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Some bounds on the neighbor-distinguishing index of graphs

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ABSTRACT

A proper edge coloring of a graph G is neighbor-distinguishing if any two adjacent vertices have distinct sets consisting of colors of their incident edges. The neighbor-distinguishing index of G is the minimum number $\chi'_a(G)$ of colors in a neighbor-distinguishing edge coloring of G.

Let G be a graph with maximum degree Δ and without isolated edges. In this paper, we prove that $\chi'_a(G) \leq 2\Delta$ if $4 \leq \Delta \leq 5$, and $\chi'_a(G) \leq 2.5\Delta$ if $\Delta \geq 6$. This improves a result in Zhang et al. (2014), which states that $\chi'_a(G) \leq 2.5\Delta + 5$ for any graph G without isolated edges. Moreover, we prove that if G is a semi-regular graph (i.e., each edge of G is incident to at least one Δ-vertex), then $\chi'_a(G) \leq \frac{5}{3}\Delta + \frac{13}{3}$.

1. Introduction

All graphs considered in this paper are finite and simple. Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G, respectively. Let $N_G(v)$ denote the set of neighbors of a vertex v in G and $d_G(v) = |N_G(v)|$ denote the degree of v in G. The vertex v is called a *k-vertex* if $d_G(v) = k$. Let $\Delta(G)$ and $\delta(G)$ denote the maximum degree and the minimum degree of a vertex in G, respectively. For a vertex $v \in V(G)$ and an integer $i \geq 1$, let $d_i(v)$ denote the number of i-vertices adjacent to v. An *edge-partition* of a graph G is a decomposition of G into subgraphs $G_1, G_2, \ldots, G_m$ such that $E(G) = \bigcup_{i=1}^{m} E(G_i)$ with $E(G_i) \cap E(G_j) = \emptyset$ for all $i \neq j$.

An *edge k-coloring* of a graph G is a function $\phi : E(G) \to \{1, 2, \ldots, k\}$ such that any two adjacent edges receive different colors. The *chromatic index*, denoted by $\chi'(G)$, of a graph G is the smallest integer k such that G has an edge k-coloring. Given an edge k-coloring ϕ of G, we use $C_\phi(v)$ to denote the set of colors assigned to those edges incident to a vertex v. The coloring ϕ is called a *neighbor-distinguishing edge coloring* (an NDE-coloring for short) if $C_\phi(u) \neq C_\phi(v)$ for any pair of adjacent vertices u and v. The *neighbor-distinguishing index* $\chi'_a(G)$ of a graph G is the smallest integer k such that G has a k-NDE-coloring. A graph G is *normal* if it contains no isolated edges. Clearly, G has an NDE-coloring if and only if G is normal. Thus, we always assume that G is normal in the following discussion.

By definition, it is easy to see that $\chi'_a(G) \geq \chi'(G) \geq \Delta(G)$ for any graph G. On the other hand, Zhang, Liu and Wang [13] proposed the following challenging conjecture, and confirmed its truth for paths, cycles, trees, complete graphs and complete bipartite graphs.

Conjecture 1. *Every connected graph G with $|V(G)| \geq 6$ has $\chi'_a(G) \leq \Delta(G) + 2$.*

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Balister et al. [2] affirmed Conjecture 1 for bipartite graphs and all graphs with $\Delta(G) = 3$. They also proved that $\chi'_a(G) \le \Delta(G) + O(\log \chi(G))$, where $\chi(G)$ is the vertex chromatic number of the graph G. This result and Brooks' Theorem imply immediately that $\chi'_a(G) \le 2\Delta(G)$ if $\Delta(G)$ is sufficiently large. Using probabilistic method, Hatami [4] showed that every graph G with $\Delta(G) > 10^{20}$ has $\chi'_a(G) \le \Delta(G) + 300$. Akbari, Bidkhori and Nosrati [1] proved that every graph G satisfies $\chi'_a(G) \le 3\Delta(G)$. Zhang, Wang and Lih [14] improved this bound to that $\chi'_a(G) \le 2.5\Delta(G) + 5$ for any graph G. For planar graphs G, Horňák, Huang and Wang [6] showed that $\chi'_a(G) \le \Delta(G) + 2$ if $\Delta(G) \ge 12$. More recently, Wang and Huang [9] further verified that if G is a planar graph with $\Delta(G) \ge 16$, then $\chi'_a(G) \le \Delta(G)+1$, and moreover $\chi'_a(G) = \Delta(G)+1$ if and only if G contains two adjacent vertices of maximum degree. This result is an extension to the result in [3], which says that if G is a planar bipartite graph with $\Delta(G) \ge 12$, then $\chi'_a(G) \le \Delta(G) + 1$. The reader is referred to [5,10–12] for other results on this direction.

In this paper, we investigate the neighbor-distinguishing index of some special graphs such as graphs with maximum degree 4 or 5 and semi-regular graphs. These results are applied to improve the upper bound of the neighbor-distinguishing index on general graphs. Here a graph G is called *semi-regular* if each edge of G is incident to at least one vertex of maximum degree. Clearly, a regular graph is a semi-regular graph, and not vice versa.

2. Graphs with $\Delta = 4$

This section is devoted to the study of the neighbor-distinguishing index of graphs with maximum degree 4.

Lemma 2.1 ([7]). *If G is a $2k$-regular graph with $k \ge 1$, then G is 2-factorizable.*

It is well-known that, given a graph G, there exists a $\Delta(G)$-regular graph H such that $G \subseteq H$. This fact, together with Lemma 2.1, implies that every graph G with $\Delta(G) = 4$ can be edge-partitioned into two subgraphs G_1 and G_2 such that $\Delta(G_i) \le 2$ for $i = 1, 2$.

