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On the vertices of a 3-partite tournament not in triangles[☆]

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ABSTRACT

Let T be a 3-partite tournament and $F_3(T)$ be the set of vertices of T not in triangles. We prove that, if the global irregularity of T, $i_g(T)$, is one and $|F_3(T)| > 3$, then $F_3(T)$ must be contained in one of the partite sets of T and $|F_3(T)| \leq \left\lfloor \frac{k+1}{4} \right\rfloor + 1$, which implies $|F_3(T)| \leq \left\lfloor \frac{n+5}{12} \right\rfloor + 1$, where k is the size of the largest partite set and n the number of vertices of T. Moreover, we give some upper bounds on the number, as well as results on the structure of said vertices within the digraph, depending on its global irregularity.

1. Introduction

Let c be a nonnegative integer. A *c-partite or multipartite tournament* is a digraph obtained from a complete c-partite graph orienting each edge. Let $N^+(x)$, $N^-(x)$, $d^+(x)$ and $d^-(x)$ denote the *out-neighborhood*, *in-neighborhood*, *out-degree* and the *in-degree* of x, respectively. A digraph D is *r-regular* if $d^+(x) = d^-(x) = r$ for every $x \in V(D)$.

Let T be a c-partite tournament. We say that a vertex v is $\overrightarrow{C_3}$*-free* if v does not lie on any directed triangle of T. Let $F_3(T)$ be the set of the $\overrightarrow{C_3}$-free vertices in a c-partite tournament and let $f_3(T)$ be its cardinality.

The structure of cycles in multipartite tournaments has been extensively studied, see for example [6,5]. In 1998, Zhou et al. [8] has proved that, if T is a regular c-partite tournament with $c \geq 4$, then T does not have $\overrightarrow{C_3}$-free vertices. In 2002, Volkmann [5] provided an infinite family of $4p$-regular 3-partite tournaments with $\overrightarrow{C_3}$-free vertices.

In 2010, Figueroa et al. [2] proved that, if T is a regular 3-partite tournament, then $F_3(T)$ must be contained in one of the partite sets of T and that $f_3(T) \leq \left\lfloor \frac{n}{9} \right\rfloor$. In 2012, Figueroa and Olsen [3] proved that $f_3(T) \leq \lfloor \frac{n}{12} \rfloor$ and showed that this bound is tight, generalizing the family of Volkmann to an infinite family of r-regular 3-partite tournaments.

A natural problem is to study the structure and cardinality of $\overrightarrow{C_3}$-free vertices in 3-partite tournaments. In order to do this, we use the notion of global irregularity of a digraph. The *global irregularity* of a digraph D is defined as $i_g(D) = \max_{x,y\in V(D)}\{\max\{d^+(x), d^-(x)\} - \min\{d^+(y), d^-(y)\}\}$. A digraph D is *regular* (*almost regular*, resp.) if $i_g(D) = 0$ ($i_g(D) \leq 1$, resp.).

The analogue of Zhou et al.'s result for almost regular multipartite tournaments was proved by Tewes et al. [4] and states that, if T is an almost regular c-partite tournament with $c \geq 5$, then T does not have $\overrightarrow{C_3}$-free vertices.

In [2] there is an example of a family of strongly connected 3-partite tournaments of order n with $i_g(T) = 2k - 2$, where k is the cardinality of the largest partite set of T, and $f_3(T) = n - 4$ such that every partite set has $\overrightarrow{C_3}$-free vertices. In this

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paper, we give partial results for the structure and size of $F_3(T)$ in 3-partite tournaments in terms of the global irregularity. We use those results to prove that, if T is an almost regular 3-partite tournament with at least three $\overrightarrow{C_3}$-free vertices, then $F_3(T)$ is an independent set and $f_3(T) \leq \lfloor \frac{n+5}{12} \rfloor + 1$.

2. Preliminaries

For general concepts we refer the reader to [1].

Throughout this article, we will use the following definitions and results. Let $X, Y \subseteq V(D)$, X *dominates* Y, denoted by $X \to Y$, if $xy \in A(D)$ for every $x \in X$ and $y \in Y$. The number of arcs from X to Y is denoted by $d(X, Y)$. Let T be a multipartite tournament and $x \in V(T)$. The partite set of T that contains x is denoted by $P(x)$.

