'_0 \cup \mathcal{I}'_1 \cup \mathcal{I}'_2 \cup \mathcal{I}'_3$.

Fig. 2. The construction of $(G'; w')$.

We note that in $g(\mathcal{I})$, the clique $\overline{K}_i$ is represented by $g(s(\max(\overline{K}_i)), t(\min(\overline{K}_i))) = (g(s(\max(K_i))), \$g(t(\min(\overline{K}_i)))) = (5i, 5i+4)$. Therefore, for every vertex $v \in \overline{K}_i$, $(5i, 5i+4) \subseteq g(I_v)$. Moreover, $g(I_v)$ does not intersect the interval $(5i-5, 5i-1)$ representing $\overline{K}_{i-1}$ because otherwise $v \in K_{i-1}$. On the other hand, $t(g(I_v)) \leq t(\max(K_i)) = 5i + 11/2$. We conclude

$$\forall v \in \overline{K}_i, (5i, 5i+4) \subseteq\ g(I_v)\ \subseteq (5i-1, 5i+11/2) \tag{1}$$

$$\forall v \in \overline{K}_i, (5i+2, 5i+6) \subseteq\ 2 + g(I_v)\ \subseteq (5i+1, 5i+15/2). \tag{2}$$

Let $\mathcal{I}'_0 = \{(0,1), (0,2), (0,3), (1,4), (2,5), (3,7), (4,8)\}$. We note that $\mathcal{I}'_0$ is a proper interval representation of G_0. Let also $\mathcal{I}'_1 = g(\mathcal{I})$, $\mathcal{I}'_2 = 2 + g(\mathcal{I})$, $\mathcal{I}'_3 = \{(5i+3, 5i+8) : i \in [1,k]\}$. As g is order preserving, $\mathcal{I}'_1$ and $\mathcal{I}'_2$ are sets of proper intervals, and $\mathcal{I}'_3$ is clearly a set of proper intervals. We leave to the reader to verify that following (1) and (2), $\mathcal{I}' \overset{def}{=} \mathcal{I}'_0 \cup \mathcal{I}'_1 \cup \mathcal{I}'_2 \cup \mathcal{I}'_3$ is a set of proper intervals (see Fig. 2(a)). G' is the proper interval graph represented by $\mathcal{I}'$. For $j \in \{1,2,3\}$, let V'_j be the vertices represented by the intervals of $\mathcal{I}'_j$, and G'_j the subgraph induced by V'_j. The number of vertices of G' is $|V'_0| + |V'_1| + |V'_2| + |V'_3| = 7 + n + n + k \leq 4n$.

We proceed with the construction of w':

- Every interval of $\mathcal{I}'_0$ corresponds to a vertex v of G_0. For a vertex $v' \in V'_0$, $w'(v') \overset{def}{=} \omega(G; w) \cdot w(v)/2$.
- Every interval in $I' \in \mathcal{I}'_1 \cup \mathcal{I}'_2$ corresponds to an interval $I_v \in \mathcal{I}$. For a vertex v' corresponding to $I' \in \mathcal{I}_1 \cup \mathcal{I}_2$, $w'(v') \overset{def}{=} w(v)$.
- For a vertex $v' \in V'_3$, $w'(v') = \omega(G; w)$.

This completes the construction of $(G'; w')$ that can clearly be computed in polynomial time. We proceed with the proof of the claimed properties:

(i) Clearly, $\chi(G'; w') \geq \omega(G'; w') \geq 3 \cdot \omega(G; w)$ where the second inequality holds because for the clique $K'_1 \overset{def}{=} \{(0,1), (0,2), (0,3)\}$ we have $w'(K'_1) = \omega(G; w) \cdot w(K)/2 = 3 \cdot \omega(G; w)$. It remains to show that $\chi(G'; w') \leq 3 \cdot \omega(G; w)$. Consider the coloring $\frac{\omega(G;w)}{2} \cdot c_0$ of the vertices V_0 (recall that c_0 is the coloring of G_0 from Proposition 3.1 depicted in Fig. 1(d)). A coloring c' of size $3 \cdot \omega(G; w)$ of $(G'; w')$ is obtained by extending $\frac{\omega(G;w)}{2} \cdot c_0$ so that the vertices of V_1 (resp. V_2, V_3) are colored using colors from $(2 \cdot \omega(G; w), 3 \cdot \omega(G; w))$ (resp. $(0, \omega(G; w))$, $(\omega(G; w), 2 \cdot \omega(G; w))$) (see Fig. 2(b)).