In order to prove the main result in this section, i.e., Theorem 2.5, we need the following three useful consequences:

Theorem 2.2 ([14]). *If a normal graph G has an edge-partition into two normal subgraphs G_1 and G_2, then $\chi'_a(G) \le \chi'_a(G_1) + \chi'_a(G_2)$.*

Theorem 2.3 ([13]). *If P is a path of length at least two, then $\chi'_a(P) \le 3$.*

Theorem 2.4 ([2]). *If G is a graph with $\Delta(G) \le 3$, then $\chi'_a(G) \le 5$.*

Suppose that ϕ is a partial NDE-coloring of a graph G using a color set C. We call two adjacent vertices u and v *conflict under* ϕ (or simply *conflict*) if $C_\phi(u) = C_\phi(v)$. An edge uv is said to be *legally* colored if its color is different from that of its neighbors and no pair of conflict vertices is produced.

Theorem 2.5. *If G is a graph with $\Delta(G) \le 4$, then $\chi'_a(G) \le 8$.*

Proof. We prove the theorem by induction on the edge number $|E(G)|$. If $|E(G)| \le 8$, the theorem holds trivially. Let G be a graph with $\Delta(G) \le 4$ and $|E(G)| \ge 9$. If $\Delta(G) \le 3$, then the result follows from Theorem 2.4. So suppose that $\Delta(G) = 4$. The proof is split into the following cases, depending on the size of $\delta(G)$.

Case 1 $\delta(G) = 1$.

Let x be a 1-vertex adjacent to a vertex y. Let $H = G - xy$. Then H is a normal graph with $\Delta(H) \le 4$ and $|E(H)| < |E(G)|$. By the induction hypothesis, H has an 8-NDE-coloring ϕ using the color set $C = \{1, 2, \dots, 8\}$. Note that $|C_\phi(y)| = d_H(y) = d_G(y) - 1 \le 3$ and y has at most $d_G(y) - 1 \le 3$ possible conflict vertices. Thus, xy has at most $|C_\phi(y)| + 3 \le 6$ forbidden colors when colored, we can color xy with a color in $C \setminus C_\phi(y)$ such that y does not conflict with its neighbors. So an 8-NDE-coloring of G is constructed.

Case 2 $\delta(G) = 2$.

Let x be a 2-vertex with neighbors y and z. Without loss of generality, assume that $2 \le d_G(y) \le d_G(z) \le 4$. There are two possibilities to be handled.

Case 2.1 $d_G(y) = 2$.

Let w denote the neighbor of y other than x. Without loss of generality, we assume that $d_G(w) \ge 3$, for otherwise we may further consider the neighbor of w other than y until a desired vertex is found. Let $H = G - wy$. Then H is a normal graph with $\Delta(H) \le 4$ and $|E(H)| < |E(G)|$. By the induction hypothesis, H has an 8-NDE-coloring ϕ with the color set $C = \{1, 2, \dots, 8\}$. We first remove the color of xy. Since w has at most three conflict vertices and y has at most one conflict vertex, we can color yw with a color $a \in C \setminus (C_\phi(w) \cup \{\phi(xz)\})$ and xy with a color in $C \setminus \{a, \phi(xz)\}$ such that neither of x, y, w conflicts with its neighbors.

Case 2.2 $d_G(y) \geq 3$.

Then $d_G(z) \geq 3$. Let $H = G - xy$. Then H is a normal graph with $\Delta(H) \leq 4$ and $|E(H)| < |E(G)|$. By the induction hypothesis, H has an 8-NDE-coloring ϕ with the color set $C = \{1, 2, \dots, 8\}$. Since y is incident to at most three edges in H and has at most three conflict vertices, x is incident to exactly one edge in H and has no conflict vertex, we color xy with a color in $C \setminus (C_\phi(y) \cup \{\phi(xz)\})$ such that y does not conflict with its neighbors.

Case 3 $\delta(G) \geq 3$.

We need to deal with the following two subcases.

Case 3.1 There is an edge uv such that $d_G(u) = d_G(v) = 3$.

Let u_1, u_2 denote the neighbors of u other than v, and v_1, v_2 the neighbors of v other than u. Then $d_G(u_i), d_G(v_i) \geq 3$ for $i = 1, 2$.

First, assume that uv does not lie on any 3-cycle. Let H denote the graph obtained from G by contracting the edge uv. Let z denote the new vertex formed by identifying u and v. Then H is a simple and normal graph with $\Delta(H) \leq 4$ and $|E(H)| < |E(G)|$. By the induction hypothesis, H has an 8-NDE-coloring ϕ using the color set $C = \{1, 2, \dots, 8\}$. Let $\phi(zu_1) = 1$, $\phi(zu_2) = 2$, $\phi(zv_1) = 3$, and $\phi(zv_2) = 4$. In G, we first color uu_1, uu_2, vv_1, vv_2 with $1, 2, 3, 4$, respectively. If uv can be legally colored, we are done. Otherwise, we may assume that $C_\phi(u_1) = \{1, 2, 5\}$, $C_\phi(u_2) = \{1, 2, 6\}$, $C_\phi(v_1) = \{3, 4, 7\}$, and $C_\phi(v_2) = \{3, 4, 8\}$. This means that $d_G(u_1) = d_G(u_2) = d_G(v_1) = d_G(v_2) = 3$. Let u_1' and u_1'' denote the neighbors of u_1 other than u. Similarly, we define u_2' and u_2'' for u_2, v_1' and v_1'' for v_1, and v_2' and v_2'' for v_2. If there exists $a \in \{3, 4\}$ such that $\{2, 5, a\} \notin \{C_\phi(u_1'), C_\phi(u_1'')\}$, we recolor uu_1 with a and color uv with 1. So assume that $C_\phi(u_1') = \{2, 3, 5\}$ and $C_\phi(u_1'') = \{2, 4, 5\}$. Similarly, $C_\phi(u_2') = \{1, 3, 6\}$ and $C_\phi(u_2'') = \{1, 4, 6\}$. Recolor uu_1 with 7, uu_2 with 8, and color uv with 1.

