Discrete Mathematics 338 (2015) 1860–1865

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

The complexity of the 3-colorability problem in the absence of a pair of small forbidden induced subgraphs

D.S. Malyshev

National Research University Higher School of Economics, 25/12 Bolshaya Pecherskaya Ulitsa, 603155, Nizhny Novgorod, Russia

ARTICLE INFO

Article history:
Received 20 October 2013
Received in revised form 18 April 2015
Accepted 20 April 2015
Available online 5 June 2015

Keywords:
3-colorability problem
Computational complexity
Polynomial-time algorithm

ABSTRACT

We completely determine the complexity status of the 3-colorability problem for hereditary graph classes defined by two forbidden induced subgraphs with at most five vertices.

1. Introduction

The coloring problem is one of classical problems on graphs. Its formulation is as follows. A *coloring* is an arbitrary mapping of colors to vertices of some graph. A graph coloring is said to be *proper* if no pair of adjacent vertices have the same color. The *chromatic number* $\chi(G)$ of a graph G is the minimal number of colors in proper colorings of G. The *coloring problem* for a given graph and a number k is to determine whether its chromatic number is at most k or not. The *k-colorability problem* is to verify whether vertices of a given graph can be properly colored with at most k colors.

A graph H is an *induced subgraph* of G if H is obtained from G by deletion of vertices. A *class* is a set of simple unlabeled graphs. A class of graphs is *hereditary* if it is closed under deletion of vertices. It is well known that any hereditary (and only hereditary) graph class $\mathcal{X}$ can be defined by a set of its forbidden induced subgraphs $\mathcal{Y}$. We write $\mathcal{X} = Free(\mathcal{Y})$ in this case, and the graphs in $\mathcal{X}$ are said to be *$\mathcal{Y}$-free*. If $\mathcal{Y} = \{G\}$, then we will write "G-free" instead of "$\{G\}$-free". If a hereditary class can be defined by a finite set of the forbidden induced subgraphs, then it is said to be *finitely defined*.

The coloring problem for G-free graphs is polynomial-time solvable if G is an induced subgraph of P_4 or $P_3 + K_1$, and it is NP-complete in all other cases [13]. The situation for the k-colorability problem is not clear, even when only one induced subgraph is forbidden. The complexity of the 3-colorability problem is known for all classes of the form $Free(\{G\})$ with $|V(G)| \leq 6$ [4]. A similar result for G-free graphs with $|V(G)| \leq 5$ was recently obtained for the 4-colorability problem [9]. On the other hand, for fixed k, the complexity status of the k-colorability problem is open for P_7-free graphs ($k = 3$), for P_6-free graphs ($k = 4$), and for $P_2 + P_3$-free graphs ($k = 5$).

When we forbid two induced subgraphs, the situation becomes more difficult. For the coloring problem, a complete classification for pairs is open, even if forbidden induced subgraphs have at most four vertices. Although, the complexity is known for some such pairs [8,15,17,18,21]. The same is true for the 3-colorability problem and the five-vertex barrier. We determine here its complexity status for all classes defined by two forbidden induced subgraphs with at most five vertices.

E-mail address: dsmalyshev@rambler.ru.

http://dx.doi.org/10.1016/j.disc.2015.04.019
0012-365X/

2. Notation

For a vertex x of a graph, $deg(x)$ means its degree, $N(x)$ is its neighborhood, $N[x]$ denotes its *closed neighborhood* (i.e. the set $N(x) \cup \{x\}$), $N_k(x)$ is the set of vertices lying at distance k from x. The formula $\Delta(G)$ is the maximum degree of vertices in G.

As usual, P_n, C_n, K_n, O_n, and $K_{p,q}$ stand respectively for the simple path with n vertices, the chordless cycle with n vertices, the complete graph with n vertices, the empty graph with n vertices, and the complete bipartite graph with p vertices in the first part and q vertices in the second. The graph *paw* is obtained from a triangle by adding a vertex and an edge incident to the new vertex and a vertex of the triangle. The graphs *fork*, *bull*, *butterfly* have the vertex set $\{x_1, x_2, x_3, x_4, x_5\}$. The edge set for *fork* is $\{x_1x_2, x_1x_3, x_1x_4, x_4x_5\}$, for *bull* is $\{x_1x_2, x_1x_3, x_2x_3, x_1x_4, x_2x_5\}$, for *butterfly* is $\{x_1x_2, x_1x_3, x_2x_3, x_1x_4, x_1x_5, x_4x_5\}$. The graph $hammer_k$ has the vertex set $\{x_1, x_2, x_3, y_1, y_2, \dots, y_k\}$ and the edges $x_1x_2, x_1x_3, x_2x_3, x_1y_1, y_1y_2, \dots, y_{k-1}y_k$. Note that $paw = hammer_1$.