Lemma 1 (*Lemma 2.1 [4]*). *If T is a c-partite tournament with partite sets $P_0, P_1, \ldots, P_{c-1}$, then $\||P_i| - |P_j|\| \leq 2i_g(T)$ for $0 \leq i, j \leq c-1$.*

Lemma 2 (*Lemma 2.1 [7]*). *If T is a multipartite tournament and x a vertex of T with $|P(x)| = p$, then* $\frac{|V(T)|-p-i_g(T)}{2} \leq \min\{d^+(x), d^-(x)\} \leq \max\{d^+(x), d^-(x)\} \leq \frac{|V(T)|-p+i_g(T)}{2}$.

Let T be a 3-partite tournament with partite sets P_0, P_1, P_2 and let $A \subseteq V(T)$ and $x \in V(T)$. For $i \in \{0, 1, 2\}$, we will use the following notation.

- $P_i^{\epsilon}(A) = \bigcap_{a \in A} N^{\epsilon}(a) \cap P_i$ with $\epsilon \in \{+, -\}$.
- $P_i^{*}(A) = P_i \setminus (P_i^{+}(A) \cup P_i^{-}(A))$.
- $P_i^{\epsilon}(x) = P_i^{\epsilon}(\{x\})$, $\epsilon \in \{+, -\}$.
- $P_i^{\epsilon,\delta}(A, x) = P_i^{\epsilon}(A) \cap P_i^{\delta}(x)$, $\epsilon \in \{+, -, *\}$, $\delta \in \{+, -\}$.

Definition 1. Let T be a 3-partite tournament with partite sets P_0, P_1, and P_2. Suppose that $A \subseteq V(T)$ is an independent set. We say that T has an A-partition if $P_i = P_i^{+}(A) \cup P_i^{-}(A)$ for some partite set P_i.

3. Tripartite tournaments with arbitrary global irregularity

In this section, we give sufficient conditions to assure that all $\overrightarrow{C_3}$-free vertices of a 3-partite tournament with arbitrary global irregularity are contained in the same partite set. We also prove an upper bound on the number of $\overrightarrow{C_3}$-free vertices under these conditions.

Remark 1. Let T be a 3-partite tournament with partite sets P_0, P_1 and P_2. Suppose that $A \subseteq F_3(T) \cap P_0$ and $x \in F_3(T) \cap (P_1^{+}(A) \cup P_1^{*}(A))$. If $P_1^{*}(A) = \emptyset$ or $P_2^{*+}(A, x) = \emptyset$, then T has the following structure.

(i) $P_1 = P_1^{+}(A) \cup P_1^{*}(A) \cup P_1^{-}(A)$, and
(ii) $P_2 = P_2^{++}(A, x) \cup P_2^{+-}(A, x) \cup P_2^{*-}(A, x) \cup P_2^{--}(A, x)$.

Proof. It is enough to prove that $P_2^{*+}(A, x) \cup P_2^{-+}(A, x) = \emptyset$.

If $P_1^{*}(A) = \emptyset$, then by definition, for each $z \in P_2^{*+}(A, x) \cup P_2^{-+}(A, x)$ there exists a vertex $y \in (A \cap N^{+}(z))$. Since $x \in P_1^{+}(A)$, we have a directed triangle $z \to y \to x$, which is a contradiction. Hence, $P_2^{*+}(A, x) \cup P_2^{-+}(A, x) = \emptyset$.

If $P_2^{*+}(A, x) = \emptyset$, it remains to prove that $P_2^{-+}(A, x) = \emptyset$.

Let $z \in P_2^{-+}(A, x)$. By definition, $A \subseteq N^{+}(z)$. For $y \in (A \cap N^{-}(x))$ we have a directed triangle $z \to y \to x$, which is a contradiction. □

The next theorem is our main result about the structure of the set $F_3(T)$ for a 3-partite tournament with arbitrary global irregularity.

Theorem 1. *Let T be a 3-partite tournament with global irregularity $i_g(T) \geq 1$ and partite sets P_0, P_1 and P_2. Suppose that $A = F_3(T) \cap P_0$ and T has an A-partition. If $|A| > \frac{3}{2} i_g(T)$, then $A = F_3(T)$.*

Proof. Suppose that $A \neq F_3(T)$. Without loss of generality, we can assume that there exists an $x \in F_3(T) \cap (P_1^{+}(A) \cup P_1^{*}(A))$. Since T has an A-partition, we have the following two cases.

