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# The b-chromatic index of graphs[☆]

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ABSTRACT

A b-coloring of the vertices of a graph is a proper coloring where each color class contains a vertex which is adjacent to at least one vertex in each other color class. The b-chromatic number of $G$ is the maximum integer $b(G)$ for which $G$ has a b-coloring with $b(G)$ colors. This problem was introduced by Irving and Manlove (1999), where they showed that computing $b(G)$ is $\mathcal{NP}$-hard in general and polynomial-time solvable for trees. A natural question that arises is whether the edge version of this problem is also $\mathcal{NP}$-hard or not. Here, we prove that computing the b-chromatic index of a graph $G$ is $\mathcal{NP}$-hard, even if $G$ is either a comparability graph or a $C_k$-free graph, and give partial results on the complexity of the problem restricted to trees, more specifically, we solve the problem for caterpillars graphs. Although solving problems on caterpillar graphs is usually quite simple, this problem revealed itself to be unusually hard. The presented algorithm uses a dynamic programming approach that combines partial solutions which are proved to exist if, and only if, a particular polyhedron is non-empty.



## 1. Introduction

Let $G$ be a simple graph[1] and suppose that we have a proper coloring of $G$ for which there exists a color class $C$ such that every vertex $v$ in $C$ is not adjacent to at least one other color class; then we can separately change the color of each vertex in $C$ to obtain a proper coloring with fewer colors. This heuristic, called here b-heuristic, can be applied iteratively, but we cannot expect to reach the chromatic number of $G$, since the coloring problem is $\mathcal{NP}$-hard.

On the basis of this idea, Irving and Manlove introduced the notion of b-coloring in [12]. Intuitively, a b-coloring is a proper coloring that cannot be improved by the b-heuristic, and the b-chromatic number $b(G)$ measures the worst possible such coloring. Finding $b(G)$ was proved to be $\mathcal{NP}$-hard in general graphs [12], and remains so even when restricted to bipartite graphs [16] or to chordal graphs [10]. However, this problem is polynomial when restricted to some graph classes, including trees [12], cographs and $P_4$-sparse graphs [3], $P_4$-tidy graphs [20], cacti [18], some power graphs [5–7], Kneser graphs [9,14],

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[1] The graph terminology used in this paper follows [2].

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some graphs with large girth [8,15,17], etc. Also, some other aspects of the problem were studied, as for example, the b-spectrum of a graph [1], and b-perfect graphs [11].

In this article, we propose to study a natural variation of the b-coloring problem, coloring the edges of a graph under the same constraints. In fact, we will investigate the b-coloring of the line graph of some classes of graphs. More formally, a *proper coloring (with k colors)* of a graph is a function $\psi : V(G) \to \{1, \ldots, k\}$ such that no two adjacent vertices have the same color (function value). For $X \subseteq V(G)$, denote by $\psi(X)$ the set $\{\psi(v) \mid v \in X\}$. We say that a vertex $v$ *realizes color* $\psi(v)$ if $\psi(N(v))$ contains every color distinct from $\psi(v)$. We also call $v$ a *b-vertex* and say that $\psi(v)$ is *realized (on v)*. A *b-coloring of G* is a proper coloring $\psi$ such that each color is realized. The *b-chromatic number of G* is the maximum integer $b(G)$ for which $G$ has a b-coloring with $b(G)$ colors.

In [12], Irving and Manlove also introduced a simple upper bound for $b(G)$, defined as follows. The *m-degree of G* is the maximum integer $k$ for which there are at least $k$ edges of degree at least $k-1$; we denote it by $m(G)$. It is easy to see that

$$\chi(G) \leq b(G) \leq m(G).$$

We mention that, up to our knowledge, only one other work has been done on this metric. In [13], Jakovac and Peterin investigate graphs whose line graphs have b-chromatic number inferior to their $m$-degree, and prove that graphs whose line graphs are cubic have b-chromatic number equal to 5, with the exception of four line graphs: $K_4$, $K_{3,3}$, the prism over $K_3$, and the cube $Q_3$. They also claim to have proved that the b-chromatic number of the line graph $G$ of a tree is either $m(G)$, or $m(G)-1$. However, in [17] the authors show that $m(G)-b(G)$ can be arbitrarily large when $G$ is the line graph of a tree. Nevertheless, in Section 3, we prove that this difference is at most 1 when the tree is a caterpillar. Despite being a simple class of graphs, the algorithm found is not quite as simple, and combines a Linear Programming model whose coefficient matrix is proved to be Totally Unimodular (TU), and a dynamic programming algorithm. For general trees, some partial results are presented in [19], and the decision problem for fixed $k$ is proved to be polynomial-time solvable [10,19]. We mention that b-coloring the line graph of a tree is equivalent to b-coloring a claw-free block graph, which are contained in the class of chordal graphs. Computing $b(G)$ when $G$ is chordal is $\mathcal{NP}$-hard and, up to now, nothing was known about the b-chromatic number of subclasses of chordal graphs. Finally, in Section 2, we prove that deciding if $b(G)$ equals $m(G)$ is $\mathcal{NP}$-complete, even if $G$ is the line graph of either a comparability graph or a $C_k$-free graph.

In the remaining text, we use the following notation and terminology. Consider a simple graph $G$, and let $u \in V(G)$. The *neighborhood of u in G* is the set of vertices adjacent to $u$, and is denoted by $N_G(u)$. The *closed neighborhood of u in G* is the set $N_G(u) \cup \{u\}$, and is denoted by $N_G[u]$. The *degree* of $u \in V(G)$ is the cardinality of $N(u)$, and is denoted by $d_G(u)$; analogously, the *degree of an edge* $e \in E(G)$ is the number of edges adjacent to $e$, and is denoted by $d_G(e)$. In every case, the subscript can be omitted if there is no ambiguity. Now, consider $X \subseteq V(G)$. Then, $N(X)$ denotes the subset $(\bigcup_{x\in X} N(x)) \setminus X$, while $N[X]$ denotes the subset $N(X) \cup X$. Also, given a proper coloring $\psi$ of $V(G)$, we denote by $\psi(X)$ the set $\{\psi(x) \mid x \in X\}$. Finally, the subset of all vertices of $G$ with degree at least $m(G)-1$ is denoted by $D(G)$ (these are the "candidates" for b-vertices).