Next, assume that uv lies on a 3-cycle uvu_1u, where $v_1 = u_1$. Let $H = G - uv$. By the induction hypothesis, H has an 8-NDE-coloring ϕ using the color set $C = \{1, 2, \dots, 8\}$. Assume that $\phi(uu_1) = 1$ and $\phi(uu_2) = 2$. Note that the number of 3-vertices that conflict with u or v is at most three. If $\phi(vu_1) \neq 2$ or $\phi(vv_2) \neq 1$, then we color uv with a color in $C \setminus (C_\phi(u) \cup C_\phi(v))$ such that both u and v do not conflict with their neighbors. Otherwise, assume that $\phi(vu_1) = 2$ and $\phi(vv_2) = 1$. Recolor uu_1 with a color $a \in C \setminus C_\phi(u_1)$ such that u_1 does not conflict with its neighbors. Since u_1 has at most four incident edges and two conflict vertices except u and v, such color a exists. Afterward, we legally color uv as above.

Case 3.2 For any edge $uv \in E(G)$, $\max\{d_G(u), d_G(v)\} = 4$.

It follows from Lemma 2.1 that G can be edge-partitioned into subgraphs G_1 and G_2 such that $\Delta(G_i) \leq 2$ for $i = 1, 2$. Since $\delta(G) \geq 3$, we further deduce that $\delta(G_i) \geq 1$ for $i = 1, 2$. Let xy be an arbitrary edge of G_1. Then $xy \in E(G)$ with $\max\{d_G(x), d_G(y)\} = 4$, say $d_G(y) = 4$. Then $d_{G_1}(x) \geq d_G(x) - 2 \geq 3 - 2 = 1$ and $d_{G_1}(y) \geq d_G(y) - 2 = 4 - 2 = 2$. This implies that xy is not an isolated edge of G_1. That is, G_1 is normal. Similarly, we can prove that G_2 is normal.

Note that G_i is the union of vertex-disjoint cycles and paths. For each cycle B in G_1, we pick out an edge $e_B \in E(B)$. Then we set

$E^* = \{e_B \mid B \text{ is a cycle of } G_1\}$,

$H_1 = G_1 - E^*$,

$H_2 = G_2 \cup E^*$.

Then $H_1 \cup H_2$ is an edge-partition of G. It is easy to affirm the following three assertions (a), (b) and (c):

(a) H_1 and H_2 are normal;
(b) H_1 is acyclic and $\Delta(H_1) \leq 2$;
(c) $\Delta(H_2) \leq 3$ (since $\Delta(G_2) \leq 2$ and E^* is a matching of G).

Now, combining Theorems 2.2–2.4, we easily obtain:

$$\chi_a'(G) = \chi_a'(H_1 \cup H_2) \leq \chi_a'(H_1) + \chi_a'(H_2) \leq 3 + 5 = 8.$$

This completes the proof. ■

3. Semi-regular graphs

The following is Vizing's celebrated result on the edge coloring [8]:

Theorem 3.1. *Every simple graph G has $\chi'(G) \leq \Delta(G) + 1$.*

Lemma 3.2. *Let G be a semi-regular graph with $\Delta(G) = \Delta \geq 5$. Then there is an edge-partition of G into normal subgraphs $G_1, G_2, \dots, G_k$ such that one of the following conditions holds.*

(1) *If $\Delta \equiv 2 \pmod 3$, then $k = \frac{1}{3}(\Delta + 1)$ and $\Delta(G_i) \leq 3$ for $1 \leq i \leq k$.*
(2) *If $\Delta \equiv 1 \pmod 3$, then $k = \frac{1}{3}(\Delta - 1)$, $\Delta(G_i) \leq 4$ for $1 \leq i \leq 2$ and $\Delta(G_i) \leq 3$ for $3 \leq i \leq k$.*
(3) *If $\Delta \equiv 0 \pmod 3$, then $k = \frac{1}{3}\Delta$, $\Delta(G_1) \leq 4$ and $\Delta(G_i) \leq 3$ for $2 \leq i \leq k$.*

Proof. By Theorem 3.1, $E(G)$ can be partitioned into $\Delta + 1$ disjoint color classes $E_1, E_2, \ldots, E_{\Delta+1}$ such that each E_i is a matching of G. Let H be a subgraph of G edge-induced by s, $3 \le s \le \Delta$, of these color classes. Obviously, $\Delta(H) \le s$. Let uv be an arbitrary edge of H. Then $uv \in E(G)$, assuming $d_G(v) \le d_G(u)$. Then $d_G(u) = \Delta$ as G is semi-regular. Note that exactly one color is not used on any edge incident to u. Therefore, $d_H(u) \ge s - 1 \ge 3 - 1 = 2$, and hence uv is not an isolated edge of H. This shows that H is a normal graph.

In the following, we simply write $E_{i,i+1,\ldots,j} = E_i \cup E_{i+1} \cup \cdots \cup E_j$, where $i < j$.

If $\Delta \equiv 2 \pmod 3$, let $k = \frac{1}{3}(\Delta + 1)$. We define $G_1 = G[E_{1,2,3}]$, $G_2 = G[E_{4,5,6}]$, …, $G_k = G[E_{\Delta-1,\Delta,\Delta+1}]$. Then $G_1, G_2, \ldots, G_k$ form an edge-partition of G satisfying the condition (1).

If $\Delta \equiv 1 \pmod 3$, let $k = \frac{1}{3}(\Delta - 1)$. We define $G_1 = G[E_{1,2,3,4}]$, $G_2 = G[E_{5,6,7,8}]$, $G_3 = [E_{9,10,11}], \ldots, G_k = G[E_{\Delta-1,\Delta,\Delta+1}]$. Then $G_1, G_2, \ldots, G_k$ form an edge-partition of G satisfying the condition (2).

If $\Delta \equiv 0 \pmod 3$, let $k = \frac{1}{3}\Delta$. We define $G_1 = G[E_{1,2,3,4}]$, $G_2 = G[E_{5,6,7}]$, $G_3 = [E_{8,9,10}], \ldots, G_k = G[E_{\Delta-1,\Delta,\Delta+1}]$. Then $G_1, G_2, \ldots, G_k$ form an edge-partition of G satisfying the condition (3). ∎

Theorem 3.3. *If G is a semi-regular graph with $\Delta(G) = \Delta \ge 2$, then $\chi'_a(G) \le \frac{5}{3}\Delta + c$, where $c = \frac{5}{3}$ if $\Delta \equiv 2 \pmod 3$, $c = \frac{13}{3}$ if $\Delta \equiv 1 \pmod 3$, and $c = 3$ if $\Delta \equiv 0 \pmod 3$.*

Proof. If $2 \le \Delta \le 4$, the result follows from Theorems 2.4 and 2.5. Assume that $\Delta \ge 5$. By Lemma 3.2, there is an edge-partition of G into normal subgraphs $G_1, G_2, \ldots, G_k$ such that one of the stated conditions (1), (2) or (3) holds.

