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Orthogonal matchings revisited

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ABSTRACT

Let G be a graph on n vertices, which is an edge-disjoint union of ms-factors, that is, s regular spanning subgraphs. Alspach first posed the problem that if there exists a matching M of m edges with exactly one edge from each 2-factor. Such a matching is called orthogonal because of applications in design theory. For $s = 2$, so far the best known result is due to Stong in 2002, which states that if $n \geq 3m-2$, then there is an orthogonal matching. Anstee and Caccetta also asked if there is a matching M of m edges with exactly one edge from each s-factor? They answered yes for $s \geq 3$. In this paper, we get a better bound and prove that if $s = 2$ and $n \geq 2\sqrt{2}m+4.5$ (note that $2\sqrt{2} \leq 2.825$), then there is an orthogonal matching. We also prove that if $s = 1$ and $n \geq 3.2m - 1$, then there is an orthogonal matching, which improves the previous bound ($3.79m$).

1. Introduction and notation

We use [4] for terminology and notations not defined here and consider simple undirected graphs only. Let $G = (V, E)$ be a graph. For a subgraph H of G, let $|H|$ denote the order of H, i.e. the number of vertices of H and let $\|H\|$ denote the size of H, that is, the number of edges of H. If a vertex u is an end vertex of an edge e, we write $u \in e$.

Let G be a graph on n vertices, which is an edge-disjoint union of ms-factors, that is, s regular spanning subgraphs. In 1988, Alspach [1] first posed the problem that if there exists a matching M of m edges with exactly one edge from each 2-factor. Such a matching is called *orthogonal* because of applications in design theory. A matching M is *suborthogonal* if there is at most one edge from each s-factor. Alspach, Heinrich and Liu [2] proved that the answer is affirmative if $n \geq 4m - 5$. Kouider and Sotteau improved this bound to $3.23m$. In 2002, Stong [17] further improved this bound and proved the following result.

Theorem 1.1 ([17]). *Let G be a $2m$-regular graph with $n \geq 3m - 2$. Then for any decomposition of $E(G)$ into m 2-factors $F_1, F_2, \ldots, F_m$, there is an orthogonal matching.*

The problem with $s = 2$ and all the 2-factors being hamiltonian cycles was raised by Caccetta and Mardiyono [5] and Chung (referred to in [12]) but apparently the extra condition is no help.

In 1998, Anstee and Caccetta [3] asked if there is a matching M of m edges with exactly one edge from each s-factor in the cases of $s = 1$ and $s \geq 3$? For $s \geq 3$, the answer is yes (see [3]).

For $s = 1$, the answer is negative: let G be a complete graph K_{m+1} (m is even) which is an edge disjoint union of m 1-factors, however, the size of maximum matching is at most $\frac{m}{2}$. Indeed, it is best possible, see [11]. But how about when we restrict ourselves to large graph? Wang, Liu and Liu [20] proved the following result.

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Theorem 1.2 ([20]). *Let G be an m-regular graph with $n \geq 3.79m$. Then for any decomposition of $E(G)$ into m 1-factors $F_1, F_2, \dots, F_m$, there is an orthogonal matching.*

In particular, if G is $K_{m,m}$ and is a union of m 1-factors $F_1, F_2, \dots, F_m$, then G corresponds to a Latin square, where entry a_{ij} is l if edge $(u_i, v_j) \in F_l$. Now our desired matching corresponds to a transversal. Hatami and Shor [9] proved that if $K_{m,m}$ is a union of m 1-factors $F_1, F_2, \dots, F_m$, then there is a matching M of p edges with at most one edge from any 1-factor with $p = m - O(\log m)^2$.

If G is assigned an arbitrary edge-coloring (not necessarily proper), then we say that G is an *edge-colored graph*. A subgraph H of an edge-colored graph G is called *rainbow* (also *heterochromatic*, *multicolored*, *polychromatic*) if its edges have distinct colors. The *minimum color degree* of G is the smallest number of distinct colors on the edges incident with a vertex over all vertices. Recently, the study of rainbow paths and cycles under minimum color degree condition has received much attention, see [6,15]. For rainbow matchings under minimum color degree condition, see [11,10,16,13,14,19].

In any decomposition of $E(G)$ into ms-factors, we can construct an edge-colored graph by giving each s-factor a color. Then a rainbow matching of G corresponds to a suborthogonal matching of G. In particular, when $s = 1$, the edge-colored graph obtained above is properly edge-colored. For rainbow matchings in properly edge-colored graphs, see [7,8,18,21].

In this paper, we improve the bounds in Theorems 1.1 and 1.2 and get the following results.

Theorem 1.3. *Let G be an m-regular graph with $n \geq 3.2m - 1$. Then for any decomposition of $E(G)$ into m 1-factors $F_1, F_2, \dots, F_m$, there is an orthogonal matching.*

Theorem 1.4. *Let G be a $2m$-regular graph with $n \geq 2\sqrt{2}m + 4.5$. Then for any decomposition of $E(G)$ into m 2-factors $F_1, F_2, \dots, F_m$, there is an orthogonal matching.*

2. Proof of main results

We prove our conclusions by contradiction. Firstly, when $m = 1$ and $m = 2$, the proof is trivial. If Theorems 1.3 and 1.4 are false, then there exists a minimal m, such that there is no a rainbow matching of size m for G. We construct an edge-colored graph by giving each 1-factor (in Theorem 1.3), 2-factor (in Theorem 1.4) a color from $\{1, 2, \dots, m\}$. For an edge $e \in E(G)$, let $c(e)$ denote the color of e. For a subgraph H of G, let $C(H) = \{c(e) \mid e \in E(H)\}$. By the minimality of m, G has a rainbow matching of size $m - 1$. For simplicity, let $p = m - 1$ and $n = |G|$. We define a *good configuration* $H_p = M_1 \cup M_2 \cup M_3 \cup F$ as follows (see Fig. 1). Note that the blue edges in the figure are colored m.

