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Some new resolvable GDDs with $k = 4$ and doubly resolvable GDDs with $k = 3$

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ABSTRACT

A doubly resolvable packing design with block size k, index λ, replication number r, and v elements is called a generalized Kirkman square and denoted by $\mathrm{GKS}_k(v; 1, \lambda; r)$. Existence of $\mathrm{GKS}_3(4u; 1, 1; 2(u-1))$s and $\mathrm{GKS}_3(6u; 1, 1; 3(u-1))$s is implied by existence of doubly resolvable group divisible designs with block size 3, index 1, and types 4^u and 6^u (i.e., (3, 1)-DRGDDs of types 4^u and 6^u). In this paper, we establish the spectra of (3, 1)-DRGDDs of types 4^u and 6^u with 15 and 31 possible exceptions, respectively. As applications, we get some new classes of permutation codes and doubly constant weight codes. We also construct 5 new resolvable GDDs with block size 4 and index 1.

1. Introduction

Let K be a set of positive integers and λ be a positive integer. A (K, λ) *group divisible design* (GDD) is a triple $(X, \mathcal{G}, \mathcal{B})$ where X is a set of points, $\mathcal{G}$ is a partition of X into subsets (called *groups*) and $\mathcal{B}$ is a collection of subsets of X with sizes in K (called *blocks*) such that (1) no two points in the same group lie in any block, and (2) any two points in different groups appear together in exactly λ blocks. If $\mathcal{G}$ contains t_i groups of size h_i for $i = 1, 2, \ldots, n$, then the GDD is said to have type $h_1^{t_1} h_2^{t_2} \ldots h_n^{t_n}$. If $K = \{k\}$, we more commonly write k instead of $\{k\}$.

A (v, K, λ)-PBD or *pairwise balanced design* is a (K, λ)-GDD of type 1^v; the parameter λ is sometimes omitted if $\lambda = 1$. Also, a (v, k, λ)-BIBD or *balanced incomplete block design* is a (k, λ)-GDD of type 1^v, and a $(k, 1)$-GDD of type h^k is usually called a *transversal design*, denoted as $\mathrm{TD}(k, h)$. It is well known that existence of a $\mathrm{TD}(k, v)$ is equivalent to existence of $k-2$ mutually orthogonal Latin squares (or MOLS) of order v. For known information on existence of TDs and MOLS, see [3].

A design D is called *resolvable* if it is possible to partition its blocks into classes $R_1, R_2, \ldots, R_t$ (called *parallel classes* or *resolution classes*) such that each point of the design lies in exactly one block of each class. The classes $R_1, R_2, \ldots, R_t$ are said to form a resolution of D. We use the notation RGDD for a resolvable group divisible design.

A design D is called *doubly resolvable* if it possesses two resolutions, with the extra property that each parallel class in the first resolution contains at most one block in common with each parallel class in the second resolution. We use the notation DRGDD for a doubly resolvable GDD.

Several authors have looked at RGDDs of type h^u. For $k = 3$ and $\lambda = 1$, existence of these designs has now been completely solved:

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Table 1
Values of $v \geq 18$ for which there is no known DRNKTS(v).

144	150	156	168	174	180	186	192	204	216	222	228	234	240	246	252
264	270	276	294	300	312	318	324	330	336	342	348	360	372	378	384
390	396	408	414	426	432	438	456	462	468	474	480	504	510	516	528
540	558	564	570	576	582	588	606	612	624	630	636	654	660	666	672

Theorem 1.1 (*[21,22]*). *There exists a* $(3, 1)$*-RGDD of type* h^u *if and only if* $u \geq 3$, $hu \equiv 0 \pmod 3$, $h(u-1)$ *is even, and* $(h, u) \notin \{(2, 3), (2, 6), (6, 3)\}$.

Recent progress has been made on existence of (4, 1)-RGDDs, but work still has to be done. The current state of affairs regarding these designs is summarized in the following theorem:

Theorem 1.2 (*[24]*). *Necessary conditions for existence of a* $(4, 1)$*-RGDD of type* h^u *are* $u \geq 4$, $hu \equiv 0 \pmod 4$, $h(u-1) \equiv 0 \pmod 3$ *and* $(h, u) \notin \{(2, 4), (2, 10), (3, 4), (6, 4)\}$. *The conditions are sufficient, except possibly in the following cases:*

(1) $h \equiv 2$ *or* $10 \pmod{12}$*: Either* (1a) $h = 2$ *and* $u \in \{34, 46, 52, 70, 82, 94, 100, 118, 130, 178, 202, 214, 238, 250, 334\}$, (1b) $h = 10$ *and* $u \in \{4, 34, 52, 94\}$, (1c) $h = 26$ *and* $u \in \{10, 70, 82\}$, *or* (1d) $h \in \{38, 58, 74, 82, 86, 94, 106\}$ *and* $u = 10$.

(2) $h \equiv 6 \pmod{12}$*:* $(h, u) \in \{(6, 6), (6, 68), (18, 38), (18, 62)\}$.

(3) $h \equiv 0 \pmod{12}$*:* $h = 36$ *and* $u \in \{14, 15, 18, 23\}$.

Much less is known about existence of doubly resolvable $(k, 1)$-GDDs of type h^u. For $k \geq 4$, there is almost no information at all available on these designs. In this paper, we examine the existence of such designs for $k = 3$ and $h = 4$ or 6. The cases $h = 1$ and 2 have already been examined in earlier papers. These DRGDDs are commonly called *doubly resolvable Kirkman triple systems* (denoted DRKTS(u)) when $h = 1$, or *doubly resolvable nearly Kirkman triple systems* (denoted DRNKTS($2u$)) when $h = 2$. The following two theorems summarize the known results for these two types of design:

Theorem 1.3 (*[2,4,9]*). *There exists a doubly resolvable* $(v, 3, 1)$*-BIBD (or a* DRKTS(v)*) for all* $v \equiv 3 \pmod 6$ *except for* $v \in \{3, 9, 15\}$ *and possibly for* $v \in \{21, 141, 153, 165, 177, 189, 231, 249, 261, 285, 357\}$.

Theorem 1.4 (*[2]*). *Let* V *be the set of* 64 *numbers listed in* Table 1. *There exists a* $(3, 1)$*-DRGDD of type* $2^{v/2}$ *(or* DRNKTS(v)*) whenever* $v \equiv 0 \pmod 6$ *and* $v \geq 18$*, except possibly for* $v \in V$. *There is no* DRNKTS(v) *when* $v = 6$ *or* 12.

Let k, λ, r, and v be positive integers. A *doubly resolvable packing design* or *generalized Kirkman square* with block size k, index λ, replication number r, and v elements, $GKS_k(v; 1, \lambda; r)$, is an $r \times r$ array S defined on a v-set X such that

(1) each cell of S is either empty or contains a k-set of X,
(2) every element of X occurs once in each row and column of S,
(3) each 2-subset of X is contained in at most λ k-element sets of S.

By simple argument, a necessary condition for the existence of a generalized Kirkman square $GKS_k(v; 1, \lambda; r)$ is $v/k \leq r \leq \lambda(v-1)/(k-1)$. When $r = \lambda(v-1)/(k-1)$, a generalized Kirkman square is a *Kirkman square*, and denoted by $KS_k(v; 1, \lambda)$. It is well known that existence of a doubly resolvable (v, k, λ)-BIBD is equivalent to existence of a Kirkman square $KS_k(v; 1, \lambda)$, see for instance [9,16,17]. There are tight connections between generalized Kirkman squares and doubly resolvable group divisible designs. It is clear that doubly resolvable group divisible designs are one kind of generalized Kirkman squares. A lot of work has been done on existence of generalized Kirkman squares with $k \in \{2, 3\}$. For $k = 2$, a generalized Kirkman square, $GKS_2(v; 1, 1; r)$, is more commonly known as a *Howell design*, and is denoted as H(r, v). A Howell design with $r = v - 1$, i.e. a $GKS_2(v; 1, 1; v-1)$ is called a *Room square* of order $v - 1$, or RS($v - 1$). Thus a Howell design is a generalization of a Room square, and a generalized Kirkman square is a generalization of a Howell design. In 1975, Mullin and Wallis [19] established the spectrum of Room squares. The existence of Howell designs has been completely determined by Stinson [23] in 1982 and Anderson et al. [6] in 1984. For $k = 3$, Lamken [16] established the spectrum for $KS_3(v; 1, 2)$s with six possible exceptions which were later solved in [4]. In [9], Colbourn et al. established the spectrum for $KS_3(v; 1, 1)$s with 23 of possible exceptions; later 11 of these were removed in [4] and one ($v = 351$) was removed in [2]. A summary of these exceptions was given earlier in Theorem 1.3. Also, in [2], the problem for the existence of doubly resolvable nearly Kirkman triple systems (or doubly resolvable (3, 1)-GDDs of type $2^{v/2}$, denoted as DRNKTS(v)) was studied. These are equivalent to $GKS_3(v; 1, 1; (v-2)/2)$s, and a summary of the 64 unknown cases is given in Theorem 1.4.

In this paper, we are mainly interested in the construction of generalized Kirkman squares with $k = 3$; a few new resolvable GDDs with $k = 4$ are also given. We will discuss the existence of $GKS_3(4u; 1, 1, 2(u-1))$s and $GKS_3(6u; 1, 1, 3(u-1))$s; these will be obtained from (3, 1)-DRGDDs of types 4^u and 6^u.

The material in this paper is organized as follows. First in Section 2, we improve Theorem 1.2, giving 5 new resolvable (4, 1)-GDDs. In Section 3, we describe some new direct constructions for (3, 1)-DRGDDs of types 4^u and 6^u. These are obtained by methods using starters and adders, and are of a similar nature to those in [2]. Our main recursive constructions use frames. In Section 4 we summarize the known existence results for (1, 1; 3)-frames of types 2^u, 4^u and 6^u. We also give some new (1, 1; 3)-frames of type 4^u and some general recursive constructions for (1, 1; 3)-frames and (3, 1)-DRGDDs.

In Sections 5 and 6, we establish the spectra of (3, 1)-DRGDDs of types 4^u and 6^u with 15 and 31 possible exceptions, respectively. Some of the recursive methods used in these sections are of a similar nature to those in [2], in particular, Lemmas 5.3 and 6.1 are quite similar to Theorem 4.6 in [2]. Finally, Section 7 gives their applications to permutation codes and doubly constant weight codes, and Section 8 gives a summary of our results.

2. Some new (4, 1)-RGDDs

In this section, we obtain five improvements on Theorem 1.2. Two of these are obtained directly, and three are obtained by a direct product construction and a filling holes construction. The direct constructions for types 2^{34} and 2^{52} in the next lemma are starter constructions, similar to those of types 2^{22}, 2^{28} and 2^{40} in [14].