Case 1. *The partite set $P_1 = P_1^{+}(A) \cup P_1^{-}(A)$.*

In this case, $x \in P_1^{+}(A)$. By Remark 1, $P_2 = P_2^{++}(A, x) \cup P_2^{+-}(A, x) \cup P_2^{*-}(A, x) \cup P_2^{--}(A, x)$. We claim that $P_1^{-}(A) \neq \emptyset$. Suppose to the contrary that $P_1^{-}(A) = \emptyset$, then $P_1 = P_1^{+}(A)$. For every $y \in A$, we have $N^{-}(y) \subseteq P_2$ and thus, as $N^{-}(y) \to y \to x$ does not have any directed triangle, $N^{-}(y) \subseteq N^{-}(x)$. Therefore, $d^{-}(x) \geq d^{-}(y) + |A| > d^{-}(y) + \frac{3}{2} i_g(T)$. By definition of global irregularity, $i_g(T) \geq d^{-}(x) - d^{-}(y) > \frac{3}{2} i_g(T)$, which is a contradiction.

We will prove that $P_2^{--}(A,x) \neq \emptyset$. If $P_2^{--}(A,x) = \emptyset$, then by Remark 1, $P_2 = P_2^{++}(A,x) \cup P_2^{+-}(A,x) \cup P_2^{*-}(A,x)$. Thus, for every $u \in P_2$ there exists a $y \in A$ such that $y \in N^-(u)$. Let $z \in P_1^-(A)$. Since $z \to y \to u$ is not a directed triangle, $u \in N^+(z)$. Therefore, $d^+(z) \geq |A| + |P_2|$ and $d^-(z) \leq |P_0| - |A|$. So, $i_g(T) \geq d^+(z) - d^-(z) \geq |P_2| - |P_0| + 2|A| > -\||P_2| - |P_0|\| + 3i_g(T) \geq i_g(T)$, by Lemma 1, a contradiction.

We claim that $P_2^{++}(A,x) \neq \emptyset$. Otherwise, by Remark 1, $P_2 = P_2^{+-}(A,x) \cup P_2^{*-}(A,x) \cup P_2^{--}(A,x)$ and we reach the similar contradiction $i_g(T) \geq d^-(x) - d^+(x) \geq |P_2| - |P_0| + 2|A| > i_g(T)$.

Let $u \in P_2^{++}(A,x)$ and $v \in P_2^{--}(A,x)$. Since $P_0^-(x) \to x \to u$ does not have any directed triangle, $P_0^-(x) \subseteq N^-(u)$. Similarly, $P_1^-(A) \to A \to u$ does not have any directed triangle, so $P_1^-(A) \subseteq N^-(u)$. Which implies that $d^-(u) \geq |P_1^-(A)| + |P_0^-(x)| + 1$. Analogously, $P_1^+(A) \subseteq N^+(v)$, $P_0^+(x) \subseteq N^+(v)$ and $d^+(u) \geq |P_1^+(A)| + |P_0^+(x)| + |A|$. By those inequalities and Lemma 2, $|P_0| + |P_1| + i_g(T) \geq d^-(u) + d^+(v) \geq |P_0| + |P_1| + \frac{3}{2}i_g(T) + 1$, a contradiction.

Case 2. *The partite set $P_2 = P_2^+(A) \cup P_2^-(A)$.*

By definition, $\emptyset = P_2^*(A) = P_2^{*+}(A,x) \cup P_2^{*-}(A,x)$ and by Remark 1, $P_2 = P_2^{++}(A,x) \cup P_2^{+-}(A,x) \cup P_2^{--}(A,x)$. Define $A^+ = A \cap N^+(x)$, $A^- = A \cap N^-(x)$.

Claim 1. *The set of vertices $P_2^{++}(A,x) = \emptyset$ or $P_2^{--}(A,x) = \emptyset$.*

Let $u \in P_2^{++}(A,x)$ then $P_0^-(x) \subseteq N^-(u)$ because $x \in F_3(T)$. For every $z \in P_1 \setminus P_1^+(A)$, there exists a $y \in A \cap N^+(z)$. Since $z \to y \to u$ is not a directed triangle, $P_1 \setminus P_1^+(A) \subseteq N^-(u)$. Therefore, $d^-(u) \geq |P_0^-(x)| + |P_1| - |P_1^+(A)| + |A^+| + 1$. Analogously, if $v \in P_2^{--}(A,x)$, then $d^+(v) \geq |P_0^+(x)| + |P_1^+(A)| + |A^-| + 1$. Which implies that $|P_0| + |P_1| + i_g(T) \geq d^-(u) + d^+(v) \geq |P_0| + |P_1| + \frac{3}{2}i_g(T) + 2$, which is a contradiction. Hence, Claim 1 has been proved.