(a) For some integer $k \geq 0$, $M_1 = \{e_i\ (e_i = u_iv_i) : i = 1, 2, \dots, k\}$ and $M_2 = \{f_i : i = 1, 2, \dots, k\}$ are two vertex-disjoint rainbow matchings of G with $c(e_i) = c(f_i)$.

(b) $M_3 = \{g_i\ (g_i = u_iv_i) : i = k + 1, \dots, p\}$ is a rainbow matching, which is vertex-disjoint from $M_1 \cup M_2$ and $c(g_i) \neq c(e_j)$ for $1 \leq j \leq k < i \leq p$.

For abbreviation, let G_1 denote the subgraph induced by $V(G)\backslash V(M_1 \cup M_2 \cup M_3)$. Without loss of generality, we assume that $C(M_1 \cup M_3) = \{1, 2, \dots, m - 1\}$.

(c) $F = \{h_i\ (h_i = v_iz_i) : i = k + 1, \dots, k + t\}$ is a matching, vertex-disjoint from $M_1 \cup M_2$, $h_i \cap M_3 = \{v_i\} \in g_i$, and $c(h_i) = m$.

We choose a good configuration $H_p = M_1 \cup M_2 \cup M_3 \cup F$ satisfying the following conditions:

(1) $k = \|M_1\|$ is maximum;
(2) subject to (1), F is maximal, that is, F covers the maximum number of vertices of M_3.

Claim 2.1. *If $u \in V(G_1)$ and $c(uv) = m$, then $v \in V(M_3)$.*

Proof. By symmetry, we may assume that $v \notin V(M_2)$. If $v \notin V(M_3)$, then $M_2 \cup M_3 \cup uv$ is an orthogonal matching of G, which is a contradiction. □

Claim 2.2. *If $u \in V(e_i \cup f_i)$ and $c(uv) = m$, where $v \notin V(M_3)$, then $v \in V(e_i \cup f_i)$.*

Proof. Suppose to the contrary that $v \notin V(e_i \cup f_i)$. By symmetry and without loss of generality, we may assume that $u, v \notin V(M_2)$. Since $c(uv) = m$, $M_2 \cup M_3 \cup uv$ is an orthogonal matching, which is a contradiction. □

If there is an edge uv such that $u, v \in V(e_i \cup f_i)$ and $c(uv) = m$, then we call $e_i \cup f_i$ a *nice pair*. Let q denote the number of nice pairs in $M_1 \cup M_2$. Without loss of generality, we assume that the nice pairs are $\{e_1 \cup f_1, \dots, e_q \cup f_q\}$ and we call $c(e_i)$ a *nice color*, for $i = 1, 2, \dots, q$. Let n_1 be the number of edges uv such that $u \in V(M_3)$, $v \in V(G)\backslash V(M_3)$ and $c(uv) = m$. Note that each vertex is incident with at least one edge with color m since each color induces a 1-factor (in Theorem 1.3) or 2-factor (in Theorem 1.4).

Claim 2.3. *We have that $V(H_p) = V(G)$.*

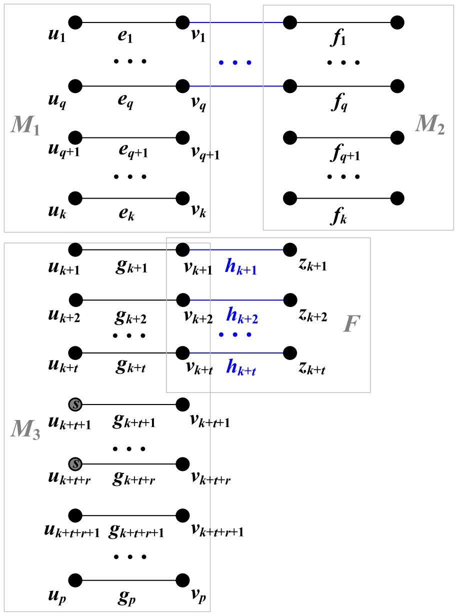

Fig. 1. A good configuration. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Proof. First we will prove this claim in Theorem 1.3. Since color m induces a 1-factor, we have a monochromatic matching $M' = \{z_iw_i \mid c(z_iw_i) = m, z_i \in V(G_1)\}$, which saturates each vertex of G_1. Note that there exists no edge in M_3 such that both of its ends are in M', otherwise, without loss of generality, we assume that $w_1 = v_{k+1}$ and $w_2 = u_{k+1}$. Then we add z_1w_1 to M_1, add z_2w_2 to M_2 and delete $u_{k+1}v_{k+1}$ from M_3. Thus we get a new good configuration with bigger k, which is a contradiction. By Claims 2.1 and 2.2, $w_i \notin V(M_1 \cup M_2)$. Hence F saturates each vertex in G_1. So $V(H_p) = V(G)$.

Next, we will prove this claim in Theorem 1.4. Define M' as before.