Lemma 2.1. *There exists a* (4, 1)*-RGDD of type* 2^u *for* $u \in \{34, 52\}$.

Proof. For $u = 34$, the point set is $Z_{66} \cup (S = \{\infty_1, \infty_2\})$ and the groups are S and $\{x, x + 33\}$ for $0 \le x \le 32$. The base blocks given form a parallel class. Other parallel classes are obtained by developing the base parallel class by the repeated addition of 3 (mod 66).

$\{0, 9, 63, 1\}$, $\{46, 49, 55, 47\}$, $\{5, 8, 20, 6\}$, $\{15, 33, 60, 22\}$,
$\{7, 19, 43, 29\}$, $\{11, 35, 41, 48\}$, $\{24, 54, 10, 37\}$, $\{16, 31, 44, 65\}$,
$\{17, 26, 21, 45\}$, $\{51, 57, 25, 2\}$, $\{40, 61, 56, 30\}$, $\{32, 50, 42, 13\}$,
$\{3, 18, 64, 38\}$, $\{4, 52, 59, 27\}$, $\{23, 62, 39, 58\}$, $\{12, 28, 53, \infty_1\}$,
$\{36, 34, 14, \infty_2\}$.

For $u = 52$, the point set is Z_{104} and the groups are $\{x, x+52\}$ for $0 \le x \le 51$. The blocks of the RGDD are obtained by developing the following 17 base blocks by the repeated addition of 2 (mod 104). Adding either $0, 4, 8, \dots, 100$ or $2, 6, 10, \dots, 102$ to any one of the four blocks in the first row produces a parallel class. Also, adding 0 and 52 (mod 104) to the last 13 blocks produces another parallel class; the remaining parallel classes are obtained by adding $2, 4, 6, \dots, 50$ to this last one.

$\{0, 14, 61, 71\}$, $\{0, 46, 43, 65\}$, $\{0, 18, 27, 53\}$, $\{0, 38, 7, 49\}$,
$\{0, 24, 72, 75\}$, $\{78, 5, 33, 53\}$, $\{4, 27, 29, 93\}$, $\{2, 94, 98, 89\}$,
$\{14, 48, 64, 92\}$, $\{13, 25, 43, 59\}$, $\{16, 84, 97, 101\}$, $\{34, 36, 35, 103\}$,
$\{38, 44, 17, 31\}$, $\{22, 11, 55, 61\}$, $\{6, 70, 80, 21\}$, $\{8, 30, 50, 71\}$,
$\{10, 15, 39, 47\}$.

Lemma 2.2. *There exist* (4, 1)*-RGDDs of types* 10^{34}, 10^{52} *and* 2^{238}.

Proof. It is well known that existence of a $(k, 1)$-RGDD of type g^u and a resolvable TD(k, s) implies existence of a $(k, 1)$-RGDD of type $(gs)^u$. See for instance, Corollary 3.4.6 in [13]. Applying this corollary with $k = 4, g = 2, s = 5$ and $u \in \{34, 52\}$ gives (4, 1)-RGDDs of types 10^{34} and 10^{52}. For type 2^{238}, start with a (4, 1)-RGDD of type 68^7 which exists by Theorem 1.2. Filling in the groups of size 68 with (4, 1)-RGDDs of type 2^{34} now gives the required RGDD. □

With these new results, Theorem 1.2 can now be updated as follows:

Theorem 2.3. *Necessary conditions for existence of a* (4, 1)*-RGDD of type* h^u *are* $u \ge 4$, $hu \equiv 0 \pmod 4$, $h(u-1) \equiv 0 \pmod 3$ *and* $(h, u) \notin \{(2, 4), (2, 10), (3, 4), (6, 4)\}$. *The conditions are sufficient, except possibly in the following cases:*

(1) $h \equiv 2$ *or* $10 \pmod{12}$*: Either* (1a) $h = 2$ *and* $u \in \{46, 70, 82, 94, 100, 118, 130, 178, 202, 214, 250, 334\}$, (1b) $h = 10$ *and* $u \in \{4, 94\}$, (1c) $h = 26$ *and* $u \in \{10, 70, 82\}$, *or* (1d) $h \in \{38, 58, 74, 82, 86, 94, 106\}$ *and* $u = 10$.

(2) $h \equiv 6 \pmod{12}$*:* $(h, u) \in \{(6, 6), (6, 68), (18, 38), (18, 62)\}$.

(3) $h \equiv 0 \pmod{12}$*:* $h = 36$ *and* $u \in \{14, 15, 18, 23\}$.

3. Direct constructions for (3, 1)-DRGDDs of types 4^u and 6^u

In this section, we will apply standard "starter–adder" method to construct some (3, 1)-DRGDDs with small orders, which will be used as input designs in recursive constructions of Section 4. Instead of listing all the blocks and the parallel classes of the desired designs, we only list the starters and adders of the first resolution class and its orthogonal resolution class. For more information on the "starter–adder" method, the reader is referred to [2,4,7,9,15].

Lemma 3.1. *There exist* (3, 1)*-DRGDDs of type* 4^u *for* $u \equiv 0 \pmod 3$ *and* $3 \le u \le 30$.

Proof. For $u = 3$, the required design comes from a TD(5, 4) [3]. The points in the last two groups can be deleted and used to define parallel classes for the two resolutions, giving a doubly resolvable TD(3, 4). For $u = 27$, a construction will be obtained later (using Lemma 5.2 with $n = 4$). For other u, the given designs are over $\mathbb{Z}_{4u-4} \cup (S = \{\infty_1, \dots, \infty_4\})$. Groups for these DRGDDs are S and $\{i, i + (u - 1), i + 2(u - 1), i + 3(u - 1)\}$ for $0 \le i \le u - 2$. An initial parallel class for the first resolution (consisting of a number of starter blocks) is given in the Appendix, Table 8; for the orthogonal resolution,

							ijbk	adfg			cleh	
								jkcl	begh			dmfi
eagj									kldm	cfhi		
	fbhk									lmea	dgij	
		gcil									mafb	ehjk
fikl			hdjm									abgc
bchd	gjlm			ieka								
	cdie	hkma			jflb							
		dejf	ilab			kgmc						
			efkg	jmbc			lhad					
				fglh	kacd			mibe				
					ghmi	lbde			ajcf			
						hiaj	mcef			bkdg		

Fig. 1. A (1, 3; 4)-frame of type 1^{13}.

the initial parallel class is obtained by adding the given adders (mod $4u - 4$) to the starter blocks, while keeping the infinite points fixed. For both the first resolution and its orthogonal resolution, the remaining $2u - 3$ parallel classes are obtained by developing the initial parallel class by the repeated addition of 2 (mod $4u - 4$) to the non-infinite points, while keeping the infinite points fixed. □

Lemma 3.2. *There exists a* $(3, 1)$*-DRGDD of type* 6^u *for* $6 \le u \le 20$.

Proof. These are constructed in a similar manner to those for $u \ge 4$ in the previous lemma. Here the given designs are over $\mathbb{Z}_{6u-6} \cup (S = \{\infty_1, \ldots, \infty_6\})$, and groups for the DRGDD are S and $\{i, i + (u-1), i + 2(u-1), i + 3(u-1), i + 4(u-1), i + 5(u-1)\}$ for $0 \le i \le u - 2$. An initial parallel class of starter blocks (or starter blocks plus their adders for the orthogonal resolution) is given in the Appendix, Table 9. The remaining $3u - 4$ parallel classes for both the first and orthogonal resolutions are obtained by repeated addition of 2 (mod $6u - 6$) to the blocks in the initial parallel class. □

4. Frames

Let V be a set of v elements. Let $G_1, G_2, \ldots, G_m$ be a partition of V into m sets. For each G_i, $i = 1, 2, \ldots, m$, let $g_i = |G_i|$, let T_i be a set of size $t_i = (\lambda g_i)/(\mu(k-1))$, and let $t = \sum_{i=1}^{m} t_i$. A $\{G_1, G_2, \ldots, G_m\}$-frame F with block size k, index λ, and latinicity μ is a square array A of side $t = (\lambda v)/(\mu(k-1))$ which satisfies the properties listed below.

(1) Each cell is either empty or contains a k-subset of V.
(2) The rows and columns of A are indexed by the elements of $\cup_{i=1}^{m} T_i$.
(3) For each $i = 1, 2, \ldots, m$, the subsquare with row and column indices from T_i is empty (i.e. the main diagonal of A consists of empty subsquares of sides $t_i \times t_i$ for $i = 1, 2, \ldots, m$). Each row of A with index from T_i contains each element of $V - G_i$ μ times. So does each column of A with index from T_i.
(4) The blocks obtained from the nonempty cells of F form a (k, λ)-GDD of type $(g_1, g_2, \ldots, g_m)$.

Such a square array is usually called a $(\mu, \lambda; k)$-frame of type $(g_1, g_2, \ldots, g_m)$. As with GDDs, when several groups have identical sizes, exponential notation is usually used to describe the type; thus a $(\mu, \lambda; k)$-frame is said to have type $s_1^{u_1} s_2^{u_2} \ldots s_\ell^{u_\ell}$ if there are u_i G_j's of cardinality s_i, $1 \le i \le \ell$. In this paper, most $(\mu, \lambda; k)$-frames will have all G_i's of the same size; in this case, if there are u G_i's and $|G_i| = h$ for all i, the array A is usually called a $(\mu, \lambda; k)$-frame of type h^u.

As an example, a $(1, 3; 4)$-frame of type 1^{13} from [1] is displayed in Fig. 1. The underlying $(13, 4, 3)$-BIBD is cyclic over Z_{13}, and the values $1, 2 \ldots 12, 0$ are relabelled as $a, b, c, \ldots, m$.

The frames of most use to us in this paper are $(1, 1; 3)$-frames of types 2^u, 4^u and 6^u. Here (or whenever $\mu = 1$), the blocks in any row or column with index from T_i form a partial parallel class missing just the points in group G_i. When giving direct constructions for these frames, we indicate how to obtain these partial parallel classes in the appropriate GDDs, and indicate how to obtain the corresponding square arrays A, but will not display these large square arrays.

We frequently use $(1, 1; 3)$-frames of types 2^u, 4^u and 6^u to obtain $(1, 1; 3)$-frames with groups of larger sizes. (Existence of a $(1, 1; 3)$-frame of type 2^{3n+1} is equivalent to that of a doubly resolvable $(6n + 3, 3, 1)$-BIBD; see for instance, Theorem 4.4 in [9]. As a result Theorem 4.1 can be considered a corollary of Theorem 1.3.) The next three theorems give the known existence results for frames of these three types.