Subcase 2.1 Suppose that $A^+ \neq \emptyset$ and $A^- \neq \emptyset$.
The set $P_2^{+-}(A,x) = \emptyset$. Otherwise, $A^+ \to P_2^{+-}(A,x) \to x$ would imply a directed triangle. Therefore, $P_2 = P_2^{++}(A,x) \cup P_2^{--}(A,x)$ and by Claim 1, we have to consider two cases: $P_2 = P_2^{--}(A,x)$ or $P_2 = P_2^{++}(A,x)$.
If $P_2 = P_2^{--}(A,x)$, then for every $y \in A^-$ and $v \in P_2$, we have $v \to y \to N^+(y)$. Thus $N^+(y) \subseteq N^+(v)$, since $N^+(y) \subseteq P_1$ and $y \in F_3(T)$, which implies the contradiction $i_g(T) \geq d^+(v) - d^+(y) \geq |A| > \frac{3}{2}i_g(T)$.
Let $P_2 = P_2^{++}(A,x)$. If $y \in A^-$ and $u \in P_2$, we can conclude that $i_g(T) \geq d^-(u) - d^-(y) \geq |A| > \frac{3}{2}i_g(T)$, another contradiction.

Subcase 2.2 Suppose that $A^+ = \emptyset$ or $A^- = \emptyset$.
Without loss of generality, we can assume that $A = A^-$.
If the partite set $P_2 = P_2^{--}(A,x)$, then $i_g(T) \geq d^-(x) - d^+(x) \geq |P_2| - |P_0| + 2|A| > i_g(T)$, which is a contradiction.
Thus, we may assume that $P_2 = P_2^{++}(A,x)$. For every $y \in A$ and $u \in P_2$, $N^-(y) \subseteq N^-(u)$, which implies the contradiction $i_g(T) \geq d^-(u) - d^-(y) > \frac{3}{2}i_g(T)$. □

In the proof of the next theorem, we use the structure of 3-partite tournaments having an $F_3(T)$-partition.

Remark 2. Let T be a 3-partite tournament with partite sets P_0, P_1 and P_2. If $F_3(T)$ is independent, and T has an $F_3(T)$-partition, then

(i) There exists a partite set P_0 such that $F_3(T) \subseteq P_0$, a partite set P_1 such that $P_1 = P_1^+(F_3(T)) \cup P_1^-(F_3(T))$, and a partite set P_2 such that $P_2 = P_2^+(F_3(T)) \cup P_2^*(F_3(T)) \cup P_2^-(F_3(T))$.
(ii) $P_1^-(F_3(T)) \to P_2^+(F_3(T)) \cup P_2^*(F_3(T))$ and $(P_2^*(F_3(T)) \cup P_2^-(F_3(T))) \to P_1^+(F_3(T))$.

Theorem 2. *Let T be a 3-partite tournament, and $F_3(T)$ be an independent subset of T with $|F_3(T)| > \frac{3}{2}i_g(T)$. If T has an $F_3(T)$-partition, then $f_3(T) \leq \lfloor \frac{s}{4} + \frac{9i_g(T)^2}{2s} + \frac{29i_g(T)}{8} \rfloor$, where s is the size of the smallest partite set of T.*

Proof. Let T be a 3-partite tournament with partite sets P_0, P_1 and P_2. Since $F_3(T)$ is independent, we can assume $F_3(T) \subseteq P_0$. Since T has an $F_3(T)$ partition, without loss of generality, we may assume that $P_1 = P_1^+(F_3(T)) \cup P_1^-(F_3(T))$.

Claim 2. $P_1^+(F_3(T)) \neq \emptyset$.

If $P_1^+(F_3(T)) = \emptyset$, then consider a vertex $x \in F_3(T)$ and $y \in P_1^-(F_3(T))$. Notice that $N^+(x) \subset N^+(y)$ and $d^+(y) \geq d^+(x) + f_3(T)$. Hence, $i_g(T) \geq d^+(y) - d^+(x) \geq f_3(T) \geq \frac{3}{2}i_g(T)$, a contradiction.

Claim 3. $P_2^+(F_3(T)) \neq \emptyset$.

Suppose to the contrary that $P_2^+(F_3(T)) = \emptyset$. Let $w \in P_1^+(F_3(T))$ and $v \in P_2^*(F_3(T)) \cup P_2^-(F_3(T))$. By Remark 2, we then have $d^-(w) \geq |P_2| + f_3(T)$. Lemma 2 now implies $\frac{|P_0|+|P_2|+i_g(T)}{2} \geq d^-(w)$. Thus, $|P_0| - |P_2| \geq 2f_3(T) - i_g(T) > 2i_g(T)$, which contradicts Lemma 1.