For each edge $g_i = u_iv_i$ in M_3, let $d_m(u_i)$ denote the number of edges u_iv such that $c(u_iv) = m$, $v \in V(G_1)$ and let $d_m(v_i)$ denote the number of edges v_iv such that $c(v_iv) = m$, $v \in V(G_1)$. Then for each edge $g_i = u_iv_i \in E(M_3)$, $d_m(u_i) + d_m(v_i) \le 2$. Otherwise we can choose a matching of size two, say $\{u_iz, v_iz'\}$, which saturates u_i and v_i. Then after adding u_iz to M_1, adding v_iz' to M_2 and deleting u_iv_i in M_3, we get a new good configuration with bigger k, which is a contradiction. So $d_m(u_i) + d_m(v_i) \le 2$, for each edge $g_i = u_iv_i \in E(M_3)$. Let $d_m(z_i)$ denote the number of edges uz_i such that $c(z_iu) = m, z_i \in V(G_1)$. We have $d_m(z_i) = 2$, for $z_i \in V(G_1)$. Recall that $u \in V(M_3)$ by Claim 2.1. Then we can get a monochromatic matching $M'' = \{z_iw_i \mid z_i \in V(G_1), w_i \in V(M_3)\}$ with color m, which saturates each vertex of G_1. So we have that $V(H_p) = V(G)$. □

Recall that $t = \|F\|$. Then

$$n = 4k + 3t + 2(p - k - t) = 2k + t + 2p. \tag{2.1}$$

Thus

$$k = \frac{n - 2p - t}{2}. \tag{2.2}$$

Without loss of generality, we assume that $M_3 \cap F = \{v_{k+1}, v_{k+2}, \ldots, v_{k+t}\}$ and $F = \bigcup_{i=k+1}^{k+t} h_i = \bigcup_{i=k+1}^{k+t} v_iz_i$. For abbreviation, let T denote the subgraph induced by $V(\bigcup_{i=k+1}^{k+t} z_i \cup M_1 \cup M_2)$. Each color in $\{c(u_iv_i) \mid i = k+1, \ldots, k+t\}$ is called a *kind* color. We have the following claim.

Claim 2.4. *Let $e = uv$ be an edge with a kind color, where $u, v \in V(T)$. Recall that $c(uv) = c(g_{k+i_0})$ for some $1 \le i_0 \le t$. Then one of the following is true:*

(a) $u, v \in V(e_i \cup f_i)$ *for* $i = 1, 2, \ldots, k$;
(b) $u = z_{k+i_0}$ *and* $v \in V(M_1 \cup M_2)$;
(c) $v = z_{k+i_0}$ *and* $u \in V(M_1 \cup M_2)$.

Proof. If the claim would not hold, by symmetry and without loss of generality, then we may assume that $u, v \notin V(M_2)$. If $uv \cap z_{k+i_0} = \emptyset$, then $M_2 \cup M_3 \cup \{uv, h_{k+i_0}\} - g_{k+i_0}$ is an orthogonal matching for G, which is a contradiction. If $u, v \in V(G_1)$ and $uv \cap z_{k+i_0} \ne \emptyset$, then after adding g_{k+i_0} to M_1, adding uv to M_2 and deleting g_{k+i_0} in M_3, we get a good configuration with bigger k, which is a contradiction. □

2.1. Proof of Theorem 1.3

Claim 2.5. *If $u \in V(G_1)$ and $c(uv)$ is a nice color, then $v \in V(M_3)$.*

Proof. By symmetry, we may assume that $v \notin V(M_2)$. If $v \notin V(M_3)$, then we may assume that $c(uv) = c(e_1)$. Let e denote the edge in $e_1 \cup f_1$ and $c(e) = m$. Then $M_2 \cup M_3 \cup \{uv, e\} - f_1$ is an orthogonal matching of G, which is a contradiction. □

Claim 2.6. *If $u \in V(e_i \cup f_i)$ and $c(uv)$ is a nice color, where $v \notin V(M_3)$, then $v \in V(e_i \cup f_i)$.*

Proof. If $v \notin V(e_i \cup f_i)$, by symmetry and without loss of generality, we may assume that $u, v \notin V(M_2)$. Since $c(uv)$ is nice, we may assume that $c(uv) = c(e_1)$. Let e denote an edge with vertices in $V(e_1 \cup f_1)$ and $c(e) = m$, then $M_2 \cup M_3 \cup \{uv, e\} - f_1$ is an orthogonal matching in G, which is a contradiction. □

By Claim 2.3, we conclude that $n \le 4k + 3(p - k) = 3p + k$. Hence

$$k \ge n - 3p. \tag{2.3}$$

By (2.2) and (2.3), we get the following claim.

Claim 2.7. *We have $n \le 4p - t$.*

Recall that q denotes the number of nice pairs in $M_1 \cup M_2$ and n_1 is the number of edges uv such that $u \in V(M_3)$, $v \in V(G)\backslash V(M_3)$ and $c(uv) = m$. We will prove that $q \ge 3$. Because $n_1 \ge |T| - 4q = 4k + t - 4q$ and $n_1 \le |M_3| = 2(p - k)$, it follows that $6k + 5t - 4t - 2p \le 4q$. Note that $k + t \le p$, which implies that $10k + 5t - 6p \le 4q$. By (2.2), it follows that $5n - 16p \le 4q$. By assumption $n \ge 3.2m - 1 = 3.2p + 2.2$, so we finally arrive at $\frac{11}{4} \le q$. Since q is an integer, we have $q \ge 3$.

Let n_2 be the number of edges uv such that $u \in V(M_3)$, $v \in V(G)\backslash V(M_3)$ and $c(uv)$ is nice or $c(uv) = m$, that is, $c(uv) \in \{c(e_1), \dots, c(e_q), m\}$. By Claims 2.2 and 2.6, each vertex $v' \in V(e_i \cup f_i)$ for some $i \in \{1, 2, \dots, q\}$, is incident with at most 3 edges $u'v'$ such $c(u'v')$ is nice or $c(u'v') = m$ and $u' \notin V(M_3)$. Similarly, each $v' \in V(e_i \cup f_i)$, where $i \in \{q+1, q+2, \dots, k\}$, is incident with at most 2 edges $u'v'$ such that $c(u'v')$ is nice and $u' \notin V(M_3)$. So we have $n_2 \ge (1+q)|T| - 12q - 8(k-q) = (1+q)(4k+t) - 8k - 4q$.