The *complement graph* $\overline{G}$ of G is a graph on the same set of vertices, and two vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. The *sum* $G_1 + G_2$ is the disjoint union of G_1 and G_2. The disjoint union of k copies of a graph G is denoted by kG. For a graph G and a set $V' \subseteq V(G)$, the formula $G \setminus V'$ denotes the subgraph of G obtained by deleting all vertices in V'.

3. Boundary graph classes

The notion of a boundary graph class is a helpful tool for the analysis of the computational complexity of graph problems in the family of hereditary graph classes. This notion was originally introduced by V.E. Alekseev for the independent set problem [1]. It was later applied for the dominating set problem [3]. A study of boundary graph classes for some graph problems was extended in the paper of Alekseev et al. [2], where the notion was formulated in its most general form. We will give the necessary definitions.

Let Π be an NP-complete graph problem. A hereditary graph class is said to be *Π-easy* if Π is polynomial-time solvable for its graphs. If the problem Π is NP-complete for graphs in a hereditary class, then this class is said to be *Π-hard*. A class of graphs is said to be *Π-limit* if this class is the limit of an infinite monotonically decreasing chain of Π-hard classes. In other words, $\mathcal{X}$ is Π-limit if there is an infinite sequence $\mathcal{X}_1 \supseteq \mathcal{X}_2 \supseteq \cdots$ of Π-hard classes, such that $\mathcal{X} = \bigcap_{k=1}^{\infty} \mathcal{X}_k$. A Π-limit class that is minimal under inclusion is said to be *Π-boundary*.

The following theorem certifies the significance of the notion of a boundary class.

Theorem 1 ([1]). *A finitely defined class is Π-hard if and only if it contains some Π-boundary class.*

This theorem shows that knowledge of all Π-boundary classes leads to a complete classification of finitely defined graph classes with respect to the complexity of Π. Two concrete classes of graphs are known to be boundary for several graph problems. The first of them is $\mathcal{S}$. It constitutes all forests with at most three leaves in each connected component. The second one is $\mathcal{T}$, which is the set of line graphs of graphs in $\mathcal{S}$. The paper [2] is a good survey about graph problems, for which either $\mathcal{S}$ or $\mathcal{T}$ is boundary.

Some classes are known to be limit and boundary for the 3-colorability problem. The set $\mathcal{F}$ of all forests and the set $\mathcal{T}'$ of line graphs of forests with degrees at most three are limit classes for it [14]. Some continuum set of boundary classes for the k-colorability problem is known for any fixed $k \geq 3$ [12,19,20].

The main result of this paper can be briefly formulated by means of $\mathcal{F}$ and $\mathcal{T}'$. Namely, if G_1 and G_2 have at most five vertices, then the 3-colorability problem is tractable for $\mathcal{X} = Free(\{G_1, G_2\})$ if $\mathcal{F} \not\subseteq \mathcal{X}$, $\mathcal{T}' \not\subseteq \mathcal{X}$, $\{G_1, G_2\} \neq \{K_{1,4}, bull\}$, $\{G_1, G_2\} \neq \{K_{1,4}, butterfly\}$, and the problem is NP-complete for all other choices of G_1 and G_2 on at most five vertices.

4. NP-completeness of the 3-colorability problem for some graph classes

The results listed above on limit classes for the 3-colorability problem together with Theorem 1 allow us to prove NP-completeness of the problem for some finitely defined classes. Namely, if $\mathcal{Y}$ is a finite set of graphs, and $\mathcal{Y} \cap \mathcal{F} = \emptyset$ or $\mathcal{Y} \cap \mathcal{T}' = \emptyset$, then the problem is NP-complete for $Free(\mathcal{Y})$. But, this idea cannot be applied to $Free(\{K_{1,4}, bull, butterfly\})$, because $K_{1,4} \in \mathcal{F}$, $bull \in \mathcal{T}'$, and $butterfly \in \mathcal{T}'$. Nevertheless, the 3-colorability problem is NP-complete for this class. To show this, we use a graph operation called diamond implantation.