Theorem 4.1 ([2,4,9]). *Necessary conditions for existence of a $(1, 1; 3)$ frame of type 2^u are $u \geq 10$ and $u \equiv 1 \pmod 3$. These conditions are sufficient, except possibly for $u \in \{10, 70, 76, 82, 88, 94, 115, 124, 130, 142, 178\}$.*

Theorem 4.2 ([2,4,9]). *There exists a $(1, 1; 3)$-frame of type 6^u for $7 \leq u \leq 14$, for $u \in \{19, 31, 49, 50, 56, 57, 58\}$, and for all $u \geq 63$.*

The frames of type 4^u in the next theorem are obtained by the starter–adder method which was also used in [2,4,7,9,15]. Those of types 4^7 and 4^{10} can also be found in [7].

Theorem 4.3. *There exists a $(1, 1; 3)$-frame of type 4^u for $u \in \{7, 10, 13, 16, 19, 22, 25, 31\}$.*

Proof. For $u \in \{7, 10, 13, 16, 22, 25\}$, a $(1, 1; 3)$-frame of type 4^u is given in Table 2 with point set Z_{4u} and groups of the form $\{2i, 2i + 1, 2i + 2u, 2i + 2u + 1\}$ for $i = 0, 1, 2, \ldots, u - 1$. In each case, $4(u - 1)/3$ starter blocks and their adders $\notin \{0, 2u\}$ are listed in Table 2. These $4(u - 1)/3$ starter blocks (or starter block plus their adders for the orthogonal partial resolution) form an initial partial parallel class missing the group $\{0, 1, 2u, 2u + 1\}$. The remaining $2u - 1$ partial parallel classes are obtained by adding $2, 4, 6, \ldots, 4u - 2 \pmod{4u}$ to the initial one.

For $u = 19$ and 31, $(1, 1; 3)$-frames of type 4^u over Z_{4u} are constructed similarly, but here, groups are of the form $\{2i, 2i + u, 2i + 2u, 2i + 3u\}$ for $i = 0, 1, 2, \ldots, u - 1 \pmod{4u}$. In each case, 8 initial starter blocks and their adders are listed in Table 3. The remaining starter blocks and their corresponding adders are obtained by multiplying each of the 8 initial starter blocks (and their adders) by 45 and 49 (mod 76) when $u = 19$, or by 33, 97, 101 and 109 (mod 124) when $u = 31$.

Finally, to obtain the square array A for each frame of type 4^u, let $T = \{0, 2, 4, \ldots, 4u - 2\}$, and $T_i = \{2i, 2i + 2u\}$ for $i = 0, 1, 2, \ldots, u - 1$. The square array A of side $2u$ will have row and column indices from T, and each starter block B_j with adder a_j will be placed in the $(0, 4u - a_j)$ cell of A. With the arithmetic done (mod $4u$), any block of the form $B_j + 2x$ $(2x \in T)$ will be placed in the $(2x, 4u - a_j + 2x)$ cell of A. In particular, the blocks $B_j + a_j$ will all lie in the first column of A (with index 0). We also note that the reason we cannot use $a_j = 0$ or $2u$ as an adder for any base block B_j is that the blocks B_j and $B_j + a_j$ would then lie in the subsquare with row and column indices from $T_0 = \{0, 2u\}$, which is not allowed. □

To obtain existence results for $(1, 1; 3)$-frames with group sizes larger than 6, we use two standard recursive constructions. The first is a 'Direct Product Construction' and is obtained by inflating a $(1, 1; 3)$-frame with a doubly resolvable $TD(3, s)$ (whose existence is equivalent to that of a $TD(5, s)$). The second is the 'Fundamental Construction' for frames, and is obtained by inflating a GDD with $(1, 1; 3)$-frames. The proofs are like those for similar constructions in [7,15,25].

Theorem 4.4. *Suppose there exist a $TD(5, s)$ and a $(1, 1; 3)$-frame of type $t_1^1 t_2^1 \ldots t_m^1$. Then there exists a $(1, 1; 3)$-frame of type $(st_1)^1 (st_2)^1 \ldots (st_m)^1$.*

Theorem 4.5. *Let $(X, \mathscr{G}, \mathscr{B})$ be a GDD, and let $w : X \to Z^+ \cup \{0\}$ be a weight function on X. Suppose that for each block $B \in \mathscr{B}$ there exists a $(1, 1; 3)$-frame of type $(w(x) \mid x \in B)$. Then there exists a $(1, 1; 3)$-frame of type $(\sum_{x \in G_i} w(x) \mid G_i \in \mathscr{G})$.*

Finally, we give a construction which is frequently used to obtain $(3, 1)$-DRGDDs from a frame by filling in the holes of the frame with smaller $(3, 1)$-DRGDDs. Its proof is similar to that of Theorem 3.6 in [16].

Theorem 4.6. *Suppose there exists a $(1, 1; 3)$-frame of type $(ht_1)(ht_2)\ldots(ht_j)$. Suppose $t > 0$, and for all $i = 1, 2, \ldots, j - 1$, there exists a $(3, 1)$-DRGDD of type $h^{t_i + t}$ (which contains a sub-$(3, 1)$-DRGDD of type h^t if $t > 1$). Then if a $(3, 1)$-DRGDD of type $h^{t_j + t}$ exists, there also exists a $(3, 1)$-DRGDD of type h^u, where $u = (\sum_{i=1}^{j} t_i) + t$. Furthermore, if the $(3, 1)$-DRGDD of type $h^{t_j + t}$ contains a sub-$(3, 1)$-DRGDD of type h^t, then so does the final $(3, 1)$-DRGDD of type h^u.*

5. Existence of $(3, 1)$-DRGDDs of type 4^u

In this section, we use the recursive constructions in the previous section to determine the existence of $(3, 1)$-DRGDDs of type 4^u with at most 15 possible exceptions. We first take care of some values of $u \leq 168$.

Lemma 5.1. *There exists a $(3, 1)$-DRGDD of type 4^u for $u \in \{36, 66, 78, 96, 120, 126, 141, 144, 153, 156, 162\}$.*

Proof. For $u = 36, 78, 120, 141, 162$, start with a $(1, 1; 3)$-frame of type 4^7 from Theorem 4.3, and apply Theorem 4.4 with $s = 5, 11, 17, 20, 23$ to obtain a $(1, 1; 3)$-frame of type $(4s)^7$. Since the required $(3, 1)$-DRGDDs of type 4^{s+1} exist by Lemma 3.1, Theorem 4.6 with $h = 4$, $t = 1$ can now be applied to obtain the desired designs. Similarly, for $u = 66, 96, 126, 144, 153, 156$, we start with $(1, 1; 3)$-frames of types 4^{13}, 4^{19}, 4^{25}, 4^{13}, 4^{19}, 4^{31} from Theorem 4.3, and apply Theorem 4.4 with $s = 5, 5, 5, 11, 8$ and 5 respectively, to obtain $(1, 1; 3)$-frames of types 20^{13}, 20^{19}, 20^{25}, 44^{13}, 32^{19}, 20^{31}. The required fill in designs, i.e. $(3, 1)$-DRGDDs of types 4^6, 4^{12} and 4^9 all exist by Lemma 3.1, so we can again apply Theorem 4.6 with $h = 4$, $t = 1$ to obtain the desired designs. □

Table 2
Starter blocks and adders for (1, 1; 3)-frames of types $4^7, 4^{10}, 4^{13}, 4^{16}, 4^{22}, 4^{25}$.

u	Starter	Adder	Starter	Adder	Starter	Adder	Starter	Adder
7	{2, 4, 26}	4	{12, 24, 7}	26	{13, 17, 19}	18	{8, 25, 5}	16
	{10, 18, 23}	2	{22, 21, 3}	24	{6, 16, 9}	10	{20, 27, 11}	12
10	{2, 8, 18}	10	{6, 11, 13}	16	{12, 16, 38}	34	{34, 29, 33}	30
	{7, 17, 31}	18	{32, 19, 25}	12	{28, 30, 3}	8	{14, 37, 5}	2
	{36, 4, 15}	38	{26, 23, 35}	22	{10, 22, 39}	4	{24, 9, 27}	6
13	{2, 4, 18}	18	{38, 6, 41}	24	{12, 16, 46}	44	{10, 34, 7}	32
	{29, 33, 45}	12	{44, 13, 15}	36	{3, 11, 21}	4	{50, 43, 49}	20
	{14, 20, 31}	30	{24, 5, 19}	16	{22, 30, 35}	46	{28, 17, 37}	6
	{32, 42, 9}	22	{40, 25, 47}	8	{48, 8, 39}	50	{36, 51, 23}	48
16	{52, 54, 2}	54	{56, 22, 63}	24	{6, 10, 26}	12	{40, 27, 29}	30
	{16, 24, 62}	38	{60, 5, 9}	8	{7, 17, 47}	2	{50, 35, 41}	10
	{53, 3, 25}	50	{38, 13, 21}	14	{28, 34, 51}	44	{30, 49, 61}	58
	{8, 18, 23}	40	{14, 43, 59}	26	{20, 42, 45}	56	{58, 37, 55}	34
	{44, 4, 15}	46	{46, 19, 39}	28	{48, 12, 11}	18	{36, 31, 57}	48
22	{10, 12, 48}	74	{74, 14, 27}	34	{42, 50, 60}	60	{26, 56, 87}	56
	{22, 34, 54}	46	{86, 32, 17}	58	{53, 57, 83}	62	{84, 36, 59}	78
	{3, 9, 43}	84	{66, 20, 37}	10	{5, 13, 25}	8	{58, 65, 67}	12
	{15, 33, 61}	80	{82, 31, 41}	66	{24, 28, 21}	30	{78, 55, 69}	16
	{40, 46, 29}	6	{62, 7, 23}	36	{76, 2, 71}	40	{72, 77, 11}	52
	{80, 8, 47}	70	{6, 49, 73}	32	{30, 52, 81}	76	{16, 19, 51}	48
	{68, 4, 79}	4	{18, 39, 75}	86	{38, 64, 63}	28	{70, 85, 35}	68
25	{52, 54, 96}	30	{30, 64, 67}	44	{34, 38, 60}	78	{90, 26, 47}	68
	{6, 36, 74}	4	{86, 32, 63}	36	{28, 40, 68}	2	{8, 56, 41}	90
	{31, 37, 65}	48	{48, 21, 23}	18	{79, 87, 25}	8	{62, 45, 49}	14
	{81, 91, 7}	38	{58, 39, 53}	22	{83, 95, 19}	98	{66, 75, 93}	62
	{14, 20, 55}	34	{10, 89, 9}	82	{16, 24, 13}	40	{70, 77, 99}	66
	{92, 2, 61}	60	{82, 5, 35}	42	{84, 98, 3}	20	{12, 27, 59}	76
	{72, 88, 33}	72	{46, 71, 11}	86	{76, 94, 43}	26	{44, 15, 57}	52
	{22, 42, 85}	64	{18, 29, 73}	54	{80, 4, 97}	10	{78, 69, 17}	56

Table 3
8 initial starter blocks and adders for (1, 1; 3)-frames of types $4^{19}, 4^{31}$.

u	Starter	Adder	Starter	Adder	Starter	Adder	Starter	Adder
19	{36, 42, 62}	4	{30, 34, 69}	2	{3, 23, 9}	46	{4, 33, 73}	30
	{48, 12, 45}	8	{2, 29, 11}	72	{46, 68, 15}	52	{40, 31, 35}	62
31	{40, 70, 74}	44	{106, 18, 53}	46	{47, 103, 117}	22	{64, 97, 45}	18
	{84, 28, 41}	8	{122, 85, 3}	56	{14, 116, 91}	20	{12, 7, 11}	66

Our main construction makes use of sub-designs. Before using it, we require a class of (3, 1)-DRGDDs of type 4^u which contain a sub-(3, 1)-DRGDD of type 4^3.