We also have $n_2 \le (1+q)|M_3| = 2(1+q)(p-k)$. Hence

$$(1+q)(4k+t) - 8k - 4q \le 2(1+q)(p-k).$$

Thus

$$k \le \frac{(1+q)(2p-t)}{6q-2} + \frac{4q}{6q-2}. \tag{2.4}$$

By (2.2), we have

$$\frac{n-2p-t}{2} \le \frac{(1+q)(2p-t)}{6q-2} + \frac{4q}{6q-2}.$$

It follows that

$$\begin{aligned} t &\ge \frac{(3q-1)n - 8qp - 4q}{2q-2} \\ &\ge \frac{(3q-1)(3.2p+2.2) - 8qp - 4q}{2q-2} \\ &= 0.8p - \frac{0.8p}{q-1} + \frac{1.3q-1.1}{q-1} \\ &\ge 0.4p + \frac{1.3q-1.1}{q-1} \end{aligned}$$

as $q \ge 3$. Hence, inequality (2.4) becomes

$$k \le \frac{0.8(1+q)p}{3q-1} + \frac{2.7q^2 - 4.2q + 1.1}{(6q-2)(q-1)}.$$

We have that $\frac{k}{t} \le \frac{2(1+q)}{3q-1} + \frac{2.7q^2-4.2q+1.1}{7.8q^2-9.2q+2.2} \le 1 + \frac{9}{26} = \frac{35}{26}$.

Now we choose a kind color, say $c(g_{k+1})$, such that the edges with this kind color in T is minimum among all kind colors. By Claim 2.4, we know that the number of edges with color $c(g_{k+1})$ is at most $\frac{4k}{t} + 1$ in T. Let n_3 be the number of edges uv

such $u \in v(M_3)$, $v \in v(T)$ and $c(uv) = c(g_{k+1})$. By Claim 2.4, $n_3 \geq |T| - 2(\frac{4k}{t} + 1) = (4k + t) - 2(\frac{4k}{t} + 1)$. We also have $n_3 \leq |M_3| - 2 = 2(p - k) - 2$. Hence

$$(4k + t) - 2\left(\frac{4k}{t} + 1\right) \leq 2(p - k) - 2.$$

Recall that $\frac{k}{t} \leq \frac{35}{26}$ as $q \geq 3$. Hence

$$4k + t - \frac{140}{13} \leq 2(p - k).$$

By (2.2), we have

$$3(n - 2p - t) \leq 2p - t + \frac{140}{13}.$$

It follows that

$$\begin{aligned} t &\geq 1.5n - 4p - \frac{70}{13} \\ &\geq 1.5 \times (3.2p + 2.2) - 4p - \frac{70}{13} \\ &\geq 0.8p - \frac{271}{130}. \end{aligned}$$

By Claim 2.7, we have that

$$3.2p + 2.2 = 3.2m - 1 \leq n \leq 4p - t < 3.2p + \frac{271}{130},$$

which is a contradiction. This completes the proof of Theorem 1.3.

2.2. Proof of Theorem 1.4

Claim 2.8. *Let $c(z_iu_j)$ be a kind color or $c(z_iu_j) = m$, where $k + 1 \leq i, j \leq k + t$. Then $i = j$.*

Proof. Otherwise, without loss of generality, we may assume that $i = k + 1$ and $j = k + 2$. If $c(z_{k+1}u_{k+2}) = m$, then we add $z_{k+1}u_{k+2}$ to M_1, add h_{k+2} to M_2 and delete g_{k+2} from M_3. Hence we get another good configuration with bigger k, which is a contradiction. If $c(z_{k+1}u_{k+2}) = c(g_{k+2})$, then we get an orthogonal matching $M_1 \cup M_3 \cup \{z_{k+1}u_{k+2}, h_{k+2}\} - g_{k+2}$, a contradiction. So we conclude that $c(z_{k+1}u_{k+2}) = c(g_{k+i})$, $i \neq 2$. Then we replace M_3 by $M_3 \cup h_{k+2} - \{g_{k+2}, g_{k+i}\}$, add g_{k+i} to M_1 and add $z_{k+1}u_{k+2}$ to M_2, and thus we get a new good configuration with bigger k, a contradiction. This completes the proof. □

Let $M_3^1 = M_3 \setminus \{g_{k+1}, g_{k+2}, \ldots, g_{k+t}\}$. For $i = \{k+1, \ldots, k+t\}$, edge vz_i is called a *special edge* if $v \in V(M_3^1)$, $c(vz_i) \neq c(g_i)$ and either $c(vz_i)$ is kind or $c(vz_i) = m$. For a vertex $v \in M_3^1$, let $d_s(v)$ denote the number of special edges vx where $x \in \bigcup_{i=k+1}^{k+t} z_i$.

Claim 2.9. *For an edge $e = uv \in E(M_3^1)$, if $d_s(u) + d_s(v) \geq 3$, then $d_s(u)d_s(v) = 0$.*

Proof. Otherwise there exist two independent special edges, say uz_{k+1}, vz_{k+2}. We divide our proof into the following cases.

Case 1: $c(uz_{k+1}) = c(vz_{k+2})$.

First suppose that $c(uz_{k+1}) = m$. Then we add uz_{k+1} to M_1, add vz_{k+2} to M_2 and delete uv from M_3, hence we get another good configuration with bigger k, which is a contradiction. Now we assume that $c(uz_{k+1})$ is kind. By the definition of special edges, without loss of generality, we may assume that $c(uz_{k+1}) = c(g_{k+3})$. Then we add uz_{k+1} to M_1, add vz_{k+2} to M_2 and replace M_3 by $M_3 \cup h_{k+3} - \{uv, g_{k+3}\}$. Thus we get a good configuration with bigger k, which is also a contradiction.

