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Note

On the non-negativity of the complete **cd**-index

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ABSTRACT

The complete **cd**-index of a Bruhat interval is a non-commutative polynomial in the variables **c** and **d**, which was introduced by Billera and Brenti and conjectured to have non-negative coefficients. For a **cd**-monomial M containing at most one **d**, i.e., $M = \mathbf{c}^i$ or $M = \mathbf{c}^i\mathbf{d}\mathbf{c}^j$ $(i, j \geq 0)$, Karu showed that the coefficient of M is non-negative. In this paper, we show that when $M = \mathbf{d}\mathbf{c}^i\mathbf{d}\mathbf{c}^j$ $(i, j \geq 0)$, the coefficient of M is non-negative.

1. Introduction

Let (W, S) be a Coxeter system and $u, v \in W$ such that $u < v$ in the Bruhat order. Billera and Brenti [2] associated the interval $[u, v]$ with a non-commutative polynomial $\tilde{\phi}_{u,v}(\mathbf{a}, \mathbf{b})$ in the variables **a** and **b**. They further proved that $\tilde{\phi}_{u,v}(\mathbf{a}, \mathbf{b})$ can be written as a polynomial in the variables **c** and **d**, where $\mathbf{c} = \mathbf{a} + \mathbf{b}$, $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$. This new polynomial $\tilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ is called the complete **cd**-index of $[u, v]$, which is a generalization of the **cd**-index of $[u, v]$ in the sense that the **cd**-index of $[u, v]$ is the highest degree terms of the complete **cd**-index of $[u, v]$.

The **cd**-index of an Eulerian poset is well studied, see, e.g., Billera [1] and the references therein. In particular, since a Bruhat interval is Eulerian and shellable, hence Gorenstein*, the coefficients of the **cd**-index of a Bruhat interval are non-negative, see Karu [6]. As an analogue, Billera and Brenti [2] conjectured that the complete **cd**-index still has non-negative coefficients, see also Billera [1, Conjecture 2].

Let the variables **a**, **b**, **c** have degree 1, and the variable **d** have degree 2. Blanco [3] showed that the monomials of lowest degree in the complete **cd**-index have non-negative coefficients. Suppose that M is a **cd**-monomial of degree n. If M contains at most one **d**, then Karu [7] showed that the coefficient of M is non-negative. Let us describe briefly Karu's construction. Denote by $B_n(u, v)$ the set of Bruhat paths of length $n + 1$ in the Bruhat graph of $[u, v]$. For a Bruhat path $x \in B_n(u, v)$, Karu assigned a weight $s_M(x)$ to the path x, which can be -1, 0 or 1. If there exists a flip on $[u, v]$ which is compatible with the given reflection order, then the coefficient of M is equal to the sum of weights of all the Bruhat paths in $B_n(u, v)$.

Moreover, Karu conjectured that $s_M(x) \neq -1$ for all intervals and all monomials M, which he called the flip condition. If such a condition holds, then we can construct a compatible flip, and so the non-negativity conjecture of the complete **cd**-index is true. When the **cd**-monomial M contains at most one **d**, Karu proved that the flip condition holds by using a result of Dyer [4]. Hence in this case the coefficient of M in the complete **cd**-index of $[u, v]$ is non-negative.

In this paper, we show that the coefficient of the **cd**-monomial $M = \mathbf{d}\mathbf{c}^i\mathbf{d}\mathbf{c}^j$ $(i, j \geq 0)$ is non-negative. Based on Karu's construction, we shall show that the number of paths with weight -1 is less than or equal to the number of paths with weight 1. To this end, we divide the involving paths into four disjoint sets according to their weights and ascent–descent sequences, and then establish two injections among these four sets of paths.

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2. Preliminary

For a Coxeter system (W, S), denote by $\ell(w)$ the length of $w \in W$. The set $T = \{wsw^{-1} \mid w \in W, s \in S\}$ is called the set of reflections of W. Let $u, v \in W$, we say that $u \prec v$ if there exists $t \in T$, such that $v = ut$ and $\ell(v) > \ell(u)$, and we say that $u < v$ if there exists a sequence of elements $u_1, u_2, \ldots, u_r \in W$ such that $u \prec u_1 \prec u_2 \prec \cdots \prec u_r \prec v$. The partial order "$<$" is called the Bruhat order of (W, S). The Bruhat graph of (W, S) is a directed graph with vertex set W and there is a directed edge from u to v if $u \prec v$.