Let G be a graph with a non-leaf vertex x. Applying a *diamond implantation* to x implies:

- an arbitrary splitting $N(x)$ into two nonempty parts A and B
- deletion of x and addition of new vertices y_1, y_2, y_3, y_4
- addition of all edges of the form y_1a, $a \in A$ and of the form y_4b, $b \in B$
- addition of the edges $y_1y_2, y_1y_3, y_2y_3, y_2y_4, y_3y_4$

Clearly, for every graph G and every non-leaf vertex in G, applying the diamond implantation preserves 3-colorability. This property and the paper [11] give the key idea of the proof of Lemma 1.

Lemma 1. *The* 3*-colorability problem is NP-complete in the class Free*$(\{K_{1,4}, \ bull, butterfly\})$.

Proof. The 3-colorability problem is known to be NP-complete for triangle-free graphs with maximum degree at most four [16]. Let us consider such a graph which is connected and has at least two vertices. We will sequentially apply the operation described above to those of its vertices with edgeless neighborhoods. In other words, if H is a current graph, then diamond implantation is applied to an arbitrary vertex of H that does not belong to any triangle. The sets A and B are formed arbitrarily with the condition $\||A| - |B|\| \leq 1$. The whole process is finite, because the number of its steps is no more than the number of vertices in the initial graph. It is easy to see that the resultant graph belongs to $Free(\{K_{1,4}, bull, butterfly\})$. Thus, the 3-colorability problem for triangle-free graphs with maximum degree at most four is polynomially reduced to the same problem for graphs in $Free(\{K_{1,4}, bull, butterfly\})$. Therefore, it is NP-complete for $Free(\{K_{1,4}, bull, butterfly\})$. ■

5. Some auxiliary results

5.1. Forbidding the hammer as an induced subgraph

Lemma 2. *If G is a connected graph in Free$(\{hammer_k\})$ and $k \geq 2$, then G is triangle-free or its diameter is at most $2k + 2$.*

Proof. Let G be a graph containing a triangle constituted by vertices x, y, z. We will show that the eccentricity of x is at most $k + 1$. This fact and the triangle inequality implies the bound for the diameter. Let P be the shortest induced path between x and some vertex of G. Suppose that P has at least $k + 3$ vertices. We enumerate all vertices of P starting from x. None of the vertices x, y, z can be adjacent to a vertex of P with a number that is greater than three, as P is not shortest otherwise. Let n be the greatest numbered vertex in P that is adjacent to at least one vertex in $\{x, y, z\}$. Note that $n \in \{2, 3\}$. It is easy to verify that two or three vertices in $\{x, y, z\}$ and the vertices of P with numbers in $\{n, n+1, \dots, k+n\}$ induce a subgraph isomorphic to a $hammer_k$. We have a contradiction. ■

5.2. The notion of quasi-twins and its significance

Recall that two vertices of a graph are said to be *twins* if they have coinciding neighborhoods. Two vertices are called *quasi-twins* if the neighborhood of one of them is included in the neighborhood of the second one. The significance of the quasi-twins notion is showed by the following lemma (without proof, since it is obvious).

Lemma 3. *If G is a graph, $x, y \in V(G)$, and $N(x) \subseteq N(y)$, then $\chi(G) = \chi(G \setminus \{x\})$.*

5.3. Forbidding the fork as an induced subgraph

A *wheel* is a graph formed by connecting a single vertex to all vertices of a cycle. If the cycle has an odd number of vertices, then the corresponding graph is said to be an *odd wheel*. A graph is said to be *odd-wheel-free* if it does not contain induced odd wheels. This property can be checked in polynomial-time. Since any odd wheel is not 3-colorable, a necessary condition for a graph to be 3-colorable is to be odd-wheel-free.

Lemma 4. *If G is a connected odd-wheel-free graph in Free$(\{fork, bull\})$ containing an induced odd cycle C of length at least seven, then the* 3*-colorability problem for G can be solved in polynomial time on $|V(G)|$.*

Proof. Let x be a vertex outside C having a neighbor on C. The vertex x must be adjacent to at least two consecutive vertices of C, since G is not *fork*-free otherwise. As G is odd-wheel-free, x cannot be adjacent to all vertices of C. Hence, there are pairwise distinct vertices $a, b, c, d \in V(C)$, such that ab, bc, cd, xa, xb are edges of G and $xc \notin E(G)$. Let e be the neighbor of a in C that is different from b. The graph G must contain the edge xe, since it is *bull*-free. If there is a vertex $y \in V(C) \setminus \{a, b, c, d, e\}$ with $xy \in E(G)$, then a, b, x, c, y induce a *bull*. If x is adjacent to only a, b, d, e, then $G \notin Free(\{fork\})$. So, x is adjacent to only a, b, e.