If (1) holds, by Theorems 2.2, 2.4 and 2.5, we have

$$\chi'_a(G) \le \sum_{i=1}^{k} \chi'_a(G_i) \le 5k = \frac{5}{3}(\Delta + 1).$$

If (2) holds, then

$$\begin{aligned}\chi'_a(G) &\le \chi'_a(G_1) + \chi'_a(G_2) + \sum_{i=3}^{k} \chi'_a(G_i)\\ &\le 8 + 8 + 5(k-2)\\ &= \frac{5}{3}(\Delta - 1) + 6\\ &= \frac{5}{3}\Delta + \frac{13}{3}.\end{aligned}$$

If (3) holds, then

$$\begin{aligned}\chi'_a(G) &\le \chi'_a(G_1) + \sum_{i=2}^{k} \chi'_a(G_i)\\ &\le 8 + 5(k-1)\\ &= 5k + 3\\ &= \frac{5}{3}\Delta + 3. \quad \blacksquare\end{aligned}$$

Corollary 3.1. *If G is a semi-regular graph with $\Delta(G) = 5$, then $\chi'_a(G) \le 10$.*

4. Graphs with $\Delta = 5$

This section focuses on studying the neighbor-distinguishing edge coloring of graphs with maximum degree 5. The main purpose is to show that if G is a graph with $\Delta(G) \le 5$, then $\chi'_a(G) \le 10$. As an application, we give a new upper bound for the neighbor-distinguishing index of a general graph.

Lemma 4.1. *Let G be a connected graph with $\Delta(G) = 5$ that is not semi-regular. Then G contains one of the following configurations:*

(A1) An edge xy with $d_G(x) \le 3$ and $d_G(y) \le 4$.

(A2) A 4-vertex v satisfies one of the following conditions:

(A2.1) $d_4(v) = 3$ and $d_5(v) = 1$;

(A2.2) $d_4(v) = 2$ and $d_5(v) = 2$.

(A3) An edge vu with $d_G(v) = d_G(u) = 4$ satisfies one of the following conditions:

(A3.1) $d_4(v) = d_4(u) = 1$ and $d_5(v) = d_5(u) = 3$;

(A3.2) $d_4(v) = 1$, $d_5(v) = 3$, and $d_4(u) = 4$.

Proof. Since G is not semi-regular, there exists an edge uv such that $d_G(u) \leq 4$ and $d_G(v) \leq 4$. If either u or v is of degree at most 3, then G contains (A1). So assume that $d_G(u) = d_G(v) = 4$. If u or v is adjacent to other vertex of degree at most 3 (different from v or u), then G contains (A1). Otherwise, $d_4(u) + d_5(u) = 4$ and $d_4(v) + d_5(v) = 4$. Note that $d_4(u) \geq 1$ and $d_4(v) \geq 1$.

If $d_4(u) = 3$, then G contains (A2.1). If $d_4(u) = 2$, then G contains (A2.2). Thus, assume that $d_4(u) = 1$. If $d_4(v) = 1$, then G contains (A3.1). If $d_4(v) = 4$, then G contains (A3.2). The similar argument works for the vertex v. Hence we further suppose that $d_4(u) = d_4(v) = 4$, i.e., both u and v are only adjacent to 4-vertices. Since $\Delta(G) = 5$, G contains a 5-vertex x. As G is connected, there exists a shortest path $P = y_0y_1y_2 \cdots y_my_{m+1}$ connecting u and x, where $u = y_0, x = y_{m+1}$, with $m \geq 1$. Let y_s be the first 5-vertex occurring on P along the direction from u to x. So $d_G(y_j) \leq 4$ for all $0 \leq j \leq s-1$ and $d_G(y_s) = 5$. If $d_G(y_1) \leq 3$, then G contains (A1). Therefore $d_G(y_1) = 4$. We can recur to conclude that $d_G(v_j) = 4$ for all $j = 2, 3, \ldots, s-1$. Now, we find a 4-vertex y_{s-1} adjacent to a 5-vertex y_s. Repeating the above process, the proof of the lemma is complete. ■

Theorem 4.2. *If G is a graph with $\Delta(G) \leq 5$, then $\chi'_a(G) \leq 10$.*

Proof. We prove the result by induction on $|E(G)|$. If $|E(G)| \leq 10$, the theorem holds trivially. Let G be a graph with $\Delta(G) \leq 5$ and $|E(G)| \geq 11$. If $\Delta(G) \leq 4$, then the theorem holds automatically from Theorems 2.4 and 2.5. So suppose that $\Delta(G) = 5$. If G is semi-regular, then $\chi'_a(G) \leq 10$ by Corollary 3.1. Thus, assume that G is not semi-regular. By Lemma 4.1, G contains one of the configurations (A1)–(A3). In the sequel, the proof is split into several cases.

Case 1 G contains (A1): an edge xy with $d_G(x) \leq 3$ and $d_G(y) \leq 4$.

We need to consider the following subcases, depending on the size of $d_G(x)$.

Case 1.1 $d_G(x) = 1$.

Let $H = G - xy$. Then H is a normal graph with $\Delta(H) \leq 5$ and $|E(H)| < |E(G)|$. By the induction hypothesis, H has a 10-NDE-coloring ϕ using the color set $C = \{1, 2, \ldots, 10\}$. Note that $|C_\phi(y)| = d_G(y) - 1 \leq 3$ and y has at most three possible conflict vertices. Thus, xy has at most $|C_\phi(y)| + 3 \leq 6$ forbidden colors, we can color xy with a color in $C \setminus C_\phi(y)$ such that y does not conflict with its neighbors.

Case 1.2 $d_G(x) = 2$.