Lemma 5.2. *For all $n \geq 4$, there exists a (3, 1)-DRGDD of type 4^{6n+3} containing a sub-(3, 1)-DRGDD of type 4^3, except possibly for $n \in N^* = \{23, 25, 27, 29, 31, 38, 41, 43, 47, 59\}$. In particular, this DRGDD exists for all integers $n \equiv 0 \pmod 4$, $n \geq 4$.*

Proof. Apply Theorem 4.4 with $s = 4$ to a (1, 1; 3)-frame of type 2^{3n+1} from Theorem 4.1. This gives a (1, 1; 3)-frame of type 8^{3n+1}. Apply Theorem 4.6 with $t = 1$, filling in each hole with 4 extra points, using a (3, 1)-DRGDD of type 4^3 (which exists by Lemma 3.1). The resulting design is a (3, 1)-DRGDD of type 4^{6n+3} containing a sub-(3, 1)-DRGDD of type 4^3. □

Our main construction for (3, 1)-DRGDDs of type 4^u is based on the following lemma.

Lemma 5.3. *Suppose there exists a TD$(7 + j, 4m)$. Let m_i $(i = 1, 2, \ldots, j)$ be integers such that $0 \leq m_i \leq m$ for $i = 1, 2, \ldots, j - 1$, and $0 \leq m_j \leq 2m$. Suppose also, there exist*

(1) *(1, 1; 3)-frames of type 6^{7+i} for $i = 0, 1, \ldots, j$,*
(2) *a (3, 1)-DRGDD of type 4^{6m+3} containing a sub-(3, 1)-DRGDD of type 4^3,*
(3) *a (3, 1)-DRGDD of type 4^{6m_i+3} containing a sub-(3, 1)-DRGDD of type 4^3 for $i = 1, 2, \ldots, j - 1$,*
(4) *a (3, 1)-DRGDD of type 4^{3m_j+3}.*

Then there exists a (3, 1)-DRGDD of type 4^u where $u = 42m + 6(\sum_{i=1}^{j-1} m_i) + 3m_j + 3$.

Proof. Truncate j groups of a TD$(7+j, 4m)$ to sizes $4m_i$ (for $i = 1, 2, \ldots, j-1$) and $2m_j$ where $0 \leq m_i \leq m$ for $1 \leq i \leq j-1$ and $0 \leq m_j \leq 2m$. This gives a $(K, 1)$-GDD of type $(4m)^7(4m_1)^1(4m_2)^1 \cdots (4m_{j-1})^1(2m_j)^1$ where $K = \{7, 8, \ldots, 7+j\}$. We use Theorem 4.5 with $w(x) = 6$ to obtain a (1, 1; 3)-frame of type $(24m)^7(24m_1)^1\,(24m_2)^1 \cdots (24m_{j-1})^1(12m_j)^1$. Now we apply Theorem 4.6, filling in the holes of this frame with 12 extra points, using the appropriate (3, 1)-DRGDDs. □

Table 4
Range for u covered by each m in Lemma 5.4.

m	$TD(7+j, 4m)$	Range for u
4	TD(14, 16)	$171 \le u \le 339$
8	TD(14, 32)	$339 \le u \le 651$
12	TD(10, 48)	$507 \le u \le 675$
16	TD(14, 64)	$675 \le u \le 1275$
28	TD(14, 112)	$1179 \le u \le 2211$
52	TD(14, 208)	$2187 \le u \le 4083$

Table 5
Values of $u \ge 3$, $u \equiv 0 \pmod 3$ for which no (3, 1)-DRGDD of type 4^u is known.

42	48	54	60	72	84	90	102	108	114	132
138	150	165	168							

Lemma 5.4. *Suppose $m \ge 4$, $m \equiv 0 \pmod 4$, $2 \le j \le 7$ and a $TD(7+j, 4m)$ exists. Then there exists a (3, 1)-DRGDD of type 4^u for $u \equiv 0 \pmod 3$ and $42m+3 \le u \le 6(7+j-1)m+27$. In particular if a $TD(14, 4m)$ exists, this (3, 1)-DRGDD exists for $42m+3 \le u \le 78m+27$.*

Proof. Apply Lemma 5.3. The conditions of the lemma imply that for some integer s in the range $[1, j-1]$, u lies in the range $[42m+6(s-1)m+3, 42m+6sm+27]$. If so, write $u = 42m+6(s-1)m+6a+3b$ where $0 \le a \le m$, $a \equiv 0 \pmod 4$, and $0 \le b \le 8$ ($\le 2m$ as $m \ge 4$). Apply Lemma 5.3, with $j = s+1$, $m_i = m$ for $i \le s-1$, $m_s = a$ and $m_{s+1} = b$. From Lemma 5.2, since both $m, a \equiv 0 \pmod 4$, there exist 3-DRGDDs of types 4^{6m+3} and 4^{6a+3}, both of which contain a sub-(3, 1)-DRGDD of type 4^3. Also since $b \le 8$, a (3, 1)-DRGDD of type 4^{3b+3} exists from Lemma 3.1. □

Table 4 gives some ranges for u for which a (3, 1)-DRGDD of type 4^u can be obtained by Lemma 5.4.

Lemma 5.5. *There exists a (3, 1)-DRGDD of type 4^u for all $u \equiv 0 \pmod 3$, $u \ge 171$.*

Proof. For $171 \le u \le 4083$, see Table 4. For $u > 4083$, there exist at least 9 consecutive integers n such that u lies in the range $[42 \cdot 4n+3, 78 \cdot 4n+27]$. At most three of these values of n are divisible by 3, two by 5, two by 7 and one by 11, so at least one of these 9 values, say n^*, will not be divisible by any of 3, 5, 7 or 11. Hence a $TD(14, 16n^*)$ exists. We can therefore apply Lemma 5.4 with $m = 4n^*$ to obtain the desired (3, 1)-DRGDD of type 4^u. □

The main result of this section can now be summarized in Theorem 5.6 which is obtained by combining Lemmas 3.1, 5.1 and 5.5.

Theorem 5.6. *Let M be the set of 15 numbers listed in Table 5. There exists a (3, 1)-DRGDD of type 4^u for $u \ge 3$ and $u \equiv 0 \pmod 3$, except possibly for $u \in M$.*

6. Existence of (3, 1)-DRGDDs of type 6^u

In this section we will determine the spectrum of (3, 1)-DRGDDs of type 6^u with at most 31 possible exceptions for u.

Lemma 6.1. *Let $1 \le j \le 7$, and $7 \le n \le 19$. Suppose there exists a $TD(7+j, n)$, and either (1) $j = 1$ or (2) $9 \le n \le 19$. Then there exists a (3, 1)-DRGDD of type 6^u for $7n+6 \le u \le (7+j)n+1$.*

Proof. We can write $u = 7n+m_1+m_2+\cdots+m_j+1$ where either $m_i = 0$ or $5 \le m_i \le n$ for $i = 1, \ldots, j$. Truncate j groups of a $TD(7+j, n)$ to sizes m_i (for $i = 1, 2, \ldots, j$) to get a $\{7, 8, \ldots, 7+j\}$-GDD of type $n^7 m_1^1 m_2^1 \cdots m_j^1$. By Theorem 4.2, there exist (1, 1; 3)-frames of type 6^{7+i} for $i = 0, 1, \ldots, 7$. Therefore, we can apply Theorem 4.5, giving weight $w(x) = 6$ to all points in the resulting GDD to obtain a (1, 1; 3)-frame of type $(6n)^7(6m_1)^1(6m_2)^1, \ldots, (6m_j)^1$. Also, there exist (3, 1)-DRGDDs of types 6^{n+1} and 6^{m_i+1} by Lemma 3.2, since by assumption, n and m_i are either zero or in the range [5, 19]. Therefore we can apply Theorem 4.6 with $h = 6$, $t = 1$ to obtain the required DRGDD. □

Lemma 6.2. *If a $(v, \{7, 8, 9\})$-PBD exists, then a (3, 1)-DRGDD of type 6^v also exists.*

Proof. Deleting one point from the PBD gives a {7, 8, 9}-GDD on $v-1$ points with group sizes in {6, 7, 8}. Applying Theorem 4.5, using (1, 1; 3)-frames of types 6^7, 6^8, 6^9, gives a (1, 1; 3) frame on $6(v-1)$ points with group sizes in {36, 42, 48}. Since (3, 1)-DRGDDs of type 6^q exist for $q = 7, 8, 9$, we can apply Theorem 4.6 (with $h = 6$, $t = 1$) to obtain the required (3, 1)-DRGDD of type 6^v. □

Lemma 6.3. *There exists a (3, 1)-DRGDD of type 6^u for $u \in \{36, 41, 43, 46, 49, 50, 51, 58, 61, 66, 67, 68\}$.*

Proof. For $u \in \{49, 58, 66, 67, 68\}$, we can apply Lemma 6.2, since a $(u, \{7, 8, 9\})$-PBD exists [18].

For $u = 36, 41, 46, 51$ and 61, we use Theorem 4.4 with $t = 6$, $m = (u-1)/5 \in \{7, 8, 9, 10, 12\}$ and $s = 5$ to obtain a (1, 1; 3)-frame of type 30^m. Note the required (1, 1; 3)-frames of type 6^m all exist by Theorem 4.2. We can now apply

Table 6
Range for u covered by each n in Lemma 6.4.

n	TD$(7+j, n)$	Range for u
7	TD(8, 7)	$55 \le u \le 57$
8	TD(8, 8)	$62 \le u \le 65$
9	TD(10, 9)	$69 \le u \le 91$
11	TD(12, 11)	$83 \le u \le 133$
16	TD(14, 16)	$118 \le u \le 225$

Table 7
Values of $u \ge 4$ for which no (3, 1)-DRGDD of type 6^u is known.

4	5	21	22	23	24	25	26	27	28	29	30	31	32	33	34
35	37	38	39	40	42	44	45	47	48	52	53	54	59	60	

Theorem 4.6 with $h = 6, t = 1$ to obtain the required (3, 1)-DRGDD of type 6^u, since a (3, 1)-DRGDD of type 6^6 exists by Lemma 3.2.