## 2. $\mathcal{NP}$-completeness

In this section, for clarity reasons, we present the problem as an edge b-coloring. Consider the adaptation of the problem to edge-coloring. The corresponding values of $b(G)$ and $m(G)$ are denoted by $b'(G)$ and $m'(G)$, respectively. We consider the following problems:

**Edge Coloring**
**Instance:** A graph $G$ and an integer $k$, $k \geq 3$.
**Question:** Is there a proper edge coloring of $G$ with $k$ colors?

**Edge B-Coloring**
**Instance:** A graph $G$.
**Question:** Is $b'(G)$ equal to $m'(G)$?

A graph is called $k$-regular if each of its vertices has degree $k$; a comparability graph is a graph whose edges can be transitively oriented; and a graph is called $C_t$-free if it has no induced cycle of length $t$. The Problem **Edge Coloring** is $\mathcal{NP}$-complete even when $G$ is a $k$-regular graph and is either a comparability graph or a $C_t$-free graphs[4]. We prove that this problem can be reduced to the Problem **Edge B-Coloring**.

**Theorem 1.** *Edge B-Coloring is $\mathcal{NP}$-complete, even if G is either a comparability graph or a $C_k$-free graph, for $k \geq 4$.*

**Proof.** Denote by $V$ the vertex set of $G$ and by $n$ the cardinality of $V$. To verify if an edge coloring is an edge b-coloring with $m'(G)$ colors can be done in polynomial time and so the problem is in $\mathcal{NP}$. We show a reduction from **Edge Coloring** of $d$-regular graphs in order to prove $\mathcal{NP}$-completeness. Also, we show that the construction is closed under the subclasses of comparability and of $C_k$-free graphs, for $k \geq 4$. Since **Edge Coloring** is $\mathcal{NP}$-complete even when restricted to instances of type $(G, d)$, where $G$ is a $d$-regular graph which is also either a comparability graph or a $C_k$-free graph [4], the theorem follows.

Consider a $d$-regular graph $G$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, \ldots, e_m\}$. Let $H$ be the graph constructed from $G$ as follows. Add vertices $w, w'$ and edges $ww', wv_1, \ldots, wv_n$ to $H$. Finally, for each $i \in \{1, \ldots, d\}$, add vertices $w_i, x_1^i, \ldots, x_n^i$ and edges $w'w_i$ and $w_ix_j^i$, for all $j \in \{1, \ldots, n\}$. Fig. 1 shows graph $H$.

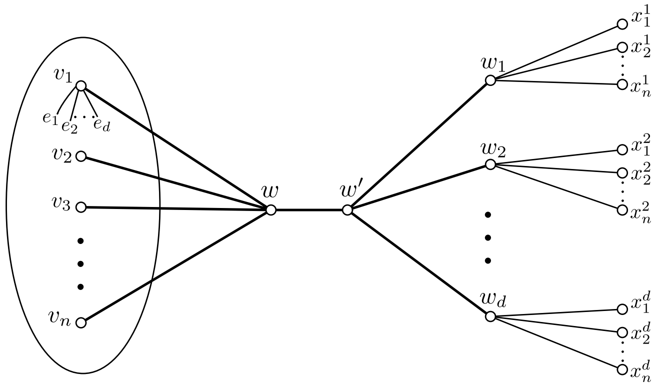


**Fig. 1.** Gadget $H$.

Observe that $d_H(wv_i) = n + d$, for each $i \in \{1, \dots, n\}$, $d_H(ww') = n + d$ and $d_H(w'w_i) = n + d$, for each $i \in \{1, \dots, d\}$. Moreover, for each $i, j \in \{1, \dots, n\}$, $i \neq j$, the degree of $v_iv_j$ in $H$ is $2d$ (which is strictly inferior to $n + d$, since $G$ is $d$-regular, i.e., $n > d$); and, for each $i \in \{1, \dots, d\}, j \in \{1, \dots, n\}$, the degree of $w_ix_j^i$ is $n$. In short, there are $n+d+1$ edges with degree $n + d$ and each other edge has degree inferior to $n + d$. Therefore, we have that $m'(H) = n + d + 1$ and there are exactly $m'(H)$ edges with degree at least $m'(H) - 1$. We claim that $G$ has an edge coloring with $d$ colors if and only if $b'(H) = m'(H)$.

Suppose $\psi$ is an edge coloring of $G$ that uses colors $\{1, \dots, d\}$. We shall construct an edge b-coloring $\psi'$ of $H$ with $n+d+1$ colors. Let $\psi'(e) = \psi(e)$, for all $e \in E(G)$, $\psi'(ww') = n + d + 1$ and $\psi'(wv_j) = j + d$, for all $j \in \{1, \dots, n\}$. Since $\psi$ is a proper edge coloring and $G$ is $d$-regular, every $v_i \in V(G)$ is incident to an edge of each color in $\{1, \dots, d\}$. Therefore, in the partial coloring $\psi'$ the edges $wv_i$ are b-edges of their respective colors. We can easily make the edge $ww'$ be the b-edge of color $n + d + 1$ and $w'w_i$ be the b-edge of color $i$ by coloring $w'w_i$ with color $i$ and coloring $w_ix_j^i$ with color $d + j$, for each $i \in \{1, \dots, d\}$ and $j \in \{1, \dots, n\}$. Then, $\psi'$ is an edge b-coloring of $H$ with $m'(H) = n + d + 1$ colors.