Let v be the second neighbor of x other than y. By the proof of Case 1.1, we may assume that $d_G(y), d_G(v) \geq 2$.

First, assume that $d_G(y) = 2$. Let w denote the neighbor of y other than x. Without loss of generality, we may assume that $d_G(w) \geq 3$. Let $H = G - wy$. Then H is a normal graph with $\Delta(H) \leq 5$ and $|E(H)| < |E(G)|$. By the induction hypothesis, H has a 10-NDE-coloring ϕ with the color set $C = \{1, 2, \ldots, 10\}$. We first remove the color of xy. Since w has at most four conflict vertices and y has at most one conflict vertex, we can legally color wy with a color in $C \setminus (C_\phi(w) \cup \{\phi(vx)\})$ and then legally color xy.

Next, assume that $d_G(y) \geq 3$. Let $H = G - xy$. Then H is a normal graph with $\Delta(H) \leq 5$ and $|E(H)| < |E(G)|$. By the induction hypothesis, H has a 10-NDE-coloring ϕ with the color set $C = \{1, 2, \ldots, 10\}$. Since y is incident to at most three edges in H and has at most three conflict vertices, x is incident to one edge in H and has at most one conflict vertex, we can color xy with a color in $C \setminus (C_\phi(x) \cup C_\phi(y))$ such that both x and y do not conflict with their neighbors.

Case 1.3 $d_G(x) = 3$.

Let s, t be the neighbors of x other than y. By the proof of Cases 1.1 and 1.2, we may assume that $d_G(s), d_G(t) \geq 3$. We have to consider the following subcases by symmetry.

Case 1.3.1 $d_G(y) = d_G(s) = d_G(t) = 3$.

Let $H = G - \{xy, xs, xt\}$. Then H is a normal graph with $\Delta(H) \leq 5$ and $|E(H)| < |E(G)|$. By the induction hypothesis, H has a 10-NDE-coloring ϕ with the color set $C = \{1, 2, \ldots, 10\}$. For an edge $e \in E(G) \setminus E(H)$, we use $L(e)$ to denote the subset of colors in C which can be legally assigned to e. Since there exist at most $2 + 2 = 4$ forbidden colors, we have $|L(e)| \geq 10 - 4 = 6$ for each $e \in \{xy, xs, xt\}$. We color xy with $a \in L(xy) \setminus C_\phi(s)$, xs with $b \in L(xs) \setminus (C_\phi(t) \cup \{a\})$, and xt with $c \in L(xt) \setminus (C_\phi(y) \cup \{a, b\})$. Since $|C_\phi(y)| = |C_\phi(s)| = |C_\phi(t)| = 2$, the coloring is available.

Case 1.3.2 $d_G(y) = d_G(s) = 3$ and $d_G(t) \geq 4$.

Let $H = G - \{xy, xs\}$. By the induction hypothesis, H has a 10-NDE-coloring ϕ with the color set $C = \{1, 2, \ldots, 10\}$. Note that there exist at most five forbidden colors for each of xy and xs. Thus, $|L(xy)|, |L(xs)| \geq 10 - 5 = 5$. We color xy with $a \in L(xy) \setminus C_\phi(s)$ and xs with $b \in L(xs) \setminus (C_\phi(y) \cup \{a\})$. Analogous to the foregoing analysis, the coloring is feasible.

Case 1.3.3 $d_G(y) = 3$ and $d_G(s), d_G(t) \geq 4$.

Let u, w be the neighbors of y other than x. By the proof of the previous subcases, we may suppose that $d_G(u), d_G(w) \geq 4$.

If xy is not on a 3-cycle, let H denote the graph obtained from G by contracting the edge xy. Let z^* denote the new vertex formed by identifying x and y. Then H is a simple and normal graph with $\Delta(H) \leq 5$ and $|E(H)| < |E(G)|$. By the induction hypothesis, H has a 10-NDE-coloring ϕ with the color set $C = \{1, 2, \ldots, 10\}$. Suppose that $\phi(z^*u) = 1$, $\phi(z^*w) = 2$, $\phi(z^*s) = 3$, and $\phi(z^*t) = 4$. In G, it suffices to color yu with 1, yw with 2, xs with 3, xt with 4, and xy with 5.

Now assume that xy is on a 3-cycle $xyux$, where $t = u$. Let $H = G - xy$. By the induction hypothesis, H has a 10-NDE-coloring ϕ with the color set $C = \{1, 2, \ldots, 10\}$. Assume that $\phi(xu) = 1$ and $\phi(xs) = 2$. If $\phi(yu) \neq 2$ or $\phi(yw) \neq 1$, it suffices to color properly xy in G. Otherwise, $\phi(yu) = 2$ and $\phi(yw) = 1$. In this case, we recolor yu with a color in $C \setminus C_\phi(u)$ such that u does not conflict with its neighbors. This is feasible since u has at most three conflict vertices. Afterward we color properly xy as above.

Case 1.3.4 $d_G(y) = 4$ and $d_G(s), d_G(t) \geq 4$.

Based on a 10-NDE-coloring ϕ of the graph $G - xy$, we legally color the edge xy. Since y has at most three conflict vertices and exactly three incident edges, and x has two incident edges and no conflict vertices, the coloring is available.

Case 2 G contains (A2.1): a 4-vertex v with $d_4(v) = 3$ and $d_5(v) = 1$.

Let v_1, v_2, v_3, v_4 be the neighbors of v with $d_G(v_1) = d_G(v_2) = d_G(v_3) = 4$ and $d_G(v_4) = 5$. Let $H = G - \{vv_1, vv_2, vv_3\}$. Then H is a normal graph with $\Delta(H) \leq 5$ and $|E(H)| < |E(G)|$. By the induction hypothesis, H has a 10-NDE-coloring ϕ with the color set $C = \{1, 2, \ldots, 10\}$. Assume that $\phi(vv_4) = 1$. Since v_i, for $1 \leq i \leq 3$, has at most three conflict vertices and three incident edges, vv_i has at most seven forbidden colors. Hence $|L(vv_i)| \geq 10 - 7 = 3$. Especially, when $1 \in C_\phi(v_i)$, we have $|L(vv_i)| \geq 4$. By symmetry, we need to consider the following possibilities.