Define T^* as $T[P_1^+(F_3(T)) \cup P_2^+(F_3(T))]$. The proof is based on counting the arcs of T^*.

Notice that

$$|A(T^*)| = |P_1^+(F_3(T))||P_2^+(F_3(T))| \\ = d(P_1^+(F_3(T), P_2^+(F_3(T)))) + d(P_2^+(F_3(T), P_1^+(F_3(T)))). \quad (1)$$

We can bound the number of arcs from $P_1^+(F_3(T))$ to $P_2^+(F_3(T))$ as follows

$$d(P_1^+(F_3(T)), P_2^+(F_3(T))) \le |P_2^+(F_3(T))| \max_{w \in P_2^+(F_3(T))} d^-_{T^*}(w).$$

Analogously, the number of arcs from $P_2^+(F_3(T))$ to $P_1^+(F_3(T))$ is bounded by

$$d(P_2^+(F_3(T)), P_1^+(F_3(T))) \le |P_1^+(F_3(T))| \max_{v \in P_1^+(F_3(T))} d^-_{T^*}(v).$$

By Remark 2, $N^-_{T^*}(w) \cup F_3(T) \cup P_1^-(F_3(T)) \subseteq N^-_T(w)$ for every $w \in P_2^+(F_3(T))$. Therefore, for every $w \in P_2^+(F_3(T))$,

$$d^-_{T^*}(w) \le d^-_T(w) - |F_3(T)| - |P_1^-(F_3(T))| \\ = d^-_T(w) - |F_3(T)| - |P_1| + |P_1^+(F_3(T))|.$$

By Remark 2, $N^-_{T^*}(v) \cup F_3(T) \cup P_2^*(F_3(T)) \cup P_2^-(F_3(T)) \subseteq N^-_T(v)$ for every $v \in P_1^+(F_3(T))$. Thus, for every $v \in P_1^+(F_3(T))$,

$$d^-_{T^*}(v) \le d^-_T(v) - |F_3(T)| - |P_2^-(F_3(T))| - |P_2^*(F_3(T))| \\ = d^-_T(v) - |F_3(T)| - |P_2| + |P_2^+(F_3(T))|.$$

By Eq. (1),

$$|P_1^+(F_3(T))||P_2^+(F_3(T))| \le |P_2^+(F_3(T))|(d^-_T(w) - f_3(T) - |P_1| + |P_1^+(F_3(T))|) \\ + |P_1^+(F_3(T))|(d^-_T(v) - f_3(T) - |P_2| + |P_2^+(F_3(T))|).$$

Let $m = |P_1^+(F_3(T))| + |P_2^+(F_3(T))|$ and $p = |P_1^+(F_3(T))|$. From the above inequality we obtain that

$$0 \le -p^2 + p(m + (|P_1| - |P_2|) + (d^-_T(v) - d^-_T(w))) + m(d^-_T(w) - |P_1| - f_3(T)). \quad (2)$$

Notice that, by Lemmas 1 and 2, $d^-_T(w) - |P_1| \le \frac{|P_0|+|P_1|+i_g(T)}{2} - |P_1| \le \frac{3i_g(T)}{2}$. Then,

$$0 \le -p^2 + p(m + 3i_g(T)) + m\left(\frac{3i_g(T)}{2} - f_3(T)\right).$$

As a consequence, the discriminant $D = (m + 3i_g(T))^2 + 4m(\frac{3i_g(T)}{2} - f_3(T))$ must be nonnegative. It follows that

$$f_3(T) \le \frac{m}{4} + \frac{9i_g(T)^2}{4m} + 3i_g(T).$$

By symmetry, we reach the same results for $P_1^-(F_3(T))$ and $P_2^-(F_3(T))$. Thus, without loss of generality, we may assume that $m \ge |P_1|/2 \ge s/2$, where s is the size of the smallest partite set of T. Since $m \le d^+(y)$ for every $y \in F_3(T)$, by Lemma 2, we obtain $m \le \frac{|P_1|+|P_2|+i_g(T)}{2} \le k + \frac{i_g(T)}{2}$, where k is the size of the largest partite set of T. Since $k \le s + 2i_g(T)$, we have proved that $f_3(T) \le \lfloor \frac{s}{4} + \frac{9i_g(T)^2}{2s} + \frac{29i_g(T)}{8} \rfloor$. □

As a corollary of Theorems 1 and 2 we have the following.