Let $u < v$ in the Bruhat order, a Bruhat path from u to v of length $n+1$ is a sequence

$$x = (u = x_0 \prec x_1 \prec x_2 \prec \cdots \prec x_n \prec x_{n+1} = v). \tag{1}$$

We label the edge $x_i \prec x_{i+1}$ by the reflection $t_i = x_i^{-1}x_{i+1}$ for $i = 0, 1, \ldots, n$. Let $B_n(u, v)$ denote all the Bruhat paths from u to v of length $n+1$. Denote by $B(u, v) = \bigcup_{n\geq 0} B_n(u, v)$ the set of all Bruhat paths from u to v.

Recall that a reflection order $(\mathcal{O}, <_T)$ is a total order defined on the set of reflections T, see [5]. The reverse of the order $\mathcal{O}$, denoted by $\overline{\mathcal{O}}$, is also a reflection order. In the sequel, we will always use the reflection order $(\mathcal{O}, <_T)$. We say that the path x in (1) is increasing (resp., decreasing), if $t_0 <_T t_1 <_T \cdots <_T t_n$ (resp., $t_n <_T t_{n-1} <_T \cdots <_T t_0$). The following result is due to Dyer [4].

Theorem 2.1 ([4]). *Let $x = (u = x_0 \prec x_1 \prec \cdots \prec x_n \prec x_{n+1} = v)$ be an increasing path in $B_n(u, v)$, and $y = (u = y_0 \prec y_1 \prec \cdots \prec y_m \prec y_{m+1} = v)$ be a decreasing path in $B_m(u, v)$. Then we have*

$$x_0^{-1}x_1 <_T y_0^{-1}y_1, \qquad y_m^{-1}y_{m+1} <_T x_n^{-1}x_{n+1}.$$

For the Bruhat path $x \in B_n(u, v)$ in (1), define the ascent–descent sequence of x by

$$\omega(x) = \beta_1\beta_2\cdots\beta_n,$$

where

$$\beta_i = \begin{cases} \mathbf{a}, & \text{if } t_{i-1} <_T t_i; \\ \mathbf{b}, & \text{if } t_i <_T t_{i-1}. \end{cases}$$

In [2], Billera and Brenti associate each interval $[u, v]$ with a non-homogeneous polynomial $\tilde{\phi}_{u,v}(\mathbf{a}, \mathbf{b})$ in the non-commutative variables $\mathbf{a}$ and $\mathbf{b}$ by summing over the ascent–descent sequences of all the Bruhat paths in $B(u, v)$. That is,

$$\tilde{\phi}_{u,v}(\mathbf{a}, \mathbf{b}) = \sum_{x\in B(u,v)} \omega(x).$$

It can be shown that $\tilde{\phi}_{u,v}(\mathbf{a}, \mathbf{b})$ is independent of the given reflection order. Moreover, $\tilde{\phi}_{u,v}(\mathbf{a}, \mathbf{b})$ can be rewritten as a polynomial in the variables $\mathbf{c}$ and $\mathbf{d}$, where $\mathbf{c} = \mathbf{a} + \mathbf{b}$, $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$. That is,

$$\tilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d}) = \tilde{\psi}_{u,v}(\mathbf{a} + \mathbf{b}, \mathbf{ab} + \mathbf{ba}) = \tilde{\phi}_{u,v}(\mathbf{a}, \mathbf{b}).$$

This new polynomial $\tilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ is called the complete **cd**-index of $[u, v]$.

Now we proceed to recall some definitions and results in [7].

For an **ab**-monomial $M(\mathbf{a}, \mathbf{b})$, denote by $\overline{M} = M(\mathbf{b}, \mathbf{a})$ the **ab**-monomial obtained by exchanging **a** and **b** in M. This operator is an involution in the non-commutative ring $\mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle$.

Definition 2.2. A flip $F = F_{u,v}$ on $[u, v]$ is an involution

$$F_{u,v} : B(u, v) \to B(u, v),$$

such that $\omega(F(x)) = \overline{\omega(x)}$ for all $x \in B(u, v)$.