- $1 \notin C_\phi(v_i)$ for all $i = 1, 2, 3$. It suffices to color vv_1 with $c_1 \in L(vv_1)$, vv_2 with $c_2 \in L(vv_2) \setminus \{c_1\}$, and vv_3 with $c_3 \in L(vv_3) \setminus \{c_1, c_2\}$.
- $1 \in C_\phi(v_i)$ for all $i = 1, 2, 3$. Then $|L(vv_i)| \geq 4$ for all $i = 1, 2, 3$. It is easy to see that there exist at least $\binom{4}{3} = 4$ ways to color vv_1, vv_2 and vv_3. Thus, we assign a color $c_i \in L(vv_i)$ to vv_i such that c_1, c_2, c_3 are mutually distinct and $\{1, c_1, c_2, c_3\} \notin \{C_\phi(v_1), C_\phi(v_2), C_\phi(v_3)\}$.
- $1 \in C_\phi(v_i)$ for $i = 1, 2$, and $1 \notin C_\phi(v_3)$. Then $|L(vv_i)| \geq 4$ for $i = 1, 2$. We color vv_1 with $c_1 \in L(vv_1) \setminus (C_\phi(v_2) \setminus \{1\})$, vv_2 with $c_2 \in L(vv_2) \setminus ((C_\phi(v_1) \setminus \{1\}) \cup \{c_1\})$, and vv_3 with $c_3 \in L(vv_3) \setminus \{c_1, c_2\}$. Noting that $|C_\phi(v_i) \setminus \{1\}| = 3 - 1 = 2$ for $i = 1, 2$, we get a legal coloring.
- $1 \in C_\phi(v_1)$, and $1 \notin C_\phi(v_i)$ for $i = 2, 3$. Then $|L(vv_1)| \geq 4$. We color vv_2 with $c_2 \in L(vv_2) \setminus (C_\phi(v_1) \setminus \{1\})$, vv_1 with $c_1 \in L(vv_1) \setminus \{c_2\}$, and vv_3 with $c_3 \in L(vv_3) \setminus \{c_1, c_2\}$.

Case 3 G contains (A2.2): a 4-vertex v with $d_4(v) = d_5(v) = 2$.

Let v_1, v_2, v_3, v_4 be the neighbors of v with $d_G(v_1) = d_G(v_2) = 4$ and $d_G(v_3) = d_G(v_4) = 5$. Let $H = G - \{vv_1, vv_2\}$. By the induction hypothesis, H has a 10-NDE-coloring ϕ with the color set $C = \{1, 2, \ldots, 10\}$. Assume that $\phi(vv_3) = 1$ and $\phi(vv_4) = 2$. It is easy to observe that $|L(vv_i)| \geq |C| - |C_\phi(v_i)| - 3 - 2 \geq 10 - 3 - 5 = 2$ for $i = 1, 2$. Moreover, $|L(vv_i)| \geq 4$ if $1, 2 \in C_\phi(v_i)$, and $|L(vv_i)| \geq 3$ if 1 or $2 \in C_\phi(v_i)$.

If $|C_\phi(v_i) \cap \{1, 2\}| \leq 1$ for $i = 1, 2$, we color vv_1 with $c_1 \in L(vv_1)$ and vv_2 with $c_2 \in L(vv_2) \setminus \{c_1\}$. Otherwise, assume that $1, 2 \in C_\phi(v_1)$. So $|L(vv_1)| \geq 4$. If $1, 2 \in C_\phi(v_2)$, then $|L(vv_2)| \geq 4$, we color vv_1 with $c_1 \in L(vv_1) \setminus C_\phi(v_2)$ and vv_2 with $c_2 \in L(vv_2) \setminus (C_\phi(v_1) \cup \{c_1\})$. If $|C_\phi(v_2) \cap \{1, 2\}| \leq 1$, then we color vv_2 with $c_2 \in L(vv_2) \setminus (C_\phi(v_1) \setminus \{1, 2\})$ and vv_1 with $c_1 \in L(vv_1) \setminus \{c_2\}$.

Case 4 G contains (A3.1): an edge vu with $d_G(v) = d_G(u) = 4$, $d_4(v) = d_4(u) = 1$, and $d_5(v) = d_5(u) = 3$.

Let x, y, z be the neighbors of v other than u with $d_G(x) = d_G(y) = d_G(z) = 5$. Let $H = G - uv$. Then H is a normal graph with $\Delta(H) \leq 5$ and $|E(H)| < |E(G)|$. By the induction hypothesis, H has a 10-NDE-coloring ϕ with the color set $C = \{1, 2, \ldots, 10\}$. Assume that $\phi(vx) = 1$, $\phi(vy) = 2$, and $\phi(vz) = 3$. If $C_\phi(u) \neq \{1, 2, 3\}$, then it suffices to color uv with a color in $C \setminus (C_\phi(u) \cup C_\phi(v))$. So suppose that $C_\phi(u) = \{1, 2, 3\}$. Now it remains to recolor vx and then the proof is reduced to the previous case.

Let x_1, x_2, x_3, x_4 be the neighbors of x other than v. If vx can be legally recolored with a color in $\{4, 5, \ldots, 10\}$, we are done. Otherwise, it is straightforward to see that at most one of 2 and 3 is in $C_\phi(x)$. Consequently, our proof is split into the following two cases by symmetry.

Case 4.1 $2 \in C_\phi(x)$ and $3 \notin C_\phi(x)$.