Now, let $\psi'$ be an edge b-coloring of $H$ that uses $n + d + 1$ colors. The edges that are incident to $w$ should receive distinct colors, therefore we may assume that $\psi'(ww') = n + d + 1$ and $\psi'(wv_i) = d + i$, for $i \in \{1, \dots, n\}$. Since there are exactly $n + d + 1$ edges of degree $n + d$ in $H$, the edges incident to $w$ must be b-edges of their respective colors. But then, since each $wv_i$ has degree exactly $n + d$ in $H$, all the edges adjacent to it should have distinct colors, $i \in \{1, \dots, n\}$. Because no edge incident to $w$ has a color in $\{1, \dots, d\}$, for a vertex $v_i \in V(G)$ with $N_G(v_i) = \{a_1, \dots, a_d\}$, we have that the edges $\{v_ia_1, \dots, v_ia_d\}$ have distinct colors in $\{1, \dots, d\}$. That implies that $\psi'$ restricted to $V$ is an edge coloring of $G$ with $d$ colors.

It remains to prove that the construction is closed under the classes of $C_k$-free graphs, $k \geq 4$, and comparability graphs. For $C_k$-free graphs, this clearly holds since the construction does not create induced cycles of length greater than 3. Now, suppose that $G$ is a comparability $d$-regular graph and let $\pi$ be a transitive orientation of its edges. We construct $H$ as explained above and extend $\pi$ to $H$ as follows: $w \to v_i$, for each $i \in \{1, \dots, n\}$; $w \to w'$; $w_i \to w'$, for each $i \in \{1, \dots, d\}$; $w_i \to x_j^i$, for each $i \in \{1, \dots, d\}$ and each $j \in \{1, \dots, n\}$. Let $\pi'$ be the orientation obtained. The set of edges in $E(H) - E(G)$ are transitively oriented, since there is no directed $P_2$. Also, because $\pi$ is a transitive orientation, and $w$ has indegree 0 in $\pi'$ and is adjacent to all vertices of $V(G)$, we get that $\pi'$ is indeed a transitive orientation. The theorem thus follows. □

## 3. Caterpillars

A *caterpillar* is a tree in which every vertex of degree at least two is contained in a path. In this section, we show that computing the b-chromatic number of the line graph of a caterpillar can be done in polynomial time.

Let $G$ be the line graph of a caterpillar. Informally, $G$ can be seen as a "path of cliques". Because $G$ is a chordal graph, if $m(G) = \omega(G)$, we know that $\chi(G) = \omega(G) = m(G) \geq b(G) \geq \chi(G)$; hence, any optimal proper coloring of $G$ is also an optimal b-coloring. Therefore, we can suppose that $m(G) > \omega(G)$, which implies that each $w \in D(G)$ is a cut vertex of $G$. We call the subgraph induced by the cut vertices the *central path of* $G$. Let $P = \langle v_1, \dots, v_n \rangle$ be the central path of $G$; we say that *a vertex* $v_i$ *is to the left of a vertex* $v_j$ (or that $v_j$ *is to the right of* $v_i$) if $i < j$; and that a vertex $w \notin P$ *is to the left of vertex* $v_j$ (or that $v_j$ *is to the right of* $w$) if there exists $v_i \in N(w)$ such that $i < j$.

In this section, we prove the following theorem.

**Theorem 2.** *If $G$ is the line graph of a caterpillar, then $b(G)$ is either $m(G)$ or $m(G) - 1$, and deciding its value can be done in polynomial time.*

We break the proof of Theorem 2 in two parts. In the next theorem, we present a polynomial algorithm that gives a b-coloring of $G$ with $m(G) - 1$ colors, thus proving that $b(G) \geq m(G) - 1$. Then, the remainder of the section is dedicated to presenting a polynomial algorithm that decides whether $b(G) = m(G)$. We note that the proof of the next theorem can be generalized so that it finds a b-coloring of $G$ with $k$ colors, for any $k \in \{\chi(G), \dots, m(G) - 1\}$.

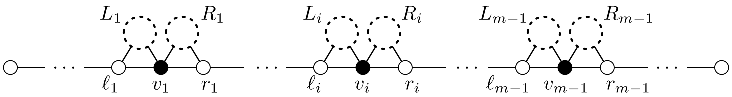


**Fig. 2.** Structure of subsets $L_i$, $R_i$, and vertices $\ell_i$, $r_i$.

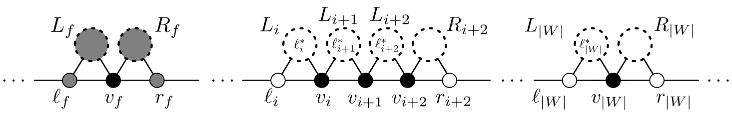


**Fig. 3.** Partial coloring when $r_i = v_{i+1}$.

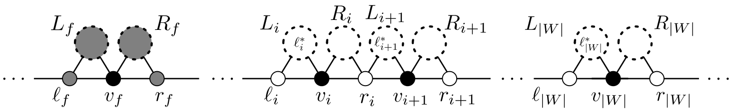


**Fig. 4.** Partial coloring when $r_i = \ell_{i+1}$.

**Theorem 3.** *If $G = (V, E)$ is the line graph of a caterpillar, then a b-coloring of $G$ with $m(G) - 1$ colors exists and can be found in polynomial time.*

**Proof.** Denote $m(G)$ by $m$. We know that $b(G) \geq \chi(G) = \omega(G)$. Thus, if $m \leq \omega(G) + 1$, the theorem follows. So, suppose that $m \geq \omega(G) + 2$. Note that if $\omega(G) = 2$, then we get $m \leq \Delta(G) + 1 = 3$. Therefore, the fact that $m \geq \omega(G) + 2$ implies that $\omega(G) \geq 3$, and that $m \geq 5$.

Let $V' = \{v_1, \ldots, v_{m-1}\}$ be any subset of $D(G)$. Suppose, without loss of generality, that $V'$ is numbered from left to right. For each $v_i$, denote by $\ell_i$ the left neighbor of $v_i$ in the central path, and by $r_i$ the right one; also, let $L_i$ be the vertices not in the central path adjacent to $\ell_i$ and $v_i$, and $R_i$ be the ones adjacent to $r_i$ and $v_i$. Observe Fig. 2. Note that if $L_i = \emptyset$, then since $d(v_i) \geq m - 1$ and $v_i$ has at most one adjacent vertex non-incident to $r_i$ (namely $\ell_i$), we have $\omega(G) \geq |R_i \cup \{v_i, r_i\}| \geq m - 1$ contradicting the fact that $\omega(G) \leq m - 2$. We can thus suppose that $L_i, R_i \neq \emptyset$, for all $i \in \{1, \ldots, m - 1\}$.