Similarly, for $u = 43$ and 50, we apply Theorem 4.4 with $t = 4, m = 7, s = 9$, and $t = 6, m = 7, s = 7$ respectively, to obtain (1, 1; 3)-frames of types 36^7 and 42^7. Now apply Theorem 4.6 to these two frames, filling in their groups with 6 extra points, using (3, 1)-DRGDDs of types 6^7 and 6^8. □

Lemma 6.4. *There exists a* (3, 1)*-DRGDD of type* 6^u *for u in the ranges* [55, 57], [62, 65], [69, 225].

Proof. Apply Lemma 6.1. Table 6 gives the range for u that can be handled by each value of n. □

Lemma 6.5. *There exists a* (3, 1)*-DRGDD of type* 6^u *for* $225 \le u \le 350$.

Proof. Write $u = 217 + (m_1 + \cdots + m_7) + 1$ where either $m_i \in \{0, 31\}$ or $7 \le m_i \le 14$ for all $i = 1, 2, \ldots, 7$. (When $225 \le u \le 316$, we can take $m_i = 0$ or $7 \le m_i \le 14$ for $1 \le i \le 7$. When $315 \le u \le 350$, we can take $m_1 = m_2 = 31$ and either $m_i = 0$ or $7 \le m_i \le 14$ for $3 \le i \le 7$.) Start with a TD(14, 31) containing a parallel class, and truncate 7 of its groups to sizes $m_1, \ldots, m_7$. Now take the (truncated) blocks in the parallel class as groups and the groups of sizes $31, m_1, \ldots, m_7$ as blocks. Since $\{m_1, m_2, \ldots, m_7\} \subset \{0\} \cup S$ where $S = \{7, 8, \ldots, 14\} \cup \{31\}$, this gives a GDD on $u - 1$ points with groups sizes in $\{7, 8, \ldots, 14\}$ and block sizes in S, Since we have (1, 1; 3)-frames of type 6^s for all $s \in S$ (see Theorem 4.2), we can apply Theorem 4.5, inflating this design by 6 to obtain a (1, 1; 3)-frame on $6(u - 1)$ points with groups sizes in $\{42, 48, \ldots, 84\}$. Finally apply Theorem 4.6 with $t = 1$, filling in each group with 6 extra points, using (3, 1)-DRGDDs of type 6^q for $8 \le q \le 15$ (these exist by Lemma 3.2). □

Lemma 6.6. *There exists a* (3, 1)*-DRGDD of type* 6^u *for all* $u \ge 343$.

Proof. This follows from Lemma 6.2, since a $(u, \{7, 8, 9\})$-PBD exists for all $u \ge 343$ [18]. □

We are now in a position to give the main result of this section.

Theorem 6.7. *Let N be the set of* 31 *numbers listed in Table 7. There exists a* (3, 1)*-DRGDD of type* 6^u *for* $u > 3$ *except possibly for* $u \in N$. *There is no* (3, 1)*-DRGDD of type* 6^3.

Proof. By Theorem 1.1, there is no (3, 1)-DRGDD of type 6^3. Combining Lemmas 3.2 and 6.3–6.6 gives the desired designs. □

7. Applications

7.1. Permutation codes

Let Z_q denote the set $\{0, 1, \ldots, q - 1\}$ (alphabet), and let Z_q^n be the set of all n-tuples (*codewords*) over Z_q, where q is a positive integer. An $(n, M, d, w)_q$ *constant weight code* (CWC) is a code $\mathscr{C} \subseteq Z_q^n$ consisting of M codewords such that (1) the Hamming weight of each codeword is exactly w and (2) the minimum Hamming distance between any 2 codewords is d. An $(n, M, d, [w_0, w_1, \ldots, w_{q-1}])_q$ *constant composition code* (CCC) is a code $\mathscr{C} \subseteq Z_q^n$ with M codewords and minimum Hamming distance d between 2 codewords such that in every codeword the element i appears exactly w_i times for every $i \in Z_q$. An $(n, M, d, [w_0, w_1, \ldots, w_{q-1}])_q$-CCC is called a *permutation code* or *permutation array*, denoted by (n, M, d)-PA if $n = q$ and $w_i = 1$ for all i. Hence, permutation codes are a special class of CWCs. CCCs are a subclass of CWCs. Permutation arrays have been applied in the design of block ciphers (see, for instance, [10]) and data transmission over power lines (see, for example, [20]). In 2004, Colbourn, Kløve, and Ling [8] developed a connection between permutation arrays and generalized Room square packings. In 2005, Ding and Yin [11] presented a link between CCCs and generalized double resolvable packing designs. As a direct corollary of their results, we have

Lemma 7.1. *If a* $\mathrm{GKS}_k(v;1,\lambda;r)$ *exists, then a* $(r,v,r-\lambda,[1,1,\ldots,1])_r$*-CCC (i.e., an* $(r,v,r-\lambda)$*-PA) also exists.*

Combining Lemma 7.1 and Theorems 5.6 and 6.7 gives the following two results about permutation codes.

Theorem 7.2. *Let* M *be the set of* 15 *numbers listed in Table 5. For* $u\geq 3$, $u\equiv 0\pmod 3$, *and* $u\notin M$, *there exists an* $(r,4u,r-1)$*-PA, where* $r=2(u-1)$.

Theorem 7.3. *Let* N *be the set of* 31 *numbers listed in Table 7. For* $u>3$ *and* $u\notin N$, *there exists an* $(r,6u,r-1)$*-PA, where* $r=3(u-1)$.

7.2. Doubly constant weight codes

A *doubly constant weight code* of length n and weight w is a constant weight binary code in which each codeword has length n and weight w, with the extra property that there are exactly w_1 ones in the first n_1 positions and w_2 ones in the last n_2 positions, where $n=n_1+n_2$ and $w=w_1+w_2$. A (w_1,n_1,w_2,n_2,d) code is a doubly constant weight code with w_1 ones in the first n_1 positions and w_2 ones in the last n_2 positions, and minimum distance d between any pair of codewords. Such codes play an important role in obtaining bounds on the sizes of constant weight codes with given minimum distance (see, [5]). We can partition the set of coordinates into two subsets A and B such that $|A|=w$ and $|B|=n-w$. We say that a word is from *configuration* (i,j) if it has weight i in the coordinates of A and weight j in the coordinates of B. A $(w_1,n_1,w_2,n_2,2(w_1+w_2-i-j+1))$ code is a *perfect* (i,j) cover if every word from configuration (i,j) is contained in exactly one codeword. In 2008, Etzion [12] shown tight connections between optimal doubly constant weight codes and some known designs such as Howell designs, Kirkman squares, and generalized Kirkman squares.

Lemma 7.4 *([12]). If there exists a generalized Kirkman square* $\mathrm{GKS}_k(v;1,1;r)$, *there exists a* $(2,2r,k,v,2k+2)$ *code which is a perfect* $(1,1)$ *cover.*

Combining Lemma 7.4 and Theorems 5.6 and 6.7 gives the following two results about doubly constant weight codes.

Theorem 7.5. *Let* M *be the set of* 15 *numbers listed in Table 5. For* $u\geq 3$, $u\equiv 0\pmod 3$, *and* $u\notin M$, *there exists a* $(2,4(u-1),3,4u,8)$ *code which is a perfect* $(1,1)$ *cover.*

Theorem 7.6. *Let* N *be the set of* 31 *numbers listed in Table 7. For* $u>3$ *and* $u\notin N$, *there exists a* $(2,6(u-1),3,6u,8)$ *code which is a perfect* $(1,1)$ *cover.*

8. Summary

In this paper, we have obtained 5 new (4, 1)-RGDDs, and examined the existence of doubly resolvable (3, 1)-GDDs of types 4^u and 6^u. For type 4^u, these exist for all $u\geq 3$, $u\equiv 0\pmod 3$ except possibly for 15 values listed in Table 5. For type 6^u, no design exists for $u=3$, but these designs exist for all $u\geq 4$, except possibly for 31 values listed in Table 7.

Doubly resolvable GDDs are a class of generalized Kirkman squares. Generalized Kirkman squares not only have their own combinatorial significance, but also have close relationships with constant composition codes and doubly constant weight codes. Therefore, the construction and existence of (3, 1)-DRGDDs or $\mathrm{GKS}_3(v;1,1;r)$s is worthy of further study. The general problem of existence of (3, 1)-DRGDDs or $\mathrm{GKS}_3(v;1,1;r)$s is far from being solved.

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Appendix

Here, using the method described in Lemmas 3.1 and 3.2, we provide starter blocks and adders for (3, 1)-DRGDDs of types 4^u (with $u\equiv 0\pmod 3$, $6\leq u\leq 30$, $u\neq 27$) and 6^u (with $6\leq u\leq 20$).

Table 8
Starter blocks and adders for (3, 1)-DRGDDs of type 4^u with $6 \le u \le 30$, $u \ne 27$ and $u \equiv 0 \pmod 3$.