Because vx cannot be legally recolored, we may suppose that $C_\phi(x_1) = \{2, 4, 5, 6, 7\}$ with $\phi(xx_1) = 2$, $C_\phi(x_2) = \{2, 4, 5, 6, 8\}$ with $\phi(xx_2) = 4$, $C_\phi(x_3) = \{2, 4, 5, 6, 9\}$ with $\phi(xx_3) = 5$, and $C_\phi(x_4) = \{2, 4, 5, 6, 10\}$ with $\phi(xx_4) = 6$. This implies that $d_G(x_i) = 5$ for all $i = 1, 2, 3, 4$. Set

$$\Omega(x_i) = \{C_\phi(t) \mid t \in N_H(x_i) \setminus \{x\}\}, \quad i = 1, 2, 3, 4.$$

Then $|\Omega(x_i)| = d_H(x_i) - 1 = 5 - 1 = 4$. If $\{1, 2, 5, 6, 8\}$ or $\{2, 3, 5, 6, 8\} \notin \Omega(x_2)$, then we recolor xx_2 with 1 or 3, and vx with 4. Thus, assume that $\{1, 2, 5, 6, 8\}, \{2, 3, 5, 6, 8\} \in \Omega(x_2)$. Similarly, we may assume that $\{1, 2, 4, 6, 9\}$, $\{2, 3, 4, 6, 9\} \in \Omega(x_3)$, and $\{1, 2, 4, 5, 10\}, \{2, 3, 4, 5, 10\} \in \Omega(x_4)$. To complete the proof, we need to consider the following two possibilities:

• Assume that $\{2, 5, 6, 7, 8\} \notin \Omega(x_2)$. If there is $p \in \{8, 10\}$ such that $\{2, 4, 6, 9, p\} \notin \Omega(x_3)$, we recolor xx_3 with p, xx_2 with 7, and vx with 4. So assume that $\{2, 4, 6, 8, 9\}, \{2, 4, 6, 9, 10\} \in \Omega(x_3)$. Similarly, we can assume that $\{2, 4, 5, 8, 10\}, \{2, 4, 5, 9, 10\} \in \Omega(x_4)$. If there is $q \in \{9, 10\}$ such that $\{2, 5, 6, 8, q\} \notin \Omega(x_2)$, we recolor xx_2 with q and vx with 7. So assume that $\{2, 5, 6, 8, 9\}, \{2, 5, 6, 8, 10\} \in \Omega(x_2)$. If there is $r \in \{3, 8, 9, 10\}$ such that $\{4, 5, 6, 7, r\} \notin \Omega(x_1)$, we recolor xx_1 with r and vx with a color in $\{8, 9, 10\}\backslash\{r\}$. Thus, assume that $\Omega(x_1) = \{\{3, 4, 5, 6, 7\}, \{4, 5, 6, 7, 8\}, \{4, 5, 6, 7, 9\}, \{4, 5, 6, 7, 10\}\}$. Now we recolor xx_1 with 1 and vx with 9.

• Assume that $\{2, 5, 6, 7, 8\} \in \Omega(x_2)$. Since $|\Omega(x_2)| \leq 4$, at least one of $\{2, 5, 6, 8, 9\}$ and $\{2, 5, 6, 8, 10\}$ does not belong to $\Omega(x_2)$, say $\{2, 5, 6, 8, 9\} \notin \Omega(x_2)$. If there is $p \in \{8, 10\}$ such that $\{2, 5, 6, 9, p\} \notin \Omega(x_3)$, we recolor xx_2 with 9, xx_3 with p, and vx with a color in $\{8, 10\} \setminus \{p\}$. So assume that $\{2, 5, 6, 8, 9\}, \{2, 5, 6, 9, 10\} \in \Omega(x_3)$. If $\{2, 4, 5, 8, 10\} \notin \Omega(x_4)$, we recolor xx_4 with 8 and vx with 9. Thus assume that $\{2, 4, 5, 8, 10\} \in \Omega(x_4)$. Analogous to the previous proof, we get that $\Omega(x_1) = \{\{3, 4, 5, 6, 7\}, \{4, 5, 6, 7, 8\}, \{4, 5, 6, 7, 9\}, \{4, 5, 6, 7, 10\}\}$. It suffices to recolor xx_2 with 9, xx_1 with 1, and vx with 10.

Case 4.2 $2, 3 \notin C_\phi(x)$.

If vx cannot be legally recolored, then we may assume that $\phi(xx_1) = 4$, $\phi(xx_2) = 5$, $\phi(xx_3) = 6$, $\phi(xx_4) = 7$, $C_\phi(x_1) = \{4, 5, 6, 7, 8\}$, $C_\phi(x_2) = \{4, 5, 6, 7, 9\}$, and $C_\phi(x_3) = \{4, 5, 6, 7, 10\}$.

If $\{1, 5, 6, 7, 8\} \notin \Omega(x_1)$, then it is enough to switch the colors of vx and xx_1. Thus, assume that $\{1, 5, 6, 7, 8\} \in \Omega(x_1)$, and similarly $\{1, 4, 6, 7, 9\} \in \Omega(x_2)$, and $\{1, 4, 5, 7, 10\} \in \Omega(x_3)$.

If there is $q \in \{2, 3\}$ such that $\{q, 5, 6, 7, 8\} \notin \Omega(x_1)$, we recolor xx_1 with q, and vx with $a \in \{9, 10\}$ such that $C_\phi(x_4) \neq \{a, q, 5, 6, 7\}$. Then the proof is reduced to Case 4.1. Thus, assume that $\{2, 5, 6, 7, 8\}, \{3, 5, 6, 7, 8\} \in \Omega(x_1)$. Similarly, we conclude that $\{2, 4, 6, 7, 9\}, \{3, 4, 6, 7, 9\} \in \Omega(x_2)$ and $\{2, 4, 5, 7, 10\}, \{3, 4, 5, 7, 10\} \in \Omega(x_3)$.

There are two subcases as follows.

Case 4.2.1 $\{5, 6, 7, 8, 9\} \notin C_\phi(x_1)$.

First, we recolor xx_1 with 9. Then we give the following detailed analysis.

• If $C_\phi(x_4) \neq \{5, 6, 7, 9, 10\}$, we recolor vx with 10.

• Assume that $C_\phi(x_4) = \{5, 6, 7, 9, 10\}$. Similar to the previous proof, we derive that $\{1, 5, 6, 9, 10\}, \{2, 5, 6, 9, 10\}, \{3, 5, 6, 9, 10\} \in \Omega(x_4)$.