Corollary 1. *Let T be a c-partite tournament. If there is an independent set $A \subseteq F_3(T)$ with more than $\frac{3}{2}i_g(T)$ vertices and T has an A-partition, then $F_3(T)$ is contained in one partite set and $f_3(T) \le \lfloor \frac{s}{4} + \frac{9i_g(T)^2}{2s} + \frac{29i_g(T)}{8} \rfloor$, where s is the size of the smallest partite set of T.*

4. Almost regular 3-partite tournaments

In this section we prove that the sufficient condition of having an $F_3(T)$-partition always holds for almost regular 3-partite tournaments and we prove the upper bound of Theorem 2 for this class of 3-partite tournaments.

Lemma 3. *If T is an almost regular 3-partite tournament and $u, v \in F_3(T)$ two non-adjacent vertices, then T has a $\{u, v\}$-partition.*

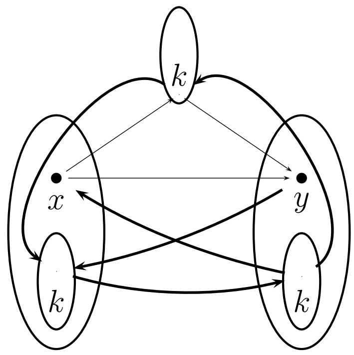

Fig. 1. 3-partite almost regular tournament with $\vec{C}_3$-vertices in two partite sets.

Proof. Let $u, v \in F_3(T) \cap P_0$. Without loss of generality, we may assume that $P_2^{-+}(u, v) \neq \emptyset$.

If $P_1^{+-}(u, v) \neq \emptyset$, then $P_2^{-+}(u, v) \to u \to P_1^{+-}(u, v) \to v \to P_2^{-+}(u, v)$ is a 4-cycle. Then u or v are in a triangle no matter the direction of the arcs between $P_2^{-+}(u, v)$ and $P_1^{+-}(u, v)$. Thus, $P_1^{+-}(u, v) = \emptyset$.

If $P_1^{-+}(u, v) \neq \emptyset$, we can prove analogously that $P_2^{+-}(u, v) = \emptyset$. In this case, since both $P_1^{-+}(u, v)$ and $P_2^{-+}(u, v)$ are empty sets, $d^-(u) + d^+(v) = |V(T)| - |V(P_0)| + \sum_{j=0}^{2} |P_j^{-+}(u, v)| \geq |V(T)| - |P_0| + 2$, which contradicts Lemma 2. So, $P_1^{-+}(u, v) = \emptyset$ and therefore, $P_1 = P_1^{+}(u, v) \cup P_1^{-}(u, v)$. □

Corollary 2. *Let T be a 3-partite almost regular tournament with at least two independent $\overrightarrow{C_3}$-free vertices. Then, $F_3(T)$ is independent and, there exists at least one partite set P such that $P = P^+(F_3(T)) \cup P^-(F_3(T))$.*

Proof. Let A be a maximal independent subset of $F_3(T)$. We assume without loss of generality that $A = F_3(T) \cap P_0$.

Claim 1. T has an A-partition.

Suppose to the contrary that $P_1 \neq P_1^+(A) \cup P_1^-(A)$ and $P_2 \neq P_2^+(A) \cup P_2^-(A)$. Then, there exist $u, v \in A$ such that $P_1^{+-}(u, v) \neq \emptyset$. By Lemma 3, T has a $\{u, v\}$-partition, therefore $P_2 = P_2^+(u, v) \cup P_2^-(u, v)$. Since $P_2 \neq P_2^+(A) \cup P_2^-(A)$, there exists $w \in A$ such that $P_2 \neq P_2^+(u, w) \cup P_2^-(u, w)$ and $P_2 \neq P_2^+(w, v) \cup P_2^-(w, v)$. Again by Lemma 3, T has a $\{u, w\}$-partition and a $\{w, v\}$-partition. That is, $P_1 = P_1^+(u, w) \cup P_1^-(u, w) = P_1^+(w, v) \cup P_1^-(w, v)$. This implies that $P_1^{+-}(u, v) \subseteq P_1^+(u, w) \cap P_1^-(v, w) \subseteq N^+(w) \cap N^-(w) = \emptyset$, which contradicts that $P_1^{+-}(u, v) \neq \emptyset$. Thus, Claim 1 is proved.

Since $|A| \geq 2 > \frac{3}{2} i_g(T)$ and T has an A-partition, by Theorem 1, $A = F_3(T)$ and therefore independent, and there exists at least one partite set P such that $P = P^+(F_3(T)) \cup P^-(F_3(T))$. □

The proof of Claim 1 of Corollary 2 is similar to the proof of Corollary 1 in [2].