Initially, for each $v_i \in V'$, we color $v_i$ with $i$ and color exactly one vertex in $L_i$, named $\ell_i^*$, as follows. If $r_i = v_{i+1}$ and $i + 2 < m$, then color $\ell_i^*$ with $i + 2$; otherwise, color $\ell_i^*$ with $i + 1$. After this, we color the remaining vertices in $N(V')$ in a way as to turn each $v_i \in V'$ into a b-vertex. For this, we color the neighborhood of each component of $G[V']$ iteratively, starting from the leftmost component. So, consider $W \subseteq V'$ as being the vertex set of a component of $G[V']$, and let $v_f$ be its leftmost vertex. We start by coloring $L_f \cup \{\ell_f\}$, then, we iterate on $W$ from left to right coloring $R_i$, for every $v_i \in W$. During the procedure, we will ensure that the properties below hold for every non-iterated $v_i$. Note that they hold initially.

**(P1)** If $r_i = v_{i+1}$ and $i + 2 < m$, then $i + 2 \in \psi(N(v_i))$; otherwise, $i + 1 \in \psi(N(v_i))$;
**(P2)** At most one color is repeated in $N(v_i)$. Furthermore, if $v_{i-1}$ has not been iterated, then no color is repeated in $N(v_i)$;
**(P3)** If $r_{i-1}\ell_i \in E(G)$ and $v_{i-1}$ has been iterated, then either $\psi(r_{i-1}) \in \psi(N(v_i))$, or $\ell_i$ is colored during the iteration of $v_{i-1}$.

From now on, we denote by $M_i$ the set of missing colors in $N(v_i)$, for each $v_i \in V'$. Observe that, by (P2), at any step of the coloring procedure, vertex $v_i$ has as many uncolored neighbors as there are colors in $M_i$. As stated before, we start by coloring the uncolored vertices in $L_f \cup \{\ell_f\}$. Note that the difficulty is in coloring vertex $\ell_f$, and it arises only when $r_{f-1}\ell_f \in E(G)$ (if $\ell_f = r_{f-1}$, we know that $\ell_f$ is colored during the iteration of $v_{f-1}$). But in this case, by (P3), we know that $\psi(r_{f-1}) \notin M_f$. Therefore, we can color $L_f \cup \{\ell_f\}$ with colors from $M_f$. If $|L_f| + 1 > |M_f|$, just greedily color the remaining vertices (recall that $m - 1 > \omega(G)$).

Now, we iterate on $W$ from left to right coloring the right neighborhood of the vertices. So, consider the iteration of $v_i \in W$; we analyze the cases below. In any of these cases, if some vertices of $R_i$ are uncolored at the end, we properly color them with colors from $M_{i+1}$ preferably, or any other color in the case where $M_{i+1} = \emptyset$.

- $r_i = v_{i+1}$: by (P1), we know that color $i + 2$ appears in $N(v_i)$, if it exists. So, at most one color of $M_i$ appears in $N(v_{i+1})$, namely the color given to $\ell_{i+2}^*$ when $r_{i+1} = v_{i+2}$. Therefore, coloring $R_i$ with colors from $M_i$ will repeat at most one color in $N(v_{i+1})$. Fig. 3 shows the partial coloring in this case, where the gray vertices are that colored, as each vertex $v_j \in W$, the black vertices, in addition to $\ell_j^*$;
- $r_i = \ell_{i+1}$: note that the difficulty is in coloring $r_i$. However, by (P2), we know that when coloring $R_i \cup \{r_i\}$, we will repeat at most one color in $N(v_{i+1})$. Fig. 4 shows this case;
- $r_i\ell_{i+1} \in E(G)$: again, the difficulty is in coloring $r_i$, since we want Property (P3) to hold. If there exists $c \in M_i \cap \psi(N(v_{i+1}))$, then color $r_i$ with $c$. Otherwise, we get that $\psi(\ell_{i+1}^*) \notin M_i$. In this case, we color $\ell_{i+1}$ with $\psi(\ell_{i+1}^*)$, and uncolor $\ell_{i+1}^*$. Since $\psi(\ell_{i+1}^*) \notin M_i$, after this we can color $r_i$ freely. Note that Properties (P1)–(P3) do not mention vertex $\ell_j^*$, for any $j$; hence, we can let $\ell_{i+1}^*$ uncolored without impacting the proof. Fig. 5 represents this case;
- $r_i \notin N(\ell_{i+1}) \cup \{\ell_{i+1}, v_{i+1}\}$: just color $R_i$ with colors from $M_i$. Fig. 6 shows this case.

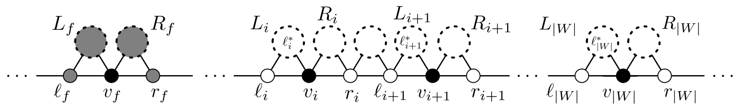


**Fig. 5.** Partial coloring when $r_i v_{i+1} \in E(G)$.

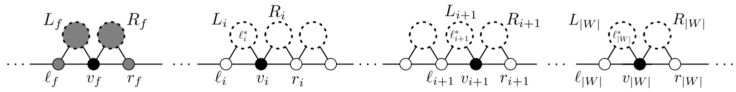


**Fig. 6.** Partial coloring when $r_i \notin N(\ell_{i+1}) \cup \{\ell_{i+1}, v_{i+1}\}$.

Finally, it remains to extend $\psi$ to a b-coloring of $G$. First, let $u$ be any uncolored vertex of the central path. We know that $u \notin N(V')$. Thus, there are at most two colored vertices adjacent to $u$, namely those pertaining to the central path which are in the neighborhood of some $v_i \in V'$. Since $m \geq 5$, there is some color $i \in \{1, \ldots, m-1\}$ with which we can color $u$. Finally, we can greedily color the vertices not in the central path since $\omega(G) \leq m-2$. □

Now, we show how to decide if a b-coloring of $G$ with $m(G)$ colors exists. We do this by testing, for a number of subsets $W \subseteq D(G)$, if the answer to the problem below is "YES". Later in the section, we show that only a linear number of subsets need to be tested.