(ii) The cliques $K'_1 = \{(0,1), (0,2), (0,3)\}$, $K'_2 = \{(0,3), (1,4), (2,5)\}$ and $K'_3 = \{(2,5), (3,7), (4,8)\}$ are critical and $K'_1 \cup K'_2 \cup K'_3 = V'_0$. We note that $(5i+3, 5i+4)$ intersects the intervals $(5i+3, 5i+8)$, $(5i, 5i+4)$, $2 + (5i, 5i+4)$. Then, $(5i+3, 5i+4)$ intersects all the intervals $g(K_i)$ and $2 + g(K_i)$. Therefore, $(5i+3, 5i+4)$ implies a clique $K'_{i+3} \overset{def}{=} \{(5i+3, 5i+8)\} \cup g(K_i) \cup (2 + g(K_i))$. Furthermore, $w'(K'_{i+3}) = w'((5i+3, 5i+8)) + w'(g(K_i)) + w'(2 + g(K_i)) = \omega(G; w) + w(K_i) + w(K_i) = 3 \cdot \omega(G; w) = \omega(G'; w')$. Therefore, K'_{i+3} is a critical clique. Moreover $\bigcup_{j=0}^{k-1} K'_{i+3+j} = V_1 \cup V_2 \cup V_3$.

(iii) Consider a coloring c' of G' of size $\omega(G'; w')$. This implies a coloring c on $(G_0; w')$ of size at most $\omega(G'; w') = \omega(G'_0; w')$. Then, by Proposition 3.1, $c = \frac{\omega(G;w)}{2} c_0$ up to duality. In the sequel we consider only the primal, as the dual is symmetric. Then, $c((3,7)) = \omega(G; w) \cdot (0,1)$ and $c((4,8)) = \omega(G; w) \cdot (1,2)$. Therefore, the interval available for c at $(5,7)$ is $\omega(G; w) \cdot (2,3)$. We conclude that all the intervals of $\mathcal{I}'_1 \cap K'_4$ are colored using sub-intervals of $\omega(G; w) \cdot (2,3)$. Then, the interval available for c at $(7,8)$ is $\omega(G; w) \cdot (0,1)$. Therefore, all the intervals of $\mathcal{I}'_2 \cap K'_4$ are colored using sub-intervals of $\omega(G; w) \cdot (0,1)$. Now, the interval available for c at $(8,9)$ is $\omega(G; w) \cdot (1,2)$. Therefore, $c(8,13) = \omega(G; w) \cdot (1,2)$. We can show by induction on k that all the intervals of $\mathcal{I}'_1$ (resp. $\mathcal{I}'_2$, $\mathcal{I}'_3$) are colored with sub-intervals of $\omega(G; w) \cdot (2,3)$ (resp. $\omega(G; w) \cdot (0,1)$, $\omega(G; w) \cdot (1,2)$). Then, there is a set of simplicial vertices $S_1 \subset K'_{k+3} \cap V'_1$ (resp. $S_2 \subset K'_{k+3} \cap V'_2$) such that $\omega(G; w) \cdot (2,3) \setminus c(S_1)$ (resp. $\omega(G; w) \cdot (0,1) \setminus c(S_2)$) consists of b disjoint intervals of size x each. We recall that $c(5k+3, 5k+8) = \omega(G; w) \cdot (1,2)$. Therefore, $S' \overset{def}{=} S_1 \cup S_2 \cup \{(5k+3, 5k+8)\}$ is a set of simplicial vertices of K'_{k+3} such that $\omega(G; w) \cdot (0,3) \setminus c(S')$ consists of $2b$ disjoint intervals of size x each. □

The following corollary can be easily shown by induction on i where $(G_0; \frac{B}{2} \cdot w_0)$ constitutes the base of the induction and the inductive step is by Lemma 3.2.

Corollary 3.1. *For every $i \in \mathbb{N}$, and $B \in \mathbb{R}$, there is a weighted proper interval graph $(G_i; \frac{B}{2} \cdot w_i)$ on at most $7 \cdot 4^i$ vertices, such that*

(i) $\omega(G_i; \frac{B}{2} \cdot w_i) = \chi(G_i; \frac{B}{2} \cdot w_i) = B \cdot 3^{i+1}$, *and*

(ii) *there is a set S_i of 2^{i+1} simplicial vertices of G_i such that in every coloring c of size $B \cdot 3^{i+1}$, $(0, B \cdot 3^{i+1}) \setminus c(S_i)$ consists of 2^{i+1} disjoint intervals, each of which having size B.*

We are now ready to prove the main result of this section.