If xx_2 and xx_3 can be, respectively, recolored legally with 10 and 8, then we recolor xx_2 with 10, xx_3 with 8, and vx with 4. Otherwise, $\{4, 6, 7, 9, 10\} \in \Omega(x_2)$, or $\{4, 5, 7, 8, 10\} \in \Omega(x_3)$. By symmetry, we consider the following two possibilities:

(i) $\{4, 6, 7, 9, 10\} \in \Omega(x_2)$. If $\{4, 5, 6, 9, 10\} \notin \Omega(x_4)$, we recolor xx_2 with 8, xx_4 with 4, and vx with 10. If $\{4, 5, 6, 9, 10\} \in \Omega(x_4)$, we recolor xx_4 with 8 and vx with 4.

(ii) $\{4, 6, 7, 9, 10\} \notin \Omega(x_2)$ and $\{4, 5, 7, 8, 10\} \in \Omega(x_3)$. If $\{4, 5, 6, 9, 10\} \notin \Omega(x_4)$, we recolor xx_2 with 10, xx_4 with 4 and vx with 8. Otherwise, we recolor xx_2 with 10, xx_4 with 8, and vx with 4.

Case 4.2.2 $\{5, 6, 7, 8, 9\} \in C_\phi(x_1)$.

First, we recolor xx_1 with 10. Then we deal with some subcases below.

• If $C_\phi(x_4) \neq \{5, 6, 7, 9, 10\}$, we recolor vx with 9.

• Assume that $C_\phi(x_4) = \{5, 6, 7, 9, 10\}$. Similar to the previous proof, we derive that $\{i, 5, 6, 9, 10\} \in \Omega(x_4)$ for $i = 1, 2, 3$.

If xx_2 and xx_3 can be, respectively, recolored legally with 8 and 9, then we recolor xx_2 with 8, xx_3 with 9, and vx with 4. Otherwise, $\{4, 6, 7, 8, 9\} \in \Omega(x_2)$, or $\{4, 5, 7, 9, 10\} \in \Omega(x_3)$. By symmetry, we consider the following two possibilities:

(i) $\{4, 5, 7, 9, 10\} \in \Omega(x_3)$. If $\{4, 5, 6, 9, 10\} \notin \Omega(x_4)$, we recolor xx_3 with 8, xx_4 with 4, and vx with 9. If $\{4, 5, 6, 9, 10\} \in \Omega(x_4)$, we recolor xx_4 with 8, and vx with 4.

(ii) $\{4, 5, 7, 9, 10\} \notin \Omega(x_3)$ and $\{4, 6, 7, 8, 9\} \in \Omega(x_2)$. If $\{4, 5, 6, 9, 10\} \notin \Omega(x_4)$, we recolor xx_3 with 9, xx_4 with 4 and vx with 8. Otherwise, we recolor xx_3 with 9, xx_4 with 8 and vx with 4.

Case 5 G contains (A3.2): an edge vu with $d_G(v) = d_G(u) = 4$, $d_4(v) = 1$, $d_5(v) = 3$, and $d_4(u) = 4$.

Let x, y, z be the neighbors of u other than v with $d_G(x) = d_G(y) = d_G(z) = 4$. Let $H = G - uv$. Then H is a normal graph with $\Delta(H) \leq 5$ and $|E(H)| < |E(G)|$. By the induction hypothesis, H has a 10-NDE-coloring ϕ with the color set $C = \{1, 2, \ldots, 10\}$. Assume that $\phi(ux) = 1$, $\phi(uy) = 2$, and $\phi(uz) = 3$.

If $C_\phi(v) \neq \{1, 2, 3\}$, then we color uv with a color $a \in C \setminus (C_\phi(u) \cup C_\phi(v))$ such that u does not conflict with its three neighbors (other than v). Since $|C| = 10$, $|C_\phi(u)| = |C_\phi(v)| = 3$, the color a exists. Assume that $C_\phi(v) = \{1, 2, 3\}$. We recolor ux with a color in $\{4, 5, \ldots, 10\} \setminus C_\phi(x)$ such that x does not conflict with its three neighbors (other than u). Then the proof is reduced to the previous case. ■

Theorem 4.3 ([14]). *Let G be a normal graph with $\Delta(G) \geq 4$. Then there is an edge-partition of G into subgraphs $G_0, G_1, \ldots, G_k$, $k \leq \lfloor \Delta(G)/2 \rfloor - 2$, such that the following statements hold.*

(1) Every G_i is a normal subgraph.

(2) $\Delta(G_i) \leq 3$ for $1 \leq i \leq k$.

(3) $\Delta(G_0) \leq 5$.

Theorem 4.4. *For a normal graph G, $\chi_a'(G) \leq 2.5\Delta(G)$.*

Proof. Since G is normal, we assume that $\Delta(G) \geq 2$. If $\Delta(G) = 2$, then $\chi_a'(G) \leq 5 = 2.5\Delta(G)$. If $\Delta(G) = 3$, then $\chi_a'(G) \leq 5 < 2.5\Delta(G)$ by Theorem 2.4. If $\Delta(G) = 4$, then $\chi_a'(G) \leq 8 < 2.5\Delta(G)$ by Theorem 2.5. If $\Delta(G) = 5$, then $\chi_a'(G) \leq 10 < 2.5\Delta(G)$ by Theorem 4.2. Now assume that $\Delta(G) \geq 6$. By Theorem 4.3, there is an edge-partition of G into subgraphs $G_0, G_1, \ldots, G_k$, $k \leq \lfloor \Delta(G)/2 \rfloor - 2$, such that the statements (1), (2) and (3) in Theorem 4.3 hold. Applying repeatedly Theorems 2.2, 2.4, 2.5 and 4.3, we have

$$\begin{aligned}
\chi_a'(G) &\leq \chi_a'(G_0) + \chi_a'(G_1) + \cdots + \chi_a'(G_k) \\
&\leq \chi_a'(G_0) + 5k \\
&\leq \chi_a'(G_0) + 5(\lfloor \Delta(G)/2 \rfloor - 2) \\
&\leq 10 + 5(\lfloor \Delta(G)/2 \rfloor - 2) \\
&\leq 2.5\Delta(G). \quad \blacksquare
\end{aligned}$$

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