The connectivity of G and its $\{fork, bull\}$-freeness imply that each element of $V(G) \setminus V(C)$ has a neighbor on C. There are exactly three such neighbors for any element of the set. Any two vertices in $V(G) \setminus V(C)$ have at most one common neighbor on the cycle, and they must be nonadjacent. Indeed, if two vertices in $V(G) \setminus V(C)$ are adjacent and have at least two common neighbors on C, then G contains a K_4, which is an odd wheel. If they are adjacent and have at most one common neighbor, then G contains an induced *bull*. If they are nonadjacent and have two or three common neighbors on the cycle, then G contains an induced *bull* or *fork*, respectively. Therefore, any two vertices in $V(G) \setminus V(C)$ must be nonadjacent and may have at most one common neighbor on the cycle.

Let $x_1, \dots, x_k$ be the vertices of C, in order. If there is a vertex in $V(G) \setminus V(C)$ adjacent to x_1, x_2, x_3, another adjacent to x_3, x_4, x_5, another adjacent to $x_5, x_6, x_7, \dots$, another adjacent to x_{k-2}, x_{k-1}, x_k, then G has no proper 3-coloring. It is trivial to verify if this is the case in polynomial time. In any case where G is not isomorphic to such a case, it is easy to see that G has a valid 3-coloring. ■

Lemma 5. *If G is a connected 3-colorable graph in $Free(\{fork, bull\})$ that contains C_5 as an induced subgraph, then the graph G has at most six vertices or it has a pair of quasi-twins.*

Proof. Let G be such a graph. Assume G has no quasi-twins. Each vertex of $H = G \setminus V(C_5)$ having a neighbor in C_5 must be adjacent to three or four consecutive vertices of the cycle, since G is $\{odd - wheel, fork, bull\}$-free. Since G is connected and *fork*-free, this implies that the cycle dominates all vertices of H. Suppose G have a vertex x with four neighbors on the cycle. We will show that x and the remaining vertex y of the cycle are quasi-twins. Suppose that there is a vertex $z \in V(G) \setminus V(C_5)$, such that $xz \notin E(G)$ and $yz \in E(G)$. It is easy to verify that the subgraph induced by $\{x, z\} \cup V(C_5)$ is not 3-colorable. Therefore, we may assume that every element of $V(H)$ has exactly three consecutive neighbors on the cycle. If some two vertices of H have exactly one common neighbor on the cycle, then G is not 3-colorable. If they have at least two common neighbors and are adjacent, then the graph contains a K_4 and is not 3-colorable. If they have exactly two common neighbors and are nonadjacent, then the graph contains an induced *bull*. Therefore, they must be nonadjacent and have three common neighbors, i.e. these vertices must be quasi-twins. Hence, H has at most one vertex. Thus, $|V(G)| \le 6$. ∎

A graph G is said to be *perfect* if it belongs to the class $Free(\{C_5, C_7, \overline{C_7}, C_9, \overline{C_9}, \ldots\})$ [6].

Lemma 6. *The 3-colorability problem for $\{fork, bull\}$-free graphs can be polynomially reduced to the same problem for perfect graphs.*

Proof. The 3-colorability problem for a class of graphs can be polynomially reduced to its odd-wheel-free part. Notice that K_4 and $\overline{C_7}$ are not 3-colorable. Hence, by Lemmas 3 and 4, the problem for $Free(\{fork, bull\})$ can be reduced in polynomial time to the same problem for graphs in $Free(\{fork, bull, K_4, \overline{C_7}, C_7, C_9, \ldots\})$ without quasi-twins. Hence, by Lemma 5, any such a graph that is connected, not C_5-free, and has at least seven vertices is not 3-colorable. Thus, the problem can be polynomially reduced to graphs in $Free(\{K_4, \overline{C_7}, C_5, C_7, C_9, \ldots\})$, and they are all perfect by the Strong Perfect Graph Theorem [6]. ∎