Case 2: $c(uz_{k+1}) \neq c(vz_{k+2})$.

First we suppose that $c(uz_{k+1}) = m$ or $c(vz_{k+2}) = m$. Without loss of generality, we may assume that $c(uz_{k+1}) = m$ and $c(vz_{k+2}) = c(g_{k+i})$, $i \neq 2$. Then we add g_{k+i} to M_1, add vz_{k+2} to M_2 and replace M_3 by $M_3 \cup uz_{k+1} - \{uv, g_{k+i}\}$. Hence we get a good configuration with bigger k, which is a contradiction. So now we assume that $c(uz_{k+1}) = c(g_{k+i})$ and $c(vz_{k+2}) = c(g_{k+j})$, where $i \neq j$. Recall that by the definition of the special edges, $i \neq 1$ and $j \neq 2$. Then we add g_{k+i} to M_1, add uz_{k+1} to M_2, and replace M_3 by $M_3 \cup h_{k+j} - \{uv, g_{k+j}\}$. Thus we obtain another good configuration with bigger k, which is a contradiction. □

Each vertex v in M_3^1 is called *special* if $d_s(v) \geq 7$. By Claim 2.9, we assume that $\{u_{k+t+1}, u_{k+t+2}, \ldots, u_{k+t+r}\}$ is the set of special vertices. A color in $\{c(g_{k+t+1}), \ldots, c(g_{k+t+r})\}$ is called a *popular color*. An edge with popular color is called a *popular edge*.

Claim 2.10. *Let uv be a popular edge such that $v \in \bigcup_{i=k+1}^{k+t} z_i$. Then $u \notin V(T)$.*

Proof. Suppose, on the contrary, there exists an edge uv such that $c(uv)$ is popular, $v \in \bigcup_{i=k+1}^{k+t} z_i$ and $u \in V(T)$. Without loss of generality, we assume that $v = z_{k+1}$ and $c(uv) = c(g_{k+t+1})$. Further, if $u \in \cup_{i=k+1}^{k+t} z_i$, then we assume that $u = z_{k+2}$ and if $u \in V(M_1 \cup M_2)$, then we assume that $u \in M_2$. Now we can choose a special edge $u_{k+t+1}w$, which is not incident with u, v such that $c(u_{k+t+1}w) \notin \{c(g_{k+1}), c(g_{k+2})\}$. If $c(u_{k+t+1}w)$ is kind, we may assume that $c(u_{k+t+1}w) = c(g_{k+i})$, where $i \in \{3, \dots, t\}$. Obviously, $w \neq z_{k+i}$. Then $M_1 \cup M_3 \cup \{uv, u_{k+t+1}w, h_{k+i}\} - \{g_{k+i}, g_{k+t+1}\}$ is an orthogonal matching for G, which is a contradiction. So now we assume that $c(u_{k+t+1}w) = m$. Then $M_1 \cup M_3 \cup \{uv, u_{k+t+1}w\} - g_{k+t+1}$ is an orthogonal matching for G, which is also a contradiction. □

Claim 2.11. *Let z_iu_j be a popular edge where $k + 1 \leq i, j \leq k + t$. Then $i = j$.*

Proof. Otherwise, without loss of generality, we may assume that $i = k + 1, j = k + 2$ and $c(z_iu_j) = c(g_{k+t+1})$. Since vertex u_{k+t+1} is special, we choose a special edge $u_{k+t+1}w$ such that $u_{k+t+1}w$ is not incident with z_i, z_j and $c(u_{k+t+1}w) \notin \{c(g_i), c(g_j)\}$. If $c(u_{k+t+1}w) = m$, then we add h_j to M_1, add $u_{k+t+1}w$ to M_2 and replace M_3 by $M_3 \cup \{z_iu_j\} - \{g_j, g_{k+t+1}\}$, and we get a new good configuration with bigger k, a contradiction. So now we assume that $c(u_{k+t+1}w) = c(g_{k+i_0})$, $i_0 \notin \{1, 2\}$. Then we add g_{k+i_0} to M_1, add $u_{k+t+1}w$ to M_2, replace M_3 by $M_3 \cup \{z_iu_j, h_j\} - \{g_j, g_{k+t+1}, g_{k+i_0}\}$. Hence we find a good configuration with bigger k, which is also a contradiction. □

Claim 2.12. *There is no popular edge between $\{z_{k+1}, z_{k+2}, \dots, z_{k+t}\}$ and $\{v_{k+t+1}, v_{k+t+2}, \dots, v_{k+t+r}\}$.*

Proof. Otherwise without loss of generality, we may assume that there exists an edge, say $z_{k+1}v_{k+t+1}$, which has a popular color. Since $d_s(u_{k+t+1}) \geq 7$, we can choose a special edge $u_{k+t+1}z$ such that $z \in \bigcup_{i=k+1}^{k+t} z_i$, $z \neq z_{k+1}$ and $c(u_{k+t+1}z) \neq c(g_{k+1})$.

Case 1: $c(z_{k+1}v_{k+t+1}) = c(g_{k+t+1})$.

First suppose that $c(u_{k+t+1}z) = m$. Then $M_1 \cup M_3 \cup \{z_{k+1}v_{k+t+1}, u_{k+t+1}z\} - g_{k+t+1}$ is an orthogonal matching for G, which is a contradiction. Now we assume that $c(u_{k+t+1}z)$ is kind. Without loss of generality, we may assume that $c(u_{k+t+1}z) = c(g_{k+2})$. Then we add g_{k+2} to M_1, add $u_{k+t+1}z$ to M_2 and replace M_3 by $M_3 \cup z_{k+1}v_{k+t+1} - \{g_{k+t+1}, g_{k+2}\}$. Thus we get a good configuration with bigger k, which is also a contradiction.