u	Starter	Adder	Starter + Adder	Starter	Adder	Starter + Adder
6	{2, 6, 8}	8	{10, 14, 16}	{14, 17, ∞_1}	10	{4, 7, ∞_1}
	{3, 5, 9}	6	{9, 11, 15}	{10, 19, ∞_2}	18	{8, 17, ∞_2}
	{4, 12, 11}	14	{18, 6, 5}	{16, 7, ∞_3}	16	{12, 3, ∞_3}
	{0, 13, 1}	0	{0, 13, 1}	{18, 15, ∞_4}	4	{2, 19, ∞_4}
9	{24, 30, 10}	12	{4, 10, 22}	{12, 21, 27}	4	{16, 25, 31}
	{11, 15, 25}	26	{5, 9, 19}	{4, 29, 9}	20	{24, 17, 29}
	{0, 2, 1}	0	{0, 2, 1}	{14, 17, ∞_1}	30	{12, 15, ∞_1}
	{18, 22, 7}	28	{14, 18, 3}	{16, 23, ∞_2}	22	{6, 13, ∞_2}
	{28, 6, 19}	24	{20, 30, 11}	{20, 31, ∞_3}	8	{28, 7, ∞_3}
	{8, 3, 5}	18	{26, 21, 23}	{26, 13, ∞_4}	14	{8, 27, ∞_4}
12	{8, 10, 26}	18	{26, 28, 0}	{6, 41, 5}	8	{14, 5, 13}
	{34, 38, 2}	22	{12, 16, 24}	{32, 35, 3}	32	{20, 23, 35}
	{24, 30, 0}	6	{30, 36, 6}	{16, 39, 11}	26	{42, 21, 37}
	{7, 17, 37}	2	{9, 19, 39}	{18, 27, 1}	16	{34, 43, 17}
	{12, 22, 19}	40	{8, 18, 15}	{42, 43, ∞_1}	42	{40, 41, ∞_1}
	{36, 21, 23}	10	{2, 31, 33}	{28, 33, ∞_2}	38	{22, 27, ∞_2}
	{4, 25, 29}	0	{4, 25, 29}	{14, 31, ∞_3}	24	{38, 11, ∞_3}
	{40, 9, 15}	36	{32, 1, 7}	{20, 13, ∞_4}	34	{10, 3, ∞_4}
15	{26, 28, 48}	10	{36, 38, 2}	{2, 45, 53}	52	{54, 41, 49}
	{16, 24, 34}	54	{14, 22, 32}	{36, 7, 17}	44	{24, 51, 5}
	{13, 15, 9}	30	{43, 45, 39}	{54, 21, 33}	48	{46, 13, 25}
	{23, 39, 1}	32	{55, 15, 33}	{14, 5, 25}	4	{18, 9, 29}
	{52, 0, 41}	16	{12, 16, 1}	{12, 27, 51}	36	{48, 7, 31}
	{40, 46, 3}	50	{34, 40, 53}	{8, 11, 37}	0	{8, 11, 37}
	{10, 22, 43}	34	{44, 0, 21}	{30, 35, ∞_1}	12	{42, 47, ∞_1}
	{4, 20, 29}	46	{50, 10, 19}	{42, 49, ∞_2}	42	{28, 35, ∞_2}
	{38, 6, 55}	24	{6, 30, 23}	{50, 47, ∞_3}	26	{20, 17, ∞_3}
	{18, 44, 19}	8	{26, 52, 27}	{32, 31, ∞_4}	28	{4, 3, ∞_4}
18	{46, 48, 6}	24	{2, 4, 30}	{66, 63, 9}	14	{12, 9, 23}
	{12, 20, 30}	6	{18, 26, 36}	{32, 51, 1}	34	{66, 17, 35}
	{24, 36, 56}	8	{32, 44, 64}	{60, 17, 37}	16	{8, 33, 53}
	{14, 28, 52}	0	{14, 28, 52}	{62, 53, 7}	22	{16, 7, 29}
	{19, 21, 25}	42	{61, 63, 67}	{8, 11, 35}	54	{62, 65, 21}
	{43, 55, 27}	28	{3, 15, 55}	{0, 5, 31}	20	{20, 25, 51}
	{38, 42, 23}	4	{42, 46, 27}	{34, 67, 29}	58	{24, 57, 19}
	{16, 22, 57}	52	{0, 6, 41}	{54, 15, 47}	64	{50, 11, 43}
	{10, 26, 33}	12	{22, 38, 45}	{64, 65, ∞_1}	62	{58, 59, ∞_1}
	{18, 40, 61}	38	{56, 10, 31}	{44, 59, ∞_2}	10	{54, 1, ∞_2}
	{2, 41, 49}	32	{34, 5, 13}	{58, 45, ∞_3}	2	{60, 47, ∞_3}
	{4, 3, 13}	36	{40, 39, 49}	{50, 39, ∞_4}	66	{48, 37, ∞_4}
21	{24, 28, 10}	58	{2, 6, 68}	{22, 50, 41}	22	{44, 72, 63}
	{32, 44, 2}	16	{48, 60, 18}	{16, 48, 63}	54	{70, 22, 37}
	{11, 13, 75}	60	{71, 73, 55}	{46, 0, 3}	10	{56, 10, 13}
	{21, 27, 65}	64	{5, 11, 49}	{62, 18, 39}	18	{0, 36, 57}
	{57, 67, 33}	42	{19, 29, 75}	{26, 51, 55}	28	{54, 79, 3}
	{59, 71, 5}	56	{35, 47, 61}	{64, 37, 45}	50	{34, 7, 15}
	{38, 40, 1}	68	{26, 28, 69}	{12, 77, 19}	12	{24, 9, 31}
	{8, 14, 47}	6	{14, 20, 53}	{56, 7, 35}	32	{8, 39, 67}
	{34, 42, 17}	24	{58, 66, 41}	{54, 23, 53}	78	{52, 21, 51}
	{66, 76, 9}	8	{74, 4, 17}	{74, 29, 61}	4	{78, 33, 65}
	{70, 6, 79}	26	{16, 32, 25}	{68, 73, ∞_1}	52	{40, 45, ∞_1}
	{78, 20, 15}	44	{42, 64, 59}	{58, 69, ∞_2}	34	{12, 23, ∞_2}
	{36, 60, 25}	2	{38, 62, 27}	{72, 43, ∞_3}	38	{30, 1, ∞_3}
	{4, 30, 31}	46	{50, 76, 77}	{52, 49, ∞_4}	74	{46, 43, ∞_4}
24	{14, 16, 72}	76	{90, 0, 56}	{28, 85, 89}	32	{60, 25, 29}
	{18, 22, 36}	8	{26, 30, 44}	{46, 57, 67}	36	{82, 1, 11}
	{32, 38, 62}	64	{4, 10, 34}	{88, 13, 25}	52	{48, 65, 77}
	{44, 60, 12}	24	{68, 84, 36}	{52, 1, 19}	72	{32, 73, 91}
	{0, 26, 64}	78	{78, 12, 50}	{24, 51, 73}	34	{58, 85, 15}
	{37, 43, 17}	0	{37, 43, 17}	{2, 55, 79}	60	{62, 23, 47}
	{27, 35, 75}	22	{49, 57, 5}	{42, 33, 61}	90	{40, 31, 59}
	{45, 59, 29}	42	{87, 9, 71}	{56, 71, 11}	88	{52, 67, 7}
	{50, 58, 31}	14	{64, 72, 45}	{70, 15, 49}	46	{24, 61, 3}
	{90, 8, 53}	10	{8, 18, 63}	{10, 41, 77}	70	{80, 19, 55}

(continued on next page)

Table 8 (*continued*)

u	Starter	Adder	Starter + Adder	Starter	Adder	Starter+Adder
	{84, 4, 91}	84	{76, 88, 83}	{20, 63, 9}	18	{38, 81, 27}
	{66, 86, 7}	28	{2, 22, 35}	{40, 23, 65}	66	{14, 89, 39}
	{54, 76, 47}	58	{20, 42, 13}	{68, 69, ∞_1}	6	{74, 75, ∞_1}
	{34, 74, 21}	12	{46, 86, 33}	{78, 81, ∞_2}	80	{66, 69, ∞_2}
	{80, 30, 39}	40	{28, 70, 79}	{82, 87, ∞_3}	26	{16, 21, ∞_3}
	{6, 3, 5}	48	{54, 51, 53}	{48, 83, ∞_4}	50	{6, 41, ∞_4}
30	{0, 2, 20}	0	{0, 2, 20}	{78, 10, 25}	110	{72, 4, 19}
	{86, 90, 6}	108	{78, 82, 114}	{58, 110, 53}	92	{34, 86, 29}
	{42, 48, 64}	64	{106, 112, 12}	{50, 106, 35}	34	{84, 24, 69}
	{12, 24, 62}	30	{42, 54, 92}	{26, 17, 23}	100	{10, 1, 7}
	{70, 84, 8}	38	{108, 6, 46}	{108, 71, 83}	76	{68, 31, 43}
	{1, 3, 43}	106	{107, 109, 33}	{74, 97, 111}	28	{102, 9, 23}
	{89, 93, 7}	104	{77, 81, 111}	{94, 61, 81}	80	{58, 25, 45}
	{5, 13, 59}	46	{51, 59, 105}	{76, 79, 101}	78	{38, 41, 63}
	{19, 29, 45}	56	{75, 85, 101}	{34, 41, 65}	54	{88, 95, 3}
	{107, 9, 47}	82	{73, 91, 13}	{72, 21, 49}	18	{90, 39, 67}
	{28, 36, 77}	16	{44, 52, 93}	{100, 37, 69}	96	{80, 17, 49}
	{4, 14, 109}	90	{94, 104, 83}	{30, 91, 11}	36	{66, 11, 47}
	{16, 40, 51}	10	{26, 50, 61}	{114, 31, 75}	112	{110, 27, 71}
	{18, 44, 87}	12	{30, 56, 99}	{102, 67, 115}	114	{100, 65, 113}
	{52, 80, 99}	72	{8, 36, 55}	{46, 63, 113}	24	{70, 87, 21}
	{66, 96, 105}	68	{18, 48, 57}	{82, 103, 39}	50	{16, 37, 89}
	{54, 88, 27}	8	{62, 96, 35}	{32, 33, ∞_1}	98	{14, 15, ∞_1}
	{112, 38, 95}	26	{22, 64, 5}	{98, 57, ∞_2}	58	{40, 115, ∞_2}
	{60, 104, 15}	88	{32, 76, 103}	{92, 85, ∞_3}	84	{60, 53, ∞_3}
	{22, 68, 73}	6	{28, 74, 79}	{56, 55, ∞_4}	42	{98, 97, ∞_4}

Table 9
Starter blocks and adders for (3, 1)-DRGDDs of type 6^u with $6 \leq u \leq 20$.