**PARTIAL BASIS**

**INSTANCE:** A GRAPH $G$ AND A SUBSET $W \subseteq D(G)$ WITH $|W| \leq m(G)$.

**QUESTION:** IS THERE A COLORING OF $G[N(W) \cup W]$ WITH $m(G)$ COLORS THAT REALIZES $|W|$ DISTINCT COLORS?

Let $\{v_1, \ldots, v_t\}$ be the vertices of $D(G)$ in the order they appear in the central path of $G$ from left to right. For each $i \in \{1, \ldots, t\}$, let $B_i$ denote the maximum cardinality of a subset $W \subseteq \{v_1, \ldots, v_i\}$ for which the answer to **PARTIAL BASIS** is "YES", and let $\mathcal{S}_i$ contain all such subsets with cardinality $B_i$. Clearly, $b(G) = m(G)$ if and only if $B_t = m(G)$. The lexicographic value of a subset $W \in \mathcal{S}_i$ is $\varrho(W) = \sum_{i=1}^{t} 2^{\phi_W[i]}$, where $\phi_W \in \{0, 1\}^{|D(G)|}$ is the incidence vector of $W$. Finally, let $S_i$ denote the subset in $\mathcal{S}_i$ of minimum lexicographic value. We prove that $S_{i-1} \subseteq S_i$, for all $i \in \{2, \ldots, t\}$. For this, we sometimes need to combine colorings of different "parts" of $G$ into a coloring of $G$ in such a way not to decrease the number of realized colors. The following tool lemma does that.

**Lemma 4.** *Let $v \in D(G)$ and $V_1, V_2$ be the vertex sets of the two components of $G - v$. Let $G_1 = G[V_1 \cup \{v\}]$, $G_2 = G[V_2 \cup \{v\}]$, and $\psi_1, \psi_2$ be partial colorings with $m(G)$ colors of $G_1, G_2$, respectively. If $\psi_i$ realizes $k_i$ colors, for $i \in \{1, 2\}$, then there exists a partial coloring $\psi$ of $G$ with $m(G)$ colors that realizes* $\min\{k_1 + k_2, m(G)\}$ *colors. Furthermore, $\varrho(\psi) \leq \varrho(\psi_1) + \varrho(\psi_2)$.*

**Proof.** Rename the colors so that colors $1, \ldots, k_1$ are realized by $\psi_1$ on vertices $\{u_1, \ldots, u_{k_1}\}$, respectively, and that $\psi_2$ realizes colors $k_1 + 1, \ldots, p$ on vertices $\{u_{k_1+1}, \ldots, u_p\}$, respectively, where $p = \min\{k_1 + k_2, m(G)\}$ (the vertices are numbered in the order they appear in the central path). We prove the lemma by recoloring $G_1$ and $G_2$ so that the color of $v$ matches; hence, the colorings can be "glued" in order to obtain the desired coloring. Let $c_1 = \psi_1(v)$, $c_2 = \psi_2(v)$ and suppose that $c_1 \neq c_2$. Also, let $L, R$ be the neighborhood of $v$ in $G_1, G_2$ not in the central path, respectively. Observe that we could easily switch the color of $v$ and some vertex of $L$ in order to make the color of $v$ in $\psi_1$ match the color of $v$ in $\psi_2$, the same argument being analogously valid for $R$. Thus, we can suppose that

$$\psi_1(L \cup \{v\}) \cap \psi_2(R \cup \{v\}) = \emptyset. \tag{1}$$

Throughout the proof we find colorings of $G_1$ and $G_2$ for which Eq. (1) does not hold and conclude that the lemma follows. Note that Eq. (1) implies:

$$|\psi_1(L) \cup \psi_2(R) \cup \{c_1, c_2\}| = d_G(v) \geq m(G) - 1. \tag{2}$$

We consider the following cases:

- *$w$ is not a b-vertex, where $w$ is the neighbor of $v$ in $G_1$ which belongs to the central path*: we suppose $\psi(w) = c_2$, as otherwise we can just change the color of $v$ in $\psi_1$ to $c_2$. Let $L'$ be the subset of neighbors of $w$ not in the central path which are not in $L$. If there exists $u \in L'$ such that $\psi_1(u) = c \notin \psi_1(L)$, then change the color of $u$ to $c_2$, the color of $w$ to $c$ and the color of $v$ to $c_2$. So, suppose otherwise (which implies $\psi_1(L') \subseteq \psi_1(L)$). If there exist $u \in L$ and $u' \in L'$ with $\psi_1(u) = \psi_1(u') = c$, then change the color of $u$ to $c_1$, the color of $u'$ and $v$ to $c_2$, and the color of $w$ to $c$. Otherwise, we get $L' = \emptyset$. We can suppose that $w$ is adjacent to $u_{k_1}$ as otherwise we can recolor $v$ with $c_2$ and $w$ with some color $c \notin \psi_1(N[w])$. Now, let $H$ be the component of $G - E(R \cup \{v, w\})$ containing $w$. If $u_{c_2} \notin V(H)$, then switch colors $c_2$ and $k_1 + 1$ in $H$, recolor $v$ with $c_2$ in $\psi_1$ and, if $u \in L$ is colored with $k_1 + 1$, then recolor it with any color not in $\psi_1(L) \cup \{k_1 + 1, c_2\}$. Otherwise, switch colors $c_2$ and $k_1$ in $H$, recolor $v$ with $c_2$ in $\psi_1$ and, if $u \in L$ is colored with $k_1$, then recolor it with any color not in $\psi_1(L) \cup \{k_1, c_2\}$.