Note that since $\tilde{\phi}_{u,v}(\mathbf{a}, \mathbf{b})$ is independent of the reflection order, when we use the orders $\mathcal{O}$ and $\overline{\mathcal{O}}$ to compute $\tilde{\phi}_{u,v}(\mathbf{a}, \mathbf{b})$ respectively, we obtain $\tilde{\phi}_{u,v}(\mathbf{a}, \mathbf{b}) = \tilde{\phi}_{u,v}(\mathbf{b}, \mathbf{a})$. That is to say, any flip $F_{u,v}$ on $[u, v]$ has no fixed points.

We fix a flip for every interval in advance. Let $1 \leq m \leq n$ and

$$x = (u = x_0 \prec x_1 \prec x_2 \prec \cdots \prec x_m \prec x_{m+1} \prec \cdots \prec x_n \prec x_{n+1} = v).$$

After applying the flip $F_{x_m,v}$ to x, we get

$$y = (u = x_0 \prec x_1 \prec x_2 \prec \cdots \prec x_m \prec y_{m+1} \prec \cdots \prec y_n \prec y_{n+1} = v).$$

If $\omega(x) = \beta_1\cdots\beta_m\cdots\beta_n$, then $\omega(y) = \beta_1\cdots\beta_{m-1}\alpha_m\overline{\beta}_{m+1}\cdots\overline{\beta}_n$, where α_m can be either **a** or **b**. Define

$$s_{m,\mathbf{a}}(x) = \begin{cases} 1, & \text{if } \beta_m = \mathbf{a}; \\ 0, & \text{otherwise.} \end{cases}$$

$$s_{m,\mathbf{b}}(x) = \begin{cases} 1, & \text{if } \beta_m = \mathbf{b}, \alpha_m = \mathbf{a}; \\ -1, & \text{if } \beta_m = \mathbf{a}, \alpha_m = \mathbf{b}; \\ 0, & \text{otherwise.} \end{cases}$$

Given a **cd**-monomial $M(\mathbf{c}, \mathbf{d})$, we can obtain a unique **ab**-monomial $M(\mathbf{a}, \mathbf{ba})$ by substituting **a** for **c** and **ba** for **d** in $M(\mathbf{c}, \mathbf{d})$. Apparently, this is a one-to-one correspondence between **cd**-monomials and **ab**-monomials in which every **b** is followed by an **a**. In the following, we shall use the letter M to denote either the **cd**-monomial $M(\mathbf{c}, \mathbf{d})$ or the **ab**-monomial $M(\mathbf{a}, \mathbf{ba})$ when the distinction is clear from the context.

Definition 2.3. Let $M(\mathbf{c}, \mathbf{d})$ be a **cd**-monomial with $M(\mathbf{a}, \mathbf{ba}) = \gamma_1\gamma_2\cdots\gamma_n$. Define

$$s_M(x) = \prod_{m=1}^{n} s_{m,\gamma_m}(x).$$

Clearly the value of $s_M(x)$ depends on the given flip.

Definition 2.4. A flip F is said to be compatible with the reflection order $\mathcal{O}$ if

$$s_M(x) = \bar{s}_M(F(x))$$

for any interval $[u, v]$, any **cd**-monomial M and any path $x \in B(u, v)$. Here $\bar{s}_M(F(x))$ is the value $s_M(F(x))$ computed by using the reverse reflection order $\overline{\mathcal{O}}$.

Theorem 2.5 *([7]). Assume that the flip F is compatible with the reflection order $\mathcal{O}$. For any* **cd***-monomial M of degree n, the coefficient of M in $\tilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ is equal to*

$$\sum_{x\in B_n(u,v)} s_M(x).$$

Remark 1. In fact, from the proof of Theorem 2.5 in [7], we see that the flip F needs only to be compatible with the reflection order $\mathcal{O}$ on all the proper sub-intervals $[w, v] \subsetneq [u, v]$ such that for a path $x \in B(w, v)$, the number of **b**'s in $\omega(x)$ is one less than the number of **b**'s in $M(\mathbf{a}, \mathbf{ba})$.

If -1 does not appear in the above sum, then the coefficient of M is clearly non-negative. Therefore Karu [7] introduced the following flip condition.