u	Starter	Adder	Starter+Adder	Starter	Adder	Starter+Adder
6	{4, 8, 20}	18	{22, 26, 8}	{26, 29, ∞_1}	24	{20, 23, ∞_1}
	{3, 5, 11}	10	{13, 15, 21}	{14, 23, ∞_2}	4	{18, 27, ∞_2}
	{21, 25, 9}	16	{7, 11, 25}	{16, 27, ∞_3}	8	{24, 5, ∞_3}
	{0, 2, 1}	0	{0, 2, 1}	{28, 15, ∞_4}	14	{12, 29, ∞_4}
	{6, 12, 19}	28	{4, 10, 17}	{22, 13, ∞_5}	6	{28, 19, ∞_5}
	{10, 18, 7}	26	{6, 14, 3}	{24, 17, ∞_6}	22	{16, 9, ∞_6}
7	{4, 8, 18}	10	{14, 18, 28}	{30, 9, 19}	0	{30, 9, 19}
	{5, 7, 27}	6	{11, 13, 33}	{32, 35, ∞_1}	28	{24, 27, ∞_1}
	{0, 2, 1}	20	{20, 22, 21}	{16, 23, ∞_2}	30	{10, 17, ∞_2}
	{34, 6, 3}	2	{0, 8, 5}	{20, 29, ∞_3}	14	{34, 7, ∞_3}
	{10, 26, 21}	16	{26, 6, 1}	{14, 31, ∞_4}	34	{12, 29, ∞_4}
	{28, 11, 15}	24	{16, 35, 3}	{22, 13, ∞_5}	18	{4, 31, ∞_5}
	{12, 25, 33}	26	{2, 15, 23}	{24, 17, ∞_6}	8	{32, 25, ∞_6}
8	{12, 16, 24}	12	{24, 28, 36}	{30, 39, 3}	16	{4, 13, 19}
	{34, 40, 18}	14	{6, 12, 32}	{26, 29, 7}	0	{26, 29, 7}
	{15, 19, 31}	28	{1, 5, 17}	{22, 27, ∞_1}	8	{30, 35, ∞_1}
	{5, 13, 23}	18	{23, 31, 41}	{6, 17, ∞_2}	4	{10, 21, ∞_2}
	{2, 4, 35}	40	{0, 2, 33}	{8, 21, ∞_3}	6	{14, 27, ∞_3}
	{10, 20, 37}	30	{40, 8, 25}	{28, 11, ∞_4}	34	{20, 3, ∞_4}
	{14, 32, 9}	2	{16, 34, 11}	{38, 25, ∞_5}	26	{22, 9, ∞_5}
	{0, 41, 1}	38	{38, 37, 39}	{36, 33, ∞_6}	24	{18, 15, ∞_6}
9	{18, 24, 36}	38	{8, 14, 26}	{14, 3, 5}	26	{40, 29, 31}
	{19, 33, 39}	32	{3, 17, 23}	{20, 37, 47}	8	{28, 45, 7}
	{25, 29, 7}	34	{11, 15, 41}	{34, 15, 27}	10	{44, 25, 37}
	{0, 2, 1}	20	{20, 22, 21}	{38, 45, ∞_1}	12	{2, 9, ∞_1}
	{12, 16, 35}	0	{12, 16, 35}	{30, 43, ∞_2}	6	{36, 1, ∞_2}
	{42, 4, 9}	44	{38, 0, 5}	{40, 13, ∞_3}	14	{6, 27, ∞_3}
	{44, 10, 21}	22	{18, 32, 43}	{46, 31, ∞_4}	36	{34, 19, ∞_4}
	{8, 28, 11}	2	{10, 30, 13}	{22, 17, ∞_5}	30	{4, 47, ∞_5}
	{32, 6, 41}	40	{24, 46, 33}	{26, 23, ∞_6}	16	{42, 39, ∞_6}
10	{6, 8, 32}	28	{34, 36, 6}	{42, 19, 27}	6	{48, 25, 33}
	{26, 30, 16}	26	{52, 2, 42}	{24, 43, 53}	52	{22, 41, 51}
	{36, 48, 14}	2	{38, 50, 16}	{18, 29, 41}	36	{0, 11, 23}
	{3, 5, 37}	32	{35, 37, 15}	{2, 17, 45}	38	{40, 1, 29}

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Table 9 (*continued*)

u	Starter	Adder	Starter+Adder	Starter	Adder	Starter+Adder
	{21, 35, 51}	50	{17, 31, 47}	{34, 39, ∞_1}	34	{14, 19, ∞_1}
	{4, 10, 7}	14	{18, 24, 21}	{40, 47, ∞_2}	46	{32, 39, ∞_2}
	{44, 52, 23}	22	{12, 20, 45}	{20, 33, ∞_3}	10	{30, 43, ∞_3}
	{12, 28, 11}	16	{28, 44, 27}	{50, 31, ∞_4}	30	{26, 7, ∞_4}
	{46, 9, 13}	0	{46, 9, 13}	{38, 25, ∞_5}	24	{8, 49, ∞_5}
	{0, 49, 1}	4	{4, 53, 5}	{22, 15, ∞_6}	42	{10, 3, ∞_6}
11	{16, 30, 54}	40	{56, 10, 34}	{4, 7, 11}	34	{38, 41, 45}
	{24, 40, 58}	26	{50, 6, 24}	{50, 23, 35}	12	{2, 35, 47}
	{9, 15, 33}	46	{55, 1, 19}	{18, 29, 45}	4	{22, 33, 49}
	{19, 27, 41}	2	{21, 29, 43}	{32, 21, 47}	44	{16, 5, 31}
	{0, 2, 1}	52	{52, 54, 53}	{34, 57, 25}	14	{48, 11, 39}
	{48, 52, 13}	38	{26, 30, 51}	{46, 51, ∞_1}	18	{4, 9, ∞_1}
	{36, 42, 17}	0	{36, 42, 17}	{28, 37, ∞_2}	30	{58, 7, ∞_2}
	{6, 14, 43}	54	{0, 8, 37}	{26, 39, ∞_3}	48	{14, 27, ∞_3}
	{10, 22, 53}	22	{32, 44, 15}	{38, 55, ∞_4}	8	{46, 3, ∞_4}
	{44, 12, 31}	28	{12, 40, 59}	{20, 59, ∞_5}	58	{18, 57, ∞_5}
	{8, 3, 5}	20	{28, 23, 25}	{56, 49, ∞_6}	24	{20, 13, ∞_6}
12	{52, 58, 0}	60	{46, 52, 60}	{8, 53, 55}	32	{40, 19, 21}
	{28, 38, 12}	10	{38, 48, 22}	{24, 25, 29}	6	{30, 31, 35}
	{18, 36, 64}	56	{8, 26, 54}	{2, 39, 45}	12	{14, 51, 57}
	{21, 31, 57}	46	{1, 11, 37}	{32, 15, 23}	58	{24, 7, 15}
	{3, 17, 41}	38	{41, 55, 13}	{10, 63, 9}	0	{10, 63, 9}
	{33, 49, 1}	16	{49, 65, 17}	{30, 51, 5}	48	{12, 33, 53}
	{20, 22, 37}	22	{42, 44, 59}	{56, 59, ∞_1}	2	{58, 61, ∞_1}
	{50, 54, 11}	18	{2, 6, 29}	{40, 47, ∞_2}	62	{36, 43, ∞_2}
	{60, 6, 19}	26	{20, 32, 45}	{34, 43, ∞_3}	28	{62, 5, ∞_3}
	{46, 4, 65}	24	{4, 28, 23}	{26, 61, ∞_4}	30	{56, 25, ∞_4}
	{14, 44, 7}	20	{34, 64, 27}	{62, 35, ∞_5}	4	{0, 39, ∞_5}
	{16, 48, 13}	34	{50, 16, 47}	{42, 27, ∞_6}	42	{18, 3, ∞_6}
13	{66, 70, 28}	60	{54, 58, 16}	{52, 8, 7}	58	{38, 66, 65}
	{42, 60, 2}	48	{18, 36, 50}	{0, 39, 43}	2	{2, 41, 45}
	{57, 59, 3}	12	{69, 71, 15}	{54, 5, 15}	52	{34, 57, 67}
	{19, 25, 65}	18	{37, 43, 11}	{62, 67, 9}	38	{28, 33, 47}
	{45, 53, 1}	50	{23, 31, 51}	{36, 11, 33}	28	{64, 39, 61}
	{46, 48, 29}	46	{20, 22, 3}	{32, 69, 27}	8	{40, 5, 35}
	{6, 12, 13}	66	{0, 6, 7}	{30, 55, 17}	0	{30, 55, 17}
	{14, 22, 63}	34	{48, 56, 25}	{38, 41, ∞_1}	44	{10, 13, ∞_1}
	{10, 20, 31}	42	{52, 62, 1}	{40, 49, ∞_2}	4	{44, 53, ∞_2}
	{34, 50, 23}	26	{60, 4, 49}	{24, 37, ∞_3}	56	{8, 21, ∞_3}
	{68, 16, 47}	16	{12, 32, 63}	{56, 71, ∞_4}	30	{14, 29, ∞_4}
	{4, 26, 61}	20	{24, 46, 9}	{64, 21, ∞_5}	6	{70, 27, ∞_5}
	{18, 44, 35}	24	{42, 68, 59}	{58, 51, ∞_6}	40	{26, 19, ∞_6}
14	{52, 54, 6}	70	{44, 46, 76}	{10, 38, 11}	56	{66, 16, 67}
	{40, 56, 16}	8	{48, 64, 24}	{28, 70, 27}	0	{28, 70, 27}
	{68, 2, 46}	72	{62, 74, 40}	{0, 23, 25}	68	{68, 13, 15}
	{1, 5, 51}	24	{25, 29, 75}	{26, 21, 31}	16	{42, 37, 47}
	{71, 77, 29}	66	{59, 65, 17}	{18, 35, 49}	38	{56, 73, 9}
	{73, 3, 61}	60	{55, 63, 43}	{12, 33, 55}	46	{58, 1, 23}
	{37, 53, 19}	32	{69, 7, 51}	{32, 69, 15}	20	{52, 11, 35}
	{20, 24, 9}	30	{50, 54, 39}	{50, 57, 17}	40	{12, 19, 57}
	{30, 36, 45}	74	{26, 32, 41}	{48, 59, ∞_1}	64	{34, 45, ∞_1}
	{58, 66, 7}	26	{6, 14, 33}	{76, 47, ∞_2}	62	{60, 31, ∞_2}
	{72, 4, 75}	6	{0, 10, 3}	{64, 43, ∞_3}	18	{4, 61, ∞_3}
	{8, 22, 63}	14	{22, 36, 77}	{60, 41, ∞_4}	12	{72, 53, ∞_4}
	{44, 62, 13}	36	{2, 20, 49}	{74, 65, ∞_5}	34	{30, 21, ∞_5}
	{14, 34, 67}	4	{18, 38, 71}	{42, 39, ∞_6}	44	{8, 5, ∞_6}
15	{34, 40, 66}	66	{16, 22, 48}	{32, 59, 63}	70	{18, 45, 49}
	{6, 16, 70}	76	{82, 8, 62}	{62, 79, 3}	58	{36, 53, 61}
	{22, 38, 4}	36	{58, 74, 40}	{14, 35, 55}	30	{44, 65, 1}
	{83, 1, 73}	22	{21, 23, 11}	{12, 27, 49}	52	{64, 79, 17}
	{75, 81, 21}	46	{37, 43, 67}	{0, 51, 77}	12	{12, 63, 5}
	{29, 45, 11}	2	{31, 47, 13}	{60, 9, 41}	78	{54, 3, 35}
	{54, 56, 25}	16	{70, 72, 41}	{24, 7, 43}	8	{32, 15, 51}
	{26, 30, 15}	60	{2, 6, 75}	{46, 69, 23}	34	{80, 19, 57}
	{36, 44, 47}	24	{60, 68, 71}	{72, 31, 71}	38	{26, 69, 25}
	{82, 10, 5}	68	{66, 78, 73}	{64, 65, ∞_1}	48	{28, 29, ∞_1}
	{20, 42, 17}	10	{30, 52, 27}	{58, 67, ∞_2}	26	{0, 9, ∞_2}
	{78, 18, 57}	20	{14, 38, 77}	{48, 61, ∞_3}	82	{46, 59, ∞_3}
	{50, 2, 37}	54	{20, 56, 7}	{76, 39, ∞_4}	0	{76, 39, ∞_4}
	{74, 28, 19}	14	{4, 42, 33}	{68, 33, ∞_5}	50	{34, 83, ∞_5}
	{52, 8, 13}	42	{10, 50, 55}	{80, 53, ∞_6}	28	{24, 81, ∞_6}