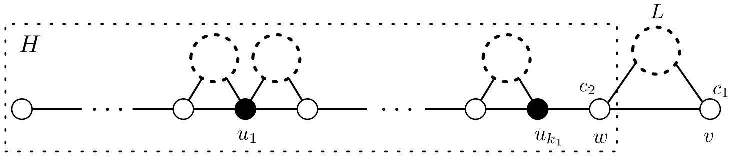


**Fig. 7.** Representation of component $G_1$, where $L' = \emptyset$, $\psi_1(v) = c_1$ and $\psi_1(w) = c_2$.

Fig. 7 shows component $G_1$ in the last step. With a symmetrical argument for $v$ in $G_2$, it remains to consider the next case.

- $u_{k_1}, u_{k_1+1} \in N(v)$: by Eq. (2), we know that at most one color of $\{1, \dots, m(G)\}$ does not appear in $\psi_1(L) \cup \psi_2(R) \cup \{c_1, c_2\}$. This includes $k_1, k_1 + 1$. We analyze the cases:
  - * *There exists* $c \in \{1, \dots, m(G)\} \setminus (\psi_1(L) \cup \psi_2(R) \cup \{c_1, c_2, k_1, k_1 + 1\})$: this implies that there must exist $x_2 \in R \cup \{v\}$ with $\psi_2(x_2) = k_1$ and $x_1 \in L \cup \{v\}$ with $\psi_1(x_1) = k_1 + 1$. In this case, we change $\psi_1$ in such a way that $v$ can become a b-vertex of color $k_1$ as follows. We switch $k_1$ and $k_1 + 1$ in $\psi_1$, and recolor $v$ with $k_1$, $x_2$ with $c_2$, and $x_1$ with $c_1$. Observe that the number of colors appearing in $R \cup L \cup \{v, u_{k_1}, u_{k_1+1}\}$ does not change, and that the only missing color in $N(v)$ is $c$. Therefore, if we switch colors $c$ and $k_1 + 1$ in $\psi_2$, we get that $v$ is now a b-vertex of color $k_1$. Additionally, note that, before the last switch, vertex $u_{k_1+1}$ was still a b-vertex of color $k_1 + 1$. Hence, if color $c$ has its b-vertex in $\psi_2$, then $u_{k_1+1}$ is now the new b-vertex of this color. Finally, note that colors 1 to $k_1 + 1$ are realized on vertices $\{u_1, \dots, u_{k_1}, v\}$, while colors $k_1 + 2$ to $p$ are realized on a subset of $\{u_{k_1+1}, \dots, u_p\}$. This implies that $\varrho(\psi) \leq \varrho(\psi_1) + \varrho(\psi_2)$.
  - * $\psi_1(L) \cup \psi_2(R) \cup \{c_1, c_2, k_1, k_1 + 1\} = \{1, \dots, m(G)\}$: first, suppose that $k_1 \notin \psi_2(R \cup \{v\})$; this implies that $k_1 + 1 \in \psi_1(L \cup \{v\})$. If $c_2 \notin \{1, \dots, k_1\}$, we can switch $c_2$ and $k_1 + 1$ in $\psi_1$, obtaining a partial b-coloring that violates Eq. (1). Therefore, we can suppose $c_2 \in \{1, \dots, k_1\}$, in which case we can switch $c_2$ and $k_1$ in $\psi_2$ with no loss, thus obtaining a partial b-coloring where $k_1 + 1 \in \psi_1(L \cup \{v\})$ and $k_1 \in \psi_2(R \cup \{v\})$. In the case where $k_1 + 1 \notin \psi_1(R \cup \{v\})$, an analogous argument can be made. Therefore, we can always suppose that $k_1 + 1 \in \psi_1(L \cup \{v\})$ and $k_1 \in \psi_2(R \cup \{v\})$, in which case we can also suppose that $c_1 \in \{k_1 + 1, \dots, p\}$ and $c_2 \in \{1, \dots, k_1\}$. Now, switch colors $k_1 + 1$ and $c_2$ in $\psi_2$, and $k_1 + 1$ and $c_1$ in $\psi_1$; let $\psi$ be the union of the obtained colorings. Observe that the only color that loses a b-vertex is color $k_1 + 1$; however, since the number of colors in $R \cup L \cup \{u_{k_1}, u_{k_1+1}\}$ does not change, we get that $v$ is a b-vertex of color $k_1 + 1$ in $\psi$. Finally, note that $\psi$ realizes colors $\{1, \dots, k_1 + 1\}$ on vertices $\{u_1, \dots, u_{k_1}, v\}$, while colors $\{k_1 + 2, \dots, c, p\}$ are realized on vertices $\{u_{k_1+2}, \dots, u_p\}$. This implies that $\varrho(\psi) \leq \varrho(\psi_1) + \varrho(\psi_2)$. □

**Lemma 5.** *For each* $i \in \{2, \dots, t\}$, $S_{i-1} \subseteq S_i$.

**Proof.** Suppose otherwise and let $f$ ($f'$) be a partial coloring of $G$ where the vertices of $S_i$ ($S_{i-1}$) realize distinct colors. Clearly, we get $v_i \in S_i$, as otherwise $B_i = B_{i-1}$ and we would get $S_i = S_{i-1}$. Furthermore, we have that $S^* = (S_i \setminus \{v_i\})$ is in $\mathcal{S}_{i-1}$, and, because $|S_{i-1}| = |S^*| = B_{i-1}$, we have $\varrho(S_{i-1}) < \varrho(S^*)$. Let $j$ be the maximum index such that $v_j \in S^* \setminus S_{i-1}$, and let $t = |S^* \cap \{v_{j+1}, \dots, c, v_{i-1}\}|$.