Case 2: $c(z_{k+1}v_{k+t+1}) \neq c(g_{k+t+1})$.

Without loss of generality, we assume that $c(z_{k+1}v_{k+t+1}) = c(g_{k+t+2})$. If $c(u_{k+t+1}z) = m$, then we add $z_{k+1}v_{k+t+1}$ to M_1, add g_{k+t+2} to M_2 and replace M_3 by $M_3 \cup u_{k+t+1}z - \{g_{k+t+1}, g_{k+t+2}\}$. Then we obtain a good configuration with larger k, which is a contradiction. So now we assume that $c(u_{k+t+1}z)$ is kind. Without loss of generality, we may assume that $c(u_{k+t+1}z) = c(g_{k+2})$. Then we add $z_{k+1}v_{k+t+1}$ to M_1, add g_{k+t+2} to M_2 and replace M_3 by $M_3 \cup \{u_{k+t+1}z, h_{k+2}\} - \{g_{k+2}, g_{k+t+1}, g_{k+t+2}\}$. Thus we get another good configuration with bigger k, which is also a contradiction. This completes our proof. □

An edge between $\{z_{k+1}, z_{k+2}, \dots, z_{k+t}\}$ and $V(M_3)$ is called an *sop* edge if it is a special edge or a popular edge.

Claim 2.13. *For an edge $uv \in E(M_3)$, if u is incident with at least two sop edges, then v can not be incident with sop edges.*

Proof. Suppose not, by Claims 2.9 and 2.12, we know that $uv = g_{k+t+r+i}$ with $i > 0$. So we can choose two independent sop edges, say uz_{k+1} and vz_{k+2}. We divide our proof into the following cases.

Case 1: uz_{k+1} is a special edge or vz_{k+2} is a special edge.

Without loss of generality, we assume that uz_{k+1} is a special edge. By Claim 2.9, we know that $c(vz_{k+2})$ is a popular color, and we may assume that $c(vz_{k+2}) = c(g_{k+t+1})$. First suppose that $c(uz_{k+1}) = m$. Then we add g_{k+t+1} to M_1, add vz_{k+2} to M_2 and replace M_3 by $M_3 \cup uz_{k+1} - \{g_{k+t+1}, uv\}$. Therefore, we get another good configuration with bigger k, which is a contradiction. Hence now we assume that $c(uz_{k+1})$ is kind, say $c(uz_{k+1}) = c(g_{k+j})$. By the definition of special edges, $j \neq 1$. Since $d_s(u_{k+t+1}) \geq 7$, we can choose a special edge $u_{k+t+1}w$ such that $w \notin \{z_{k+1}, z_{k+2}, z_{k+j}\}$ and $c(u_{k+t+1}w) \neq c(g_{k+j})$. Let $c(u_{k+t+1}w) = c(g_{k+i_0})$. If $i_0 \notin \{1, 2\}$, then we add g_{k+j} to M_1, add uz_{k+1} to M_2 and replace M_3 by $M_3 \cup \{vz_{k+2}, h_{k+i_0}, u_{k+t+1}w\} - \{g_{k+i_0}, uv, g_{k+j}, g_{k+t+1}\}$. Thus we get a new good configuration with bigger k, which is also a contradiction. If $i_0 \in \{1, 2\}$, say $i_0 = 1$, then we add g_{k+j} to M_1, add uz_{k+1} to M_2 and replace M_3 by $M_3 \cup \{vz_{k+2}, h_{k+j}, u_{k+t+1}w\} - \{g_{k+1}, uv, g_{k+j}, g_{k+t+1}\}$. Hence we also get a new good configuration with bigger k, which is a contradiction.

Case 2: uz_{k+1} and vz_{k+2} are popular edges.

First we assume that $c(uz_{k+1}) = c(vz_{k+2})$. Further without loss of generality we assume that $c(uz_{k+1}) = c(g_{k+t+1})$. Note that $uv = g_{k+t+r+i} \neq g_{k+t+1}$ or else we can add uz_{k+1} to M_1, add vz_{k+1} to M_2 and replace M_3 by $M_3 - uv$, hence we obtain a good configuration with bigger k, which is a contradiction. So $uv \neq g_{k+t+1}$. Then we choose a special edge $u_{k+t+1}w$ such that $w \notin \{z_{k+1}, z_{k+2}\}$ and $c(u_{k+t+1}w) \notin \{g_{k+1}, g_{k+2}\}$, since $d_s(u_{k+t+1}) \geq 7$, such w exists. If $c(u_{k+t+1}w) = m$, then we add uz_{k+1} to M_1, add vz_{k+2} to M_2 and replace M_3 by $M_3 \cup u_{k+t+1}w - \{g_{k+t+1}, uv\}$. So we assume that $c(u_{k+t+1}w)$ is kind, say, $c(u_{k+t+1}w) = c(g_{k+i_0})$. Now we add uz_{k+1} to M_1, add vz_{k+2} to M_2 and replace M_3 by $M_3 \cup \{u_{k+t+1}w, h_{k+i_0}\} - \{g_{k+t+1}, g_{k+i_0}, uv\}$. Then we obtain a new good configuration with bigger k, which is a contradiction.