As a corollary of Remark 2 and Corollary 2 we have the following theorem.

Theorem 3. *An almost regular 3-partite tournament T, with $f_3(T) > 3$ and partite sets P_0, P_1 and P_2 has the following structure:*

(i) *$F_3(T)$ is entirely contained in one partite set (say P_0).*
(ii) *There exists one partite set (say P_1) such that $F_3(T) \to P_1^+$, $P_1^- \to F_3(T)$ and $P_1 = P_1^+ \cup P_1^-$, where $P^+ := P_1^+(F_3(T))$ and $P^- := P_1^-(F_3(T))$.*
(iii) *If $P_2^+ = P_2^+(F_3(T))$, $P_2^- = P_2^-(F_3(T))$ and $P_2^* = P_2 \setminus (P_2^+ \cup P_2^-)$, then $(P_2^* \cup P_2^-) \to P_1^+$ and $P_1^- \to (P_2^+ \cup P_2^*)$.*

The digraph in Fig. 1 is a 3-partite tournament T, with $f_3(T) = 2$ and $F_3(T)$ has vertices in two partite sets.

Theorem 4. *If T is an almost regular 3-partite tournament with $f_3(T) > 3$, and k is the cardinality of the largest partite set of T, then $f_3(T) \leq \lfloor \frac{k+1}{4} \rfloor + 1 \leq \lfloor \frac{n+5}{12} \rfloor + 1$.*

Proof. Let T be an almost regular 3-partite tournament such that $f_3(T) > 3$. By Corollary 2, T has an $F_3(T)$-partition. Let $v \in P_1^+(F_3(T))$ and $w \in P_2^+(F_3(T))$. Following the proof of Theorem 2 and due to inequality (2), we have that

$$0 \geq p^2 - p(m + (|P_1| - |P_2|) + (d^-(v) - d^-(w))) + m(f_3(T) + |P_1| - d^-(w)),$$

where $m = |P_1^+(F_3(T))| + |P_2^+(F_3(T))|$ and $p = |P_1^+(F_3(T))|$.

Let k be the size of the largest partite set. It is not difficult to see that, if T is an almost regular 3-partite tournament, there are at least two partite sets with the same cardinality. Therefore, we have 12 cases depending on the cardinality of the partite sets P_0, P_1 and P_2 of T (see Table 1).

In every case, we find bounds x_1, x_2, x_3 such that $d^-(v) \leq x_1$ and $x_2 \leq d^-(w) \leq x_3$. Let $b = |P_1| - |P_2| + x_1 - x_2$, $c = |P_1| - x_3$ and $g(p) = p^2 - p(m + b) + m(f_3(T) + c)$. Since $b \geq |P_1| - |P_2| + d^-(v) - d^-(w)$ and $c \leq |P_1| - d^-(w)$,

$$0 \geq p^2 - p(m + (|P_1| - |P_2|) + (d^-(v) - d^-(w))) + m(f_3(T) + |P_1| - d^-(w)) \geq g(p).$$

Table 1
$f_3(T)$ in an almost regular tripartite tournament.