Theorem 3.3. *It is* Np*-Complete in the strong sense to determine for a given weighted proper interval graph $(G; w)$, whether $\chi(G; w) = \omega(G; w)$.*

Proof. The proof is by reduction from 3-Partition. Let $X = \{x_1, \ldots, x_{3m}\}$ be an instance of 3-Partition with $\sum_{k=1}^{3m} x_k = m \cdot B$. We choose i such that $2^i < m \leq 2^{i+1}$, i.e. $i < \log m$. We construct a graph G by adding to G_i a clique K on $2^{i+1} + 2m$ vertices and connecting every vertex of K to every vertex of S_i. Then, $S_i \cup K$ is a maximal clique of G.

The weight function w is the extension of $\frac{B}{2} \cdot w_i$ to the vertices of K such that (a) for an arbitrary subset $K' = \{v_1, \ldots, v_{3m}\}$ of K, $w(v_k) = x_k$ and (b) $w(v) = B$ for every $v \in K'' \stackrel{def}{=} K \setminus K'$. Then $w(K \cup S_i) = \sum_{k=1}^{3m} x_k + B \cdot (2^{i+1} - m) + \frac{B}{2} w_i(S_i) = B \cdot 2^{i+1} + \frac{B}{2} w_i(S_i)$. We recall that $(0, \frac{B}{2}\omega(G_i; w_i)) \setminus c_i(S_i)$ is a set of 2^{i+1} disjoint intervals of size B, i.e. $\frac{B}{2} w_i(S_i) = \frac{B}{2}\omega(G_i; w_i) - B \cdot 2^{i+1}$. Therefore $w(K \cup S_i) = \frac{B}{2}\omega(G_i; w_i)$. As this is the only maximal clique of G that is not in G_i, we conclude $\omega(G; w) = \frac{B}{2}\omega(G_i; w_i)$.

We have $|V(G)| = |V(G_i)| + |K| \leq 7 \cdot 4^i + 2^{i+1} + 2m \leq 7 \cdot m^2 + 3 \cdot m$, moreover, $\omega(G; w) = \omega(G_i; \frac{B}{2} w_i) = B \cdot 3^{i+1} < 3B \cdot m^{\log 3}$. Therefore, the reduction is polynomial and whenever the sizes of the numbers of X are bounded by some polynomial in m, the sizes of the numbers in w are bounded by some polynomial in m too.

It remains to show that $\chi(G; w) = \omega(G; w)$ if and only if X is a Yes instance, i.e. it can be partitioned into m triples each of which summing up to B. We show the only if direction. Assume $\chi(G; w) = \omega(G; w)$ and let c be a coloring of size $\omega(G; w) = \frac{B}{2}\omega(G_i; w_i)$ of $(G; w)$. This induces a coloring c_i of size $\frac{B}{2}\omega(G_i; w_i)$ of $(G_i; w_i)$. Then $(0, \omega(G; w)) \setminus c_i(S_i)$ consists of 2^{i+1} disjoint intervals of size B each. Therefore, all the vertices of K are colored with sub-intervals of these intervals. Every vertex of K'' fills such an interval completely. Then, the vertices of K' are colored with sub-intervals of the remaining m intervals. This coloring induces a partition of X into sets $\{X_1, \ldots, X_m\}$. The sum of the elements of every set is at most B, as the corresponding vertices are colored with sub-intervals of an interval of size B. As the overall sum is $m \cdot B$, every such sum is exactly B. Therefore X is a Yes instance. □

4. Approximation algorithm

In this section we present a simple 2-approximation algorithm for the problem of finding the interval chromatic number of a proper interval graph. We start with definitions of terms used in this section.

For a given number W, we denote a *circular interval I* modulo W as $(s(I), t(I))_W$ where $s(I), t(I) \in [0, W]$. Whenever $s(I) < t(I)$, I is an (ordinary) interval between $s(I)$ and $t(I)$, i.e. $(s(I), t(I))_W = (s(I), t(I))$. Otherwise, it is the union of two intervals. More precisely, $(s(I), t(I))_W = (s(I), W) \cup (0, t(I)) \cup \{0\}$. Clearly, an interval is a circular interval.

A *circular interval coloring* of a weighted graph is a coloring c that assigns to every vertex a circular interval modulo W_c, so that the circular intervals assigned to two adjacent vertices do not overlap. The *circular interval chromatic number* of a weighted graph $(G; w)$ is the minimum size of a circular interval coloring of it, and we denote it by $\chi_C(G; w)$.

A *non-contiguous coloring* of a weighted graph is a coloring c that assigns to every vertex a set of disjoint intervals, so that the sets assigned to two adjacent vertices do not overlap. The *chromatic number* of a weighted graph $(G; w)$ is the minimum size of a coloring of it, and we denote it by $\chi_N(G; w)$.