Next we assume that $c(uz_{k+1}) \neq c(vz_{k+2})$. Without loss of generality, we assume that $c(uz_{k+1}) = c(g_{k+t+1})$ and $c(vz_{k+2}) = c(g_{k+t+2})$, further we assume that $uv \neq g_{k+t+2}$. Then we choose a special edge $u_{k+t+2}w$ such that $w \notin$

$\{z_{k+1}, z_{k+2}\}$ and $c(u_{k+t+2}w) \notin \{c(g_{k+1}), c(g_{k+2})\}$. Since $d_s(u_{k+t+2}) \geq 7$, such w exists. Assume that $c(u_{k+t+2}w) = c(g_{k+i_0})$. Now we add uz_{k+1} to M_1, add g_{k+t+1} to M_2 and replace M_3 by $M_3 \cup \{u_{k+t+2}w, h_{k+i_0}\} - \{g_{k+t+2}, g_{k+i_0}, uv\}$. Then we obtain a new good configuration with bigger k, which is a contradiction. □

Let n_4 denote the number of special edges and popular edges between $\bigcup_{i=k+1}^{k+t} z_i$ and $V(M_3^1)$. By Claim 2.4, each z_i can send 2 edges colored $c(g_i)$ to $M_1 \cup M_2$. By Claim 2.8, $c(z_iu_i)$ can be a kind color or equal m. Moreover each z_i can send at most t edges colored m or popular colors to $\bigcup_{i=k+1}^{k+t} v_i$. So we have

$$n_4 \geq 2(1+r+t)t - 3t - t^2.$$

On the other hand, for each $i \in \{k+t+1\ldots, k+t+r\}$, u_i sends at most t special or popular edges to $\bigcup_{i=k+1}^{k+t} z_i$. By the definition of special vertices, for each $i \in \{k+t+r+1\ldots, p\}$, those vertices in $V(g_i)$ send at most 6 special edges to $\bigcup_{i=k+1}^{k+t} z_i$. Moreover, each vertex in $V(\bigcup_{i=k+t+r+1}^{p} g_i)$ sends at most r popular edges to $\bigcup_{i=k+1}^{k+t} z_i$. Combining with Claims 2.12 and 2.13, we have

$$n_4 \leq rt + 6(p-k-t-r) + 2r(p-k-t-r).$$

Hence

$$\begin{aligned} p &\geq k + \frac{3t}{2} + \frac{t^2-4t}{2r+6} + r + 3 - 3 \\ &\geq \frac{n}{2} - p + t + \frac{t^2-4t}{2r+6} + r + 3 - 3 \\ &\geq \frac{n}{2} - p + t + \sqrt{2}t - 3\sqrt{2} - 3. \end{aligned}$$

So we have

$$p \geq \frac{n}{4} + \frac{\left(\sqrt{2}+1\right)t}{2} - \frac{3\sqrt{2}+3}{2}. \tag{2.5}$$

Recall that q is the number of nice pairs in $M_1 \cup M_2$. We will prove that $q \geq 4$. Otherwise $q \leq 3$. Recall that n_1 is the number of edges uv such that $u \in V(M_3)$, $v \in V(G)\backslash V(M_3)$ and $c(uv) = m$. By Claims 2.1 and 2.2, $n_1 \geq 2|T| - 8q \geq 2(4k+t) - 24$. On the other hand, $n_1 \leq 2|M_3| = 4(p-k)$. It follows that $2(4k+t) - 24 \leq 4(p-k)$. That is, $k \leq \frac{p}{3} - \frac{t}{6} + 2$. By (2.2), we have

$$\frac{n-2p-t}{2} \leq \frac{p}{3} - \frac{t}{6} + 2.$$

It follows that $t \geq \frac{3n}{2} - 4p - 6$. By (2.5), we have that

$$p \geq \frac{\left(3\sqrt{2}+4\right)n}{4} - 2\left(\sqrt{2}+1\right)p - \frac{9\left(\sqrt{2}+1\right)}{2} \geq p + 3 + \frac{7\sqrt{2}}{8},$$

which is a contradiction. So $q \geq 4$.

Claim 2.14. *Let $e = uv$ be an edge with a nice color, where $u, v \in V(T)$. Without loss of generality, we assume that $c(uv) = c(e_{i_0})$ for some $i_0 \leq q$ and further we assume that $c(w_{i_0}^1 w_{i_0}^2) = m$, where $w_{i_0}^1 \in e_{i_0}$ and $w_{i_0}^2 \in f_{i_0}$. Then one of the following is true:*

(a) *$u, v \in V(e_i \cup f_i)$ for some $i \in \{1, 2, \ldots, k\}$;*
(b) *uv is incident with $w_{i_0}^1$ or $w_{i_0}^2$.*

Proof. Otherwise, by symmetry and without loss of generality, we may assume that $u, v \notin V(M_2)$. Then $M_2 \cup M_3 \cup \{uv, w_{i_0}^1 w_{i_0}^2\} - e_{i_0}$ is an orthogonal matching, which is a contradiction. □

Let n_5 be the number of edges uv such that $u \in V(M_3)$, $v \in V(T)$ and $c(uv)$ is nice or $c(uv) = m$ ($c(uv) \in \{c(e_1), \ldots, c(e_q), m\}$). Call a color *fine* if it is nice or equal m. For each $i \in \{1, 2, \ldots, q\}$, there are at most 6 edges with fine colors in the subgraph induced by $V(e_i \cup f_i)$. For each $i \in \{q+1, q+2, \ldots, k\}$, there are at most 4 edges with fine colors in the subgraph induced by $V(e_i \cup f_i)$. By Claim 2.14(b), for each $i \in \{1, 2, \ldots, q\}$, the vertices in $V(e_i \cup f_i)$ can send at most 2 edges with $c(e_i)$ to $V(T) \setminus V(e_i \cup f_i)$. So we have $n_5 \geq 2(1+q)|T| - 12q - 4q - 8(k-q) = 2(1+q)(4k+t) - 8k - 8q$.