Case	$\|P_0\|$	$\|P_1\|$	$\|P_2\|$	b	c	$g(p) = p^2 - p(m+b) + m(f_3+c)$	Δ_P	$f_3(T) \leq \left\lfloor \frac{\Delta_P+2b}{4} \right\rfloor - c$
1	$k-2$	$k-2$	k	-1	0	$p^2 - p(m-1) + mf_3$	$k-1$	$\lfloor \frac{k-3}{4} \rfloor \leq \lfloor \frac{n-5}{12} \rfloor$
2	$k-2$	k	$k-2$	1	1	$p^2 - p(m+1) + m(f_3+1)$	$k-1$	$\lfloor \frac{k+1}{4} \rfloor - 1 \leq \lfloor \frac{n+7}{12} \rfloor - 1$
3	$k-2$	k	k	0	1	$p^2 - pm + m(f_3+1)$	k	$\lfloor \frac{k}{4} \rfloor - 1 \leq \lfloor \frac{n+2}{12} \rfloor - 1$
4	$k-1$	$k-1$	k	0	0	$p^2 - pm + mf_3$	k	$\lfloor \frac{k}{4} \rfloor \leq \lfloor \frac{n+2}{12} \rfloor$
5	$k-1$	k	$k-1$	1	0	$p^2 - p(m+1) + mf_3$	k	$\lfloor \frac{k+2}{4} \rfloor \leq \lfloor \frac{n+8}{12} \rfloor$
6	$k-1$	k	k	1	0	$p^2 - p(m+1) + mf_3$	k	$\lfloor \frac{k+2}{4} \rfloor \leq \lfloor \frac{n+7}{12} \rfloor$
7	k	$k-2$	$k-2$	0	-1	$p^2 - pm + m(f_3-1)$	$k-2$	$\lfloor \frac{k-2}{4} \rfloor + 1 \leq \lfloor \frac{n-4}{12} \rfloor + 1$
8	k	$k-2$	k	-1	-1	$p^2 - p(m-1) + m(f_3-1)$	$k-1$	$\lfloor \frac{k-3}{4} \rfloor + 1 \leq \lfloor \frac{n-7}{12} \rfloor + 1$
9	k	$k-1$	$k-1$	1	-1	$p^2 - p(m+1) + m(f_3-1)$	$k-1$	$\lfloor \frac{k+1}{4} \rfloor + 1 \leq \lfloor \frac{n+5}{12} \rfloor + 1$
10	k	$k-1$	k	0	-1	$p^2 - pm + m(f_3-1)$	k	$\lfloor \frac{k}{4} \rfloor + 1 \leq \lfloor \frac{n+1}{12} \rfloor + 1$
11	k	k	$k-2$	1	0	$p^2 - p(m+1) + mf_3$	$k-1$	$\lfloor \frac{k+1}{4} \rfloor \leq \lfloor \frac{n+5}{12} \rfloor$
12	k	k	$k-1$	1	0	$p^2 - p(m+1) + mf_3$	k	$\lfloor \frac{k+2}{4} \rfloor \leq \lfloor \frac{n+7}{12} \rfloor$

Thus, the discriminant of $g(p)$ is nonnegative, that is $(m+b)^2 - 4m(f_3(T)+c) \geq 0$. Therefore, $f_3(T) \leq \frac{(m+b)^2}{4m} - c$. Since $f_3(T)$ is an integer, it follows that

$$f_3(T) \leq \left\lfloor \left\lfloor \frac{m+2b}{4} \right\rfloor + \frac{3}{4} + \frac{b^2}{4m} - c \right\rfloor = \left\lfloor \frac{m+2b}{4} \right\rfloor - c,$$

because $m > 1$ and $|b| \leq 1$ (see Table 1). Let $\Delta_P = \left\lceil \frac{|P_1|+|P_2|}{2} \right\rceil$. By the definition of m, $m \leq \Delta_P$ and therefore,

$$f_3(T) \leq \left\lfloor \frac{\Delta_P + 2b}{4} \right\rfloor - c.$$

We calculate b and c only for two cases, but the calculus of the rest of the cases is similar.

Case 2. $|P_0| = k-2$, $|P_1| = k$ and $|P_2| = k-2$.

Since T is almost regular, for every $v \in P_1$ and $w \in P_2$, $x_1 = d^-(v) = d^+(v) = k-2$ and $x_2 = x_3 = d^-(w) = d^+(w) = k-1$. Hence, $b = 1$, $c = 1$, $g(p) = p^2 - p(m+1) + m(f_3+1)$, and $\Delta_P = k-1$. Therefore,

$$f_3(T) \leq \left\lfloor \frac{\Delta_P + 2b}{4} \right\rfloor - c = \left\lfloor \frac{k+1}{4} \right\rfloor - 1 = \left\lfloor \frac{n+7}{12} \right\rfloor - 1,$$

because, in this case, $n = 3k-4$,

Case 9. $|P_0| = k$ and $|P_1| = |P_2| = k-1$.

Since T is almost regular, for every $v \in P_1$ and $w \in P_2$, $x_1 = k \geq d^-(v)$, $x_2 = k-1$ and $x_3 = k$. Hence, $b = 1$, $c = -1$, $g(p) = p^2 - p(m+1) + m(f_3-1)$, and $\Delta_P = k-1$. In this case, $n = 3k-4$ and therefore,

$$f_3(T) \leq \left\lfloor \frac{\Delta_P + 2b}{4} \right\rfloor - c = \left\lfloor \frac{k+1}{4} \right\rfloor + 1 = \left\lfloor \frac{n+5}{12} \right\rfloor + 1.$$

In Table 1, we depict the corresponding value of $b = \Delta_P + x_1 - x_2$, $c = |P_1| - x_3$, the polynomial $g(p)$ and the bound of $f_3(T)$ for each case.

Hence, we obtain that $f_3(T) \leq \lfloor \frac{k+1}{4} \rfloor + 1 \leq \lfloor \frac{n+5}{12} \rfloor + 1$ in every case. □

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