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Table 9 (*continued*)

u	Starter	Adder	Starter+Adder	Starter	Adder	Starter+Adder
16	{40, 42, 74}	64	{14, 16, 48}	{12, 36, 31}	66	{78, 12, 7}
	{50, 64, 2}	0	{50, 64, 2}	{82, 28, 51}	80	{72, 18, 41}
	{58, 84, 14}	38	{6, 32, 52}	{10, 48, 13}	10	{20, 58, 23}
	{39, 41, 19}	8	{47, 49, 27}	{26, 66, 37}	34	{60, 10, 71}
	{29, 43, 79}	56	{85, 9, 45}	{16, 21, 25}	82	{8, 13, 17}
	{57, 75, 11}	70	{37, 55, 81}	{46, 87, 3}	36	{82, 33, 39}
	{49, 65, 17}	26	{75, 1, 43}	{54, 89, 7}	58	{22, 57, 65}
	{53, 77, 15}	72	{35, 59, 87}	{34, 23, 33}	46	{80, 69, 79}
	{20, 24, 1}	18	{38, 42, 19}	{72, 35, 47}	54	{36, 89, 11}
	{0, 6, 63}	88	{88, 4, 61}	{68, 61, 5}	16	{84, 77, 21}
	{78, 86, 27}	78	{66, 74, 15}	{70, 83, ∞_1}	74	{54, 67, ∞_1}
	{8, 18, 9}	22	{30, 40, 31}	{56, 73, ∞_2}	68	{34, 51, ∞_2}
	{32, 44, 69}	24	{56, 68, 3}	{60, 81, ∞_3}	2	{62, 83, ∞_3}
	{22, 38, 71}	48	{70, 86, 29}	{62, 45, ∞_4}	28	{0, 73, ∞_4}
	{76, 4, 55}	40	{26, 44, 5}	{80, 67, ∞_5}	86	{76, 63, ∞_5}
	{30, 52, 59}	84	{24, 46, 53}	{88, 85, ∞_6}	30	{28, 25, ∞_6}
17	{36, 38, 30}	32	{68, 70, 62}	{32, 75, 87}	40	{72, 19, 31}
	{92, 6, 42}	44	{40, 50, 86}	{34, 31, 45}	8	{42, 39, 53}
	{40, 52, 74}	38	{78, 90, 16}	{76, 21, 39}	6	{82, 27, 45}
	{4, 22, 60}	4	{8, 26, 64}	{26, 53, 73}	84	{14, 41, 61}
	{46, 72, 18}	80	{30, 56, 2}	{48, 13, 35}	12	{60, 25, 47}
	{23, 29, 65}	60	{83, 89, 29}	{24, 91, 19}	82	{10, 77, 5}
	{57, 85, 27}	24	{81, 13, 51}	{50, 25, 51}	42	{92, 67, 93}
	{16, 20, 89}	18	{34, 38, 11}	{8, 61, 95}	92	{4, 57, 91}
	{82, 0, 3}	20	{6, 20, 23}	{86, 15, 55}	90	{80, 9, 49}
	{88, 12, 69}	0	{88, 12, 69}	{44, 63, 11}	10	{54, 73, 21}
	{70, 94, 49}	26	{0, 24, 75}	{62, 1, 47}	86	{52, 87, 37}
	{68, 2, 9}	46	{18, 48, 55}	{56, 71, ∞_1}	68	{28, 43, ∞_1}
	{66, 14, 59}	52	{22, 66, 15}	{58, 81, ∞_2}	16	{74, 1, ∞_2}
	{10, 41, 43}	22	{32, 63, 65}	{54, 83, ∞_3}	30	{84, 17, ∞_3}
	{28, 33, 37}	66	{94, 3, 7}	{84, 67, ∞_4}	88	{76, 59, ∞_4}
	{80, 93, 5}	74	{58, 71, 79}	{90, 79, ∞_5}	50	{44, 33, ∞_5}
	{64, 7, 17}	78	{46, 85, 95}	{78, 77, ∞_6}	54	{36, 35, ∞_6}
18	{0, 4, 12}	0	{0, 4, 12}	{100, 44, 39}	60	{58, 2, 99}
	{22, 28, 54}	10	{32, 38, 64}	{96, 42, 95}	66	{60, 6, 59}
	{30, 40, 68}	22	{52, 62, 90}	{48, 98, 1}	90	{36, 86, 91}
	{52, 70, 92}	4	{56, 74, 96}	{38, 71, 77}	56	{94, 25, 31}
	{3, 5, 53}	14	{17, 19, 67}	{76, 85, 97}	54	{28, 37, 49}
	{31, 35, 75}	34	{65, 69, 7}	{20, 79, 93}	80	{100, 57, 71}
	{25, 33, 99}	78	{1, 9, 75}	{90, 7, 23}	100	{88, 5, 21}
	{73, 83, 13}	32	{3, 13, 45}	{84, 51, 69}	44	{26, 95, 11}
	{9, 29, 55}	6	{15, 35, 61}	{24, 27, 49}	24	{48, 51, 73}
	{6, 8, 21}	62	{68, 70, 83}	{88, 65, 89}	36	{22, 101, 23}
	{32, 46, 43}	46	{78, 92, 89}	{66, 11, 41}	52	{16, 63, 93}
	{18, 34, 61}	16	{34, 50, 77}	{78, 101, 37}	42	{18, 41, 79}
	{62, 82, 17}	64	{24, 44, 81}	{50, 57, ∞_1}	30	{80, 87, ∞_1}
	{80, 2, 67}	18	{98, 20, 85}	{58, 87, ∞_2}	58	{14, 43, ∞_2}
	{56, 86, 47}	88	{42, 72, 33}	{94, 59, ∞_3}	38	{30, 97, ∞_3}
	{36, 72, 15}	40	{76, 10, 55}	{10, 81, ∞_4}	74	{84, 53, ∞_4}
	{74, 14, 63}	68	{40, 82, 29}	{64, 45, ∞_5}	84	{46, 27, ∞_5}
	{16, 60, 91}	50	{66, 8, 39}	{26, 19, ∞_6}	28	{54, 47, ∞_6}
19	{0, 2, 42}	0	{0, 2, 42}	{92, 36, 43}	86	{70, 14, 21}
	{52, 56, 86}	106	{50, 54, 84}	{78, 63, 69}	14	{92, 77, 83}
	{44, 50, 66}	104	{40, 46, 62}	{96, 9, 17}	72	{60, 81, 89}
	{46, 54, 74}	98	{36, 44, 64}	{62, 93, 103}	24	{86, 9, 19}
	{1, 3, 35}	102	{103, 105, 29}	{106, 65, 85}	58	{56, 15, 35}
	{41, 45, 67}	100	{33, 37, 59}	{80, 51, 75}	36	{8, 87, 3}
	{37, 49, 89}	70	{107, 11, 51}	{102, 107, 27}	94	{88, 93, 13}
	{99, 7, 21}	54	{45, 61, 75}	{84, 19, 57}	22	{106, 41, 79}
	{30, 40, 39}	28	{58, 68, 67}	{32, 59, 101}	92	{16, 43, 85}
	{6, 18, 91}	4	{10, 22, 95}	{38, 77, 13}	56	{94, 25, 69}
	{10, 24, 47}	10	{20, 34, 57}	{14, 25, 71}	84	{98, 1, 47}
	{4, 28, 105}	74	{78, 102, 71}	{64, 31, 79}	82	{38, 5, 53}
	{8, 34, 97}	66	{74, 100, 55}	{94, 33, 83}	32	{18, 65, 7}
	{16, 48, 61}	64	{80, 4, 17}	{88, 5, ∞_1}	68	{48, 73, ∞_1}
	{60, 98, 95}	6	{66, 104, 101}	{68, 11, ∞_2}	52	{12, 63, ∞_2}
	{76, 12, 29}	20	{96, 32, 49}	{26, 81, ∞_3}	50	{76, 23, ∞_3}
	{58, 104, 15}	76	{26, 72, 91}	{90, 53, ∞_4}	46	{28, 99, ∞_4}
	{82, 22, 23}	8	{90, 30, 31}	{72, 55, ∞_5}	42	{6, 97, ∞_5}
	{20, 70, 73}	62	{82, 24, 27}	{100, 87, ∞_6}	60	{52, 39, ∞_6}

(continued on next page)

Table 9 (*continued*)

u	Starter	Adder	Starter+Adder	Starter	Adder	Starter+Adder
20	{0, 2, 46}	0	{0, 2, 46}	{58, 110, 33}	38	{96, 34, 71}
	{24, 28, 78}	112	{22, 26, 76}	{44, 100, 13}	68	{112, 54, 81}
	{40, 48, 74}	110	{36, 44, 70}	{86, 43, 47}	92	{64, 21, 25}
	{56, 66, 80}	108	{50, 60, 74}	{82, 59, 67}	40	{8, 99, 107}
	{26, 38, 68}	104	{16, 28, 58}	{96, 101, 111}	10	{106, 111, 7}
	{1, 3, 55}	100	{101, 103, 41}	{72, 79, 97}	72	{30, 37, 55}
	{11, 17, 45}	102	{113, 5, 33}	{10, 19, 41}	74	{84, 93, 1}
	{61, 73, 93}	106	{53, 65, 85}	{8, 51, 75}	82	{90, 19, 43}
	{49, 63, 99}	24	{73, 87, 9}	{106, 39, 65}	12	{4, 51, 77}
	{103, 5, 35}	26	{15, 31, 61}	{16, 69, 109}	84	{100, 39, 79}
	{30, 36, 25}	32	{62, 68, 57}	{88, 29, 71}	20	{108, 49, 91}
	{6, 22, 57}	80	{86, 102, 23}	{70, 37, 81}	8	{78, 45, 89}
	{14, 32, 91}	34	{48, 66, 11}	{76, 89, 23}	6	{82, 95, 29}
	{34, 54, 27}	70	{104, 10, 97}	{112, 21, 77}	54	{52, 75, 17}
	{20, 42, 83}	78	{98, 6, 47}	{104, 7, ∞_1}	98	{88, 105, ∞_1}
	{64, 92, 113}	64	{14, 42, 63}	{94, 9, ∞_2}	58	{38, 67, ∞_2}
	{52, 84, 85}	42	{94, 12, 13}	{18, 87, ∞_3}	62	{80, 35, ∞_3}
	{90, 12, 15}	44	{20, 56, 59}	{60, 31, ∞_4}	86	{32, 3, ∞_4}
	{62, 102, 53}	30	{92, 18, 83}	{4, 105, ∞_5}	36	{40, 27, ∞_5}
	{50, 98, 95}	88	{24, 72, 69}	{108, 107, ∞_6}	2	{110, 109, ∞_6}

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