Definition 2.6. The flip condition holds for the interval $[u, v]$ and monomial M if for every $x \in B(u, v)$ the following is satisfied. If $s_{m,\gamma_m}(x) = -1$ for some m, then there exists $l > m$ such that $s_{l,\gamma_l}(x) = 0$.

Definition 2.7. Let $M = \gamma_1\gamma_2\cdots\gamma_n$ be an **ab**-monomial of length n. Define

$$T_M(u, v) = \{x \in B_n(u, v) \mid s_{m,\gamma_m}(x) = 1,\ \text{for all } 1 \le m \le n\}.$$

From Theorem 2.5 we have

Corollary 2.8. *If the flip condition holds for the interval $[u, v]$ and monomial M, then the coefficient of M in $\tilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ is equal to $|T_M(u, v)|$ and hence is non-negative.*

Remark 2. If $M(\mathbf{c}, \mathbf{d})$ contains at most one **d**, then the flip condition holds by Theorem 2.1. Therefore, in this case the coefficient of M is non-negative. Moreover, if the flip condition holds for the interval $[u, v]$, we can construct a compatible flip on $[u, v]$. More precisely, by induction on the number of **d**'s and Remark 1, we can apply Theorem 2.5 to deduce that $|T_M(u, v)| = |\overline{T}_M(u, v)|$, where $\overline{T}_M(u, v)$ is the set $T_M(u, v)$ constructed by using the reverse reflection order $\overline{\mathcal{O}}$. Now we can obtain a compatible flip on $[u, v]$ by giving a bijection between $T_M(u, v)$ and $\overline{T}_M(u, v)$.

3. Main result

In this section, we aim to show that for a **cd**-monomial $M(\mathbf{c}, \mathbf{d})$ containing two **d**'s and starting with **d**, the coefficient of M is non-negative.

Theorem 3.1. *Suppose that $M = \mathbf{dc}^i\mathbf{dc}^j$ $(i, j \ge 0)$. Then the coefficient of M in the complete* **cd***-index of the Bruhat interval $[u, v]$ is non-negative.*

Proof. Notice that when a **cd**-monomial contains at most one **d**, there exists a compatible flip. According to Remarks 1 and 2, the coefficient of M in the complete **cd**-index of $[u, v]$ can be still computed by using Theorem 2.5.

Now we aim to show that the number of paths $x \in B_n(u, v)$ with $s_M(x) = 1$ is no less than that of $s_M(x) = -1$, where $n = i + j + 4$. Let us first analyze how can $s_M(x)$ be equal to 1 or -1. Let

$$M' = \mathbf{a}^{i+1}\mathbf{b}\mathbf{a}^{j+1}, \quad i, j \geq 0.$$

By Theorem 2.1, $s_{M'}(x) \neq -1$ for any interval and any flip. Then we must have

$$s_M(x) = 1 \Longleftrightarrow s_{m,\mathbf{a}}(x) = s_{m,\mathbf{b}}(x) = 1, \quad \text{for all } 1 \leq m \leq n.$$

$$s_M(x) = -1 \Longleftrightarrow \begin{cases} s_{m,\mathbf{a}}(x) = 1, & \text{for all } \mathbf{a}\text{'s of } M; \\ s_{m,\mathbf{b}}(x) = -1, & \text{for the first } \mathbf{b} \text{ of } M; \\ s_{m,\mathbf{b}}(x) = 1, & \text{for the second } \mathbf{b} \text{ of } M. \end{cases}$$

Thus no matter $s_M(x) = 1$ or -1, we have $s_{m,\mathbf{b}}(x) = 1$ for the second **b** of M. Therefore we can concentrate only on the paths $x \in B_{n-1}(u, v)$ such that $s_{M'}(x) = 1$, and then extend these paths in $B_{n-1}(u, v)$ to $B_n(u, v)$.