Clearly, an interval coloring is a circular interval coloring which is in turn a non-contiguous coloring. Moreover, the total size of the intervals assigned to a critical clique K is $w(K) = \omega(G; w)$. Therefore,

$$\omega(G; w) \leq \chi_N(G; w) \leq \chi_C(G; w) \leq \chi(G; w)$$

for any weighted graph $(G; w)$.

Lemma 4.1.

$$\chi_C(G; w) \leq \chi(G; w) \leq 2 \cdot \chi_C(G; w)$$

for every weighted graph $(G; w)$.

Proof. Given a circular interval coloring c of size $W = \chi_C(G; w)$ of $(G; w)$, we claim that the coloring c' defined as

$$c'(v) \stackrel{def}{=} \begin{cases} W + c(v) & \text{if } 0 \in c(v) \\ c(v) & \text{otherwise} \end{cases}$$

is an interval coloring of size at most $2W$ implying the claim. Indeed, vertices are colored with subintervals of $(0, W)$ or subintervals of $(W, 2W)$. Assume that for two adjacent vertices v, v' of G we have $c'(v) \cap c'(v') \neq \emptyset$. Then, either both of

$c'(v), c'(v')$ are subintervals of $(0, W)$ or both are subintervals of $(W, 2W)$. In the first case $c'(v) = c(v)$ and $c'(v') = c(v')$, therefore $c(v) \cap c(v') \neq \emptyset$, contradicting the fact that c is a circular interval coloring. In the latter case $c'(v) = W + c(v)$ and $c'(v') = W + c(v')$, leading to the same contradiction. □

Corollary 4.1. *A ρ-approximation algorithm to determine $\chi_C(G; w)$ implies a 2ρ-approximation algorithm to determine $\chi(G; w)$.*

Lemma 4.2. [17]

$$\chi_C(G; w) = \chi_N(G; w)$$

whenever G is a proper interval graph.

Proof. Algorithm ProperToCircular presented in [17] converts a non-contiguous coloring of a proper interval graph to a circular interval coloring of the same size, thus proving the claim. We provide here the algorithm for completeness.

Algorithm 2 ProperToCircular

Require: c is a non-contiguous coloring of $(G; w)$.
Ensure: c' is a circular interval coloring of $(G; w)$.
Ensure: $size(c') = size(c)$

```
λ ← 0.
for all v ∈ V(G) ordered by their start vertices of I_v do
    c'(v) ← (λ, λ + size(c(v))).        ▷ Additions modulo W_c
    λ ← λ + size(c(v)).
end for
```

□

Therefore we conclude:

Theorem 4.1. *There is a 2-approximation algorithm to determine the interval chromatic number $\chi(G; w)$ of a weighted proper interval graph $(G; w)$.*

Proof. As G is an interval graph $\chi_N(G; w)$ can be calculated in polynomial-time. Moreover, $\chi_N(G; w) = \chi_C(G; w)$ by Lemma 4.2. Finally, by Corollary 4.1 this optimal algorithm implies a 2-approximation algorithm to determine $\chi(G; w)$. □

5. Summary

In this work we showed that the determination of the interval chromatic number of a weighted proper interval graph is Np-Complete in the strong sense. This implies that the family of proper interval graphs is not included in the family of superperfect graphs. This strengthens the almost 40 year old result of Larry Stockmeyer that showed strong NP-completeness for interval graphs. A natural question to ask is whether there are sub-families of proper interval graphs where the same result holds.

Our proofs imply an infinite family of non-superperfect proper interval graphs: Let K be a complete graph with at least two vertices, and G_0 the graph in Fig. 1(a). The 1-clique-sum of G_0 and K where the vertex $7 \in V(G_0)$ is identified with any vertex of K is not superperfect. This is because one can assign to the vertices of $V(K) \setminus V(G_0)$ weights from a No instance of Partition and complete the weights of $V(G_0)$ as in the proof of Lemma 3.1.

On the positive side, we have shown that the problem is 2-approximable for proper interval graphs using a simple argument. For interval graphs, the problem is mentioned to be Apx-Hard in [13] (without proof), and the best known approximation algorithm is a $(2 + \epsilon)$-approximation. We believe that using similar techniques to [8] and [3] the ratio of 2 can be improved. Besides improving the constant, a natural research direction in this respect is to investigate the complexity class of the problem in proper interval graphs. Namely, is the problem Apx-Hard in this graph family, or does it admit a Ptas?

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