We also have $n_5 \leq 2(1+q)|M_3| = 4(1+q)(p-k)$. Hence

$$2(1+q)(4k+t) - 8k - 8q \leq 4(1+q)(p-k).$$

Thus

$$k \leq \frac{(1+q)(2p-t)}{6q+2} + \frac{4q}{6q+2}.$$

By (2.2),

$$\frac{n-2p-t}{2} \leq \frac{(1+q)(2p-t)}{6q+2} + \frac{4q}{6q+2}.$$

It follows that

$$\begin{aligned} t &\geq \frac{(3q+1)n-(8q+4)p-4q}{2q} \\ &\geq \frac{(3q+1)(2\sqrt{2}p+2\sqrt{2}+4.5)-(8q+4)p-4q}{2q} \\ &\geq \frac{0.24pq-0.59p+8.9q}{q} \\ &\geq 0.24p+8.9-\frac{0.59p}{q} \\ &= 0.09p+8.9, \end{aligned}$$

as $q \geq 4$. Hence

$$\begin{aligned} k &\leq \frac{(1+q)(2p-t)}{6q+2} + \frac{4q}{6q+2} \\ &\leq \frac{0.955pq+0.955p}{3q+1} \\ &\leq \frac{1.20pq}{3q+1}. \end{aligned}$$

It holds that $\frac{k}{t} \leq \frac{1.20pq}{0.09p(3q+1)} \leq 4.5$.

Now we choose a kind color, say $c(g_{k+1})$, such that the number of edges with this kind color in T is minimum among all kind colors. By Claim 2.4, we know that the number of edges with color $c(g_{k+1})$ is at most $\frac{4k}{t}+2$ in T. Let n_3 be the number of edge uv such that $u \in V(M_3)$, $v \in V(T)$ and $c(uv) = c(g_{k+1})$. Thus $n_3 \geq 2|T| - 2(\frac{4k}{t}+2) = 2(4k+t) - 2(\frac{4k}{t}+2)$. We also have $n_3 \leq 2|M_3| - 2 = 4(p-k) - 2$. Hence

$$2(4k+t) - 2\left(\frac{4k}{t}+2\right) \leq 4(p-k) - 2.$$

Recall that $\frac{k}{t} \leq 4.5$. Hence

$$4k+t-19 \leq 2(p-k).$$

By (2.2),

$$3(n-2p-t) \leq 2p-t+19.$$

It follows that $t \geq 1.5n - 4p - \frac{19}{2}$. By (2.5), we have that

$$p \geq \frac{(3\sqrt{2}+4)n}{4} - 2(\sqrt{2}+1)p - \frac{25(\sqrt{2}+1)}{4} \geq p + \frac{10-7\sqrt{2}}{8} > p + \frac{1}{80},$$

which is a contradiction. This completes the proof of Theorem 1.4.

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References

[1] B. Alspach, Problem 89, Discrete Math. 69 (1988) 106.
[2] B. Alspach, K. Heinrich, G. Liu, Orthogonal claimorizations of graphs, in: Contemporary Design Theory: A Collection of Surveys, Wiley, New York, 1992.
[3] R.P. Anstee, L. Caccetta, Orthogonal matchings, Discrete Math. 179 (1998) 37–47.
[4] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Macmillan Press[M], New York, 1976.
[5] L. Caccetta, S. Mardiyono, Premature sets of one-claimors, Australas. J. Combin. 5 (1992) 229–252.
[6] H. Chen, X. Li, Long heterochromatic paths in edge-colored graphs, Electron. J. Combin. 12 (2005) P33.
[7] J. Diemunsch, M. Ferrara, A. Lo, C. Moffatt, F. Pfender, P.S. Wenger, Rainbow matchings of size $\delta(G)$ in properly edge-colored graphs, Electron. J. Combin. 19 (2) (2012) P52.
[8] A. Gyárfás, G.N. Sárközy, Rainbow matchings and cycle-free partial transversals of Latin squares, Discrete Math. 327 (2014) 96–102.
[9] P. Hatami, P.W. Shor, A lower bound for the length of a partial transversal in a Latin square, revised version, J. Combin. Theory Ser. A 115 (7) (2008) 1103–1113.
[10] A. Kostochka, F. Pfender, M. Yancey, Large Rainbow Matchings in Large graphs, arXiv preprint, arXiv:1204.3193, 2012.
[11] A. Kostochka, M. Yancey, Large rainbow matchings in edge-colored graphs, Combin. Probab. Comput. 21 (2012) 255–263.
[12] M. Kouider, D. Sotteau, On the existence of a matching orthogonal to a 2-claimorization, Discrete Math. 73 (1989) 301–304.
[13] T.D. LeSaulnier, C. Stocker, P.S. Wenger, D.B. West, Rainbow matching in edge-colored graphs, Electron. J. Combin. 17 (2010) P26.
[14] H. Li, G. Wang, Color degree and heterochromatic matchings in edge-colored bipartite graphs, Util. Math. 77 (2008) 145–154.
[15] H. Li, G. Wang, Color degree and heterochromatic cycles in edge-colored graphs, Electron. J. Combin. 33 (8) (2012) 1958–1964.
[16] A. Lo, T. Tan, A note on large rainbow matchings in edge-coloured graphs, Graphs Combin. 30 (2014) 389–393.
[17] R. Stong, Orthogonal matchings, Discrete Math. 256 (2002) 515–518.
[18] G. Wang, Rainbow matchings in properly edge colored graphs, Electron. J. Combin. 18 (2011) P162.
[19] G. Wang, H. Li, Heterochromatic matchings in edge-colored graphs, Electron. J. Combin. 15 (2008) R138.
[20] G. Wang, B. Liu, G. Liu, A note on rainbow matchings, Discrete Math. Algorithms Appl. 5 (4) (2013) 1350028.
[21] G. Wang, J. Zhang, G. Liu, Existence of rainbow matchings in properly edge-colored graphs, Front. Math. China 7 (3) (2012) 543–550.