Let $f_1$ be equal to $f$ restricted to $G[N[\{v_{j+1}, \dots, c, v_i\}]]$, and $f_2$ be equal to $f'$ restricted to $G[N[\{v_1, \dots, v_{j-1}\}]]$. Note that, by the definition of $j$ and the fact that $f_2$ is a restriction of $f'$, we get $\varrho(f_1) + \varrho(f_2) = \varrho(S_{i-1}) + 2^i$. But also, note that $f_1$ realizes $f + 1$ colors and $f_2$ realizes $B_{i-1} - t$ colors. By Lemma 4, there exists a coloring $f^*$ of $G$ that realizes $t + 1 + B_{i-1} - t = B_i$ colors with the further property that the lexicographic value of $f^*$ is at most $\varrho(f_1) + \varrho(f_2) = \varrho(S_{i-1}) + 2^i < \varrho(S^*) + 2^i = \varrho(S_i)$, a contradiction. □

Now, our solution works as follows: compute $S_1$ (it suffices to verify if $v_1$ has degree at least $m(G) - 1$); then, for each $i \in \{2, \dots, t\}$, if the answer to **Partial Basis** applied to $(G, S_{i-1} \cup \{v_i\})$ is "yes", then $S_i$ equals $S_{i-1} \cup \{v_i\}$; otherwise, $S_i$ equals $S_{i-1}$. Therefore, we only need to solve subproblem **Partial Basis** at most $t$ times. To solve **Partial Basis**, we use an integer programming model and prove that its coefficient matrix is TU (totally unimodular). A matrix is called TU if the determinant of any of its square submatrices is either $-1$, 0, or 1; and it is well known that if $A$ is a TU matrix and $b$ is an integer vector, then all vertex solutions of $\max c^T x \ : \ Ax \leq b$ are integer. Let $W \subseteq D(G)$ be a subset with $k$ vertices, and denote by $m$ the value $m(G)$. We want to decide if $W$ can realize $k$ distinct colors in a partial coloring with $m$ colors. We first color each $w \in W$ with a distinct color in $\{1, \dots, c, k\}$, and denote the vertex colored with $i$ by $w_i$. Let $\mathcal{C}$ be the subset containing every maximal clique $C \subseteq N(W)$ of $G$ not containing vertices of the central path. For each $C \in \mathcal{C}$ and each color $i \in \{1, \dots, m\}$, define:

$$x_{C,i} = \begin{cases} 1, & \text{if color } i \text{ appears in the clique } C; \\ 0, & \text{otherwise.} \end{cases}$$

Additionally, let $\mathcal{V}$ be the set of vertices in the central path that are in $N(W)$ but not in $W$. For each $v \in \mathcal{V}$ and each color $i \in \{1, \dots, m\}$, define:

$$y_{v,i} = \begin{cases} 1, & \text{if vertex } v \text{ is colored with color } i; \\ 0, & \text{otherwise.} \end{cases}$$

For each $w_i \in W$, we denote the clique in $\mathcal{C}$ to its left (right) by $C_{\ell_i}$ ($C_{r_i}$), and the left (right) neighbor of $w_i$ within the central path by $\ell_i$ ($r_i$). Note that $\ell_i \in \mathcal{V}$ if and only if $\ell_i \notin W$, the same being valid for $r_i$. The constraints are:

$$x_{C_{\ell_i},j} + x_{C_{r_i},j} + y_{\ell_i,j} + y_{r_i,j} = 1, \quad \forall i \in \{1,\ldots,k\},\ \forall j \in \{1,\ldots,m\} \setminus \{i\} \tag{3}$$

$$\sum_{j=1}^{m} x_{C,j} \le |C|, \quad \forall C \in \mathcal{C} \tag{4}$$

$$\sum_{j=1}^{m} y_{v,j} \le 1, \quad \forall v \in \mathcal{V}. \tag{5}$$

Denote by $P(G, W)$ the polytope defined by the constraints above. Constraints (3) ensure that all the colors appear in $N[w_i]$, for all $i \in \{1,\ldots,k\}$; Constraints (4) ensure that no more than $|C|$ colors appear in $C$; and Constraints (5) ensure that $v$ receives at most one color. Clearly, if $\psi$ is a partial coloring that realizes $k$ colors on $W$, then a correspondent integer point of $P(G, W)$ can be obtained. However, because there is no constraint to ensure that the obtained coloring is proper, it is not clear if an integer point of $P(G, W)$ also produces the desired coloring. To settle this, we use Lemma 4. We also ensure that the produced coloring does not have a larger lexicographic value.

**Lemma 6.** *If $(x, y)$ is an integer point in $P(G, W)$, then there exists a coloring $\psi$ of $G$ with $m$ colors that realizes $|W|$ colors. Furthermore, the lexicographic value of $\psi$ is no larger than the lexicographic value of $W$.*

**Proof.** Let $\psi$ be the partial coloring related to $(x, y)$. We suppose that $\psi$ is not proper, as otherwise there is nothing to do. If there is a conflict in a vertex $v$ that is not in the central path, it is easy to see that the partial coloring obtained from $\psi$ just by uncoloring $v$ still realizes the same number of colors. Also, there is no conflict between vertices of $W$ since each is colored with a distinct color. Thus, every conflict involves some vertex of the central path that is not in $W$. So, let $u_1,\ldots,u_p$ be all the vertices in the central path that have some conflict and suppose that they are ordered as they appear in the central path. Let $V'_0,\ldots,V'_p$ be the vertex sets of the components of $G - \{u_1,\ldots,u_p\}$, and let $G_0 = G[V'_0 \cup \{u_1\}]$, $G_p = G[V'_p \cup \{v_p\}]$ and, for each $i \in \{2,\ldots,p-1\}$, let $G_i = G[V_i \cup \{u_i, u_{i+1}\}]$. Also, for each $i \in \{0,\ldots,p\}$, let $\psi_i$ be the coloring $\psi$ restricted to $G_i$. Now, for each $u_i$, if $u_i$ has a conflict with a vertex in $\psi_j, j \in \{i-1, i\}$, we recolor $u_i$ in $\psi_j$ with any color not in $\psi_j(N(u_i))$; this is possible because $m > \omega(G)$ and $u_i$ participates in exactly one clique in $G_j$. Note that this does not change the number of realized colors in $\psi_j$. Finally, let $G^*$ be initially $G^0$; we apply Lemma 4 to $G^*$ and $G_1$ in order to obtain a proper partial coloring that realizes the sum of the amount of colors realized by $\psi_0$ and $\psi_1$, and we can do this in such a way as to not increase the lexicographic value. We then increase $G^*$ to be $G_0 \cup G_1$ and apply the same argument to $G^*$ and $G_2$, and so on, until we obtain the desired partial coloring. □

Finally, we prove that one can find an integer point of $P(G, W)$ in polynomial time, if one exists.