Choose all $w \in [u, v]$ such that there exists a Bruhat path from w to v of length $n - 1$. Recall that

$$T_{M'}(w, v) = \{x \in B_{n-1}(w, v) \mid s_{M'}(x) = 1\}.$$

For a path

$$x = (u = x_0 \prec x_1 \prec \cdots \prec x_n \prec x_{n+1} = v)$$

in $B_n(u, v)$, let

$$x' = (x_1 \prec x_2 \prec \cdots \prec x_n \prec x_{n+1} = v).$$

Define

$$A = \{x \in B_n(u, v) \mid x' \in \cup_w T_{M'}(w, v), \omega(x) = \mathbf{b}M', s_{1,\mathbf{b}}(x) = 1\},$$
$$B = \{x \in B_n(u, v) \mid x' \in \cup_w T_{M'}(w, v), \omega(x) = \mathbf{b}M', s_{1,\mathbf{b}}(x) = 0\},$$

and

$$C = \{x \in B_n(u, v) \mid x' \in \cup_w T_{M'}(w, v), \omega(x) = \mathbf{a}M', s_{1,\mathbf{b}}(x) = -1\},$$
$$D = \{x \in B_n(u, v) \mid x' \in \cup_w T_{M'}(w, v), \omega(x) = \mathbf{a}M', s_{1,\mathbf{b}}(x) = 0\}.$$

Then we need to show that $|C| \leq |A|$.

Firstly, we claim that

$$|C| + |D| \leq |A| + |B|. \tag{2}$$

To prove (2), we construct an injection

$$\rho : C \uplus D \to A \uplus B.$$

Suppose that

$$x = (u = x_0 \prec x_1 \prec x_2 \prec \cdots \prec x_n \prec x_{n+1} = v)$$

is a path in $C \uplus D$. Then $\omega(x) = \mathbf{a}M' = \mathbf{a}\mathbf{a}^{i+1}\mathbf{b}\mathbf{a}^{j+1}$, $i, j \geq 0$. So we have $t_0 <_T t_1 <_T t_2$, where $t_k = x_k^{-1}x_{k+1}$, $k = 0, 1, 2$. Let

$$\rho(x) = F_{u,x_2}(x) = (u = x_0 \prec x_1' \prec x_2 \prec \cdots \prec x_n \prec x_{n+1} = v).$$

By Theorem 2.1, we have $t_0 <_T t_0'$ and $t_1' <_T t_1 <_T t_2$, where $t_0' = x_0^{-1}x_1'$ and $t_1' = (x_1')^{-1}x_2$. It is easy to check that $\rho(x) \in A \uplus B$. Since a flip on an interval is an involution without fixed points, we see that ρ is indeed an injection. Hence (2) holds.

Secondly, we claim that

$$|B| + |C| \leq |A| + |D|. \tag{3}$$

To prove (3), we consider the paths in sets A, B, C and D after applying the flip $F_{u,v}$ on $[u, v]$. Let

$$A' = \{y \in B_n(u, v) \mid y = F_{u,v}(x), x \in A\}.$$

Clearly, there is a one to one correspondence between the sets A and A'. Define B', C' and D' similarly. Note that

$$\omega(y) = \begin{cases} \mathbf{a}\mathbf{b}^{i+1}\mathbf{a}\mathbf{b}^{j+1}, & \text{if } y \in A' \uplus D'; \\ \mathbf{b}\mathbf{b}^{i+1}\mathbf{a}\mathbf{b}^{j+1}, & \text{if } y \in B' \uplus C'. \end{cases}$$

Now we construct an injection

$$\eta : B' \uplus C' \to A' \uplus D'.$$

Suppose that

$$y = (u = y_0 \prec y_1 \prec y_2 \prec \cdots \prec y_n \prec y_{n+1} = v)$$

is a Bruhat path in $B' \uplus C'$. Define

$$\eta(y) = F_{u,y_2}(y) = (u = y_0 \prec y_1' \prec y_2 \prec \cdots \prec y_n \prec y_{n+1} = v).$$

Similarly, by Theorem 2.1 again, one can check that $\eta(y) \in A' \uplus D'$. Then $|B'| + |C'| \le |A'| + |D'|$. Thus (3) follows.

Combining (2) and (3), we obtain $|C| \le |A|$. This completes the proof. □

Remark 3. By using the method in the proof of Theorem 3.1 and Lemma 3.5 in [7], Kalle Karu pointed out to us that when $M(\mathbf{c}, \mathbf{d}) = \mathbf{cdc}^i\mathbf{dc}^j$ $(i, j \ge 0)$, one can show that the coefficient of M is non-negative. The techniques of this paper can probably be extended to prove the non-negativity for all monomials containing two **d**'s.

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