**Theorem 7.** *The coefficient matrix $Q$ that defines $P(G, W)$ is TU.*

**Proof.** It is known that a matrix $A$ is TU if and only if every subset $J$ of row indexes can be partitioned into $J_1, J_2$ in a way that, for every column $j$:

$$\left| \sum_{i \in J_1} a_{i,j} - \sum_{i \in J_2} a_{i,j} \right| \le 1. \tag{6}$$

We prove that this holds for our matrix. We call a constraint of Type (3) a *mixed row*; a constraint of Type (4), a *clique row*; and a constraint of Type (5) a *vertex row*. We index a row $r$ by its correspondent pair $(w, j)$, $w \in W$, $j \in \{1,\ldots,m(G)\}$, when $r$ is a mixed row, or by its correspondent clique when $r$ is a clique row, or by its correspondent vertex when $r$ is a vertex row.

Let $J$ be any subset of row indexes. For each $i \in \{1,\ldots,k\}$, we denote the set of rows $J \cap \{r_{\ell_i}, r_{r_i}, r_{C_{\ell_i}}, r_{C_{r_i}}\}$ by $V_i$, and the set $J \cap \{r_{(w_i,j)}, j \in \{1,\ldots,m(G)\}\}$ by $M_i$. We will iterate on $w_1,\ldots,w_k$ in this order and, at each step, decide where to put the rows involving variables related to $w_i, r_i, \ell_i, C_{r_i}, C_{\ell_i}$. First, put $V_1 \cap J$ in $J_1$ and $M_1 \cap J$ in $J_2$. Then, for $i = 2,\ldots,m(G)$, let $\phi$ be the structure which is adjacent to both $w_i$ and $w_{i-1}$, if it exists (i.e., $\phi$ is either $r_{i-1} = \ell_i$, or $C_{r_{i-1}} = C_{\ell_i}$, or does not exist).

- If $\phi$ exists, then suppose, without loss of generality, that $M_{i-1} \cap J \subseteq J_1$. By the construction, we know that if $r_\phi \in J$, then $r_\phi \in J_2$. If $r_\phi \in J$, then put $V_i \cap J$ in $J_2$ and $M_i \cap J$ in $J_1$; otherwise, put $M_i \cap J$ in $J_2$ and $V_i \cap J$ in $J_1$.
- If $\phi$ is not determined, then put $V_i \cap J$ in $J_1$ and $M_i \cap J$ in $J_2$.

Now, consider a variable $x_{C,j} \in J$ and let $w, w' \in V(G)$ be the neighbors of $C$ in the central path. Note that at most three rows of $Q$ contain $x_{C,j}$, namely $r_{(w,j)}$, $r_{(w',j)}$, and $r_C$. By definition, $W \cap \{w, w'\} \neq \emptyset$; thus, suppose that $w = w_i \in W$. By the construction, we know that $\{r_{(w_i,j)}\} \cap J \subseteq M_i$ and $\{r_C\} \cap J \subseteq V_i$ are put in different parts. Therefore, if $r_C \in J$, then (6) holds for $x_{C,j}$. So, suppose that $r_C \notin J$. We can also suppose that $w' = w_{i-1} \in W$ and that $r_{(w_{i-1},j)}, r_{(w_i,j)} \in J$, as otherwise Eq. (6) again follows. In the construction, we get $\phi = C$ in the iteration of $w_i$ and that $M_i$ is put in a part different from the one containing $M_{i-1}$. Then, Eq. (6) follows for $x_{C,j}$. A similar argument can be done for variables $y_{v,j}$. □

Finally, it remains to extend the obtained partial coloring $\psi$ to the rest of $G$. This can be easily done as follows. Let $v$ be an uncolored vertex. If $v \notin D(G)$, just color $v$ with any color that does not appear in its neighborhood. Otherwise, let $V_1, V_2$ be the vertex sets of the components of $G - v$, and let $G_i = G[V_i \cup \{v\}]$, for $i \in \{1, 2\}$. Also, let $\psi_i$ be $\psi$ restricted to $G_i$, for $i \in \{1, 2\}$. Since $m > \omega(G)$, we can color $v$ in $\psi_1$ and in $\psi_2$; then, we join these partial colorings using Lemma 4.

## 4. Conclusion

We have proved that deciding whether the line graph of $G$ can be b-colored with at least $k$ colors is an $\mathcal{NP}$-complete problem, even if $G$ is a comparability graph or a $C_t$-free graph, for $t \geq 4$. Furthermore, we prove that if $G$ is a caterpillar, then $b(L(G)) \geq m(G) - 1$ and we give a polynomial algorithm to find an optimal b-coloring of $L(G)$. We mention that, up to our knowledge, apart from the seminal result by Irving and Manlove about trees, this is the only positive result concerning the b-chromatic number of a subclass of chordal graphs. Unfortunately, the method presented in Section 3 cannot be generalized for trees because the related coefficient matrix is not TU. Also, in [17], Maffray and Silva show that the difference $m(G) - b(G)$ can be arbitrarily large when $G$ is the line graph of a tree. This means that Theorem 2 does not hold for these graphs. Finally, we mention that in [19], Silva showed some partial results for trees. In particular, she shows that if $G$ is a tree and $W \subseteq D(L(G))$ is a subset with $m(G)$ vertices such that every vertex $v \in V(L(G)) \setminus W$ having more than one neighbor in $W$ is a cut-vertex, then $b(G) = m(G)$. The complexity of the problem for line graphs of general trees remains open.

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