Lemma 7. *Let $G \in Free(\{fork\})$ be a 3-colorable graph containing a triangle (x, y, z). Then, for any 3-coloring of G there is a color c such that the graph $H = G \setminus ((N(x) \cap N(y)) \cup (N(x) \cap N(z)) \cup (N(y) \cap N(z)))$ has at most $k + 6$ vertices of the color c, where $k = |V(G) \setminus (N(x) \cup N(y) \cup N(z))|$.*

Proof. Denote $N(x) \setminus (N(y) \cup N(z))$ by $N'(x)$, $N(y) \setminus (N(x) \cup N(z))$ by $N'(y)$, and $N(z) \setminus (N(x) \cup N(y))$ by $N'(z)$. If at most one of the sets $N'(x), N'(y), N'(z)$ is nonempty, then there is a color class of H having at most k vertices, since the nonempty set must induce a bipartite subgraph in G. If there are at least two nonempty sets, then each of them induces a complete subgraph. Without loss of generality, suppose that there are nonadjacent vertices $a, b \in N'(x)$ and a vertex $c \in N'(y)$. If $ac \notin E(G)$ and $bc \notin E(G)$, then a, b, x, y, c induce a *fork*. If $ac \in E(G)$ and $bc \notin E(G)$ or vice versa, then a, b, x, z, c induce a *fork*. Finally, if $ac \in E(G)$ and $bc \in E(G)$, then a, b, c, y, z induce a *fork*. We have a contradiction. Since G is 3-colorable, $\max(|N'(x)|, |N'(y)|, |N'(z)|) \le 2$. Hence, $|V(H)| \le k + 6$. Therefore, each color class of H has at most $k + 6$ elements. ∎

Lemma 8. *If $G \in Free(\{C_3+O_2\}) \cup Free(\{paw+K_1\}) \cup Free(\{hammer_2\}) \cup Free(\{C_3+K_2\})$ is a connected $\{fork, K_4\}$-free graph and some its vertices x, y, z, v form an induced copy of a paw with triangle (x, y, z), then $|V(G) \setminus (N(x) \cup N(y) \cup N(z))| \le 9$.*

Proof. Denote the set $N(x) \cup N(y) \cup N(z)$ by N. If G is $C_3 + O_2$-free, then $|V(G) \setminus N| \le 3$, since any O_2-free graph is complete.

Let G be $paw + K_1$-free. The vertex v must be adjacent to every vertex in $V(G) \setminus N$, and this set does not contain two nonadjacent vertices, otherwise G would not be fork-free. Hence, $|V(G) \setminus N| \le 2$.

Let G be $hammer_2$-free. Due to this fact and the connectivity of G, each element of $V(G) \setminus N$ must be adjacent to a vertex in N. Any two vertices in $V(G) \setminus N$ cannot have a common neighbor in N, as G is $\{fork, K_4, hammer_2\}$-free. If $|V(G) \setminus N| \ge 2$, then there are distinct vertices $a_1, b_1 \in V(G) \setminus N$ and distinct vertices $a_2, b_2 \in N \setminus \{x, y, z\}$, such that $a_1a_2 \in E(G)$ and $b_1b_2 \in E(G)$. As G is $hammer_2$-free, then $a_2, b_2 \in (N(x) \cap N(y)) \cup (N(x) \cap N(z)) \cup (N(y) \cap N(z))$. If a_2, b_2 belong to the same set among $N(x) \cap N(y), N(x) \cap N(z), N(y) \cap N(z)$, then $a_2b_2 \notin E(G)$ and G contains *fork* as an induced subgraph. If they belong to distinct sets, then $a_1b_1 \notin E(G)$ (as G is $hammer_2$-free), $a_2b_2 \notin E(G)$ (as G is *fork*-free), and G has an induced copy of $hammer_2$. Thus, $|V(G) \setminus N| \le 1$.

Now let $G \in Free(\{C_3 + K_2\})$. Any two vertices of $V(G) \setminus N$ are nonadjacent, since G is $C_3 + K_2$-free. Each vertex in N has at most one neighbor in $V(G) \setminus N$, since G is $\{fork, K_4\}$-free. If a vertex in $V(G) \setminus N$ has a neighbor in one of the sets $N(x) \cap N(y), N(x) \cap N(z), N(y) \cap N(z)$, then it must be adjacent to all vertices of this set. Hence, for each of the sets $N(x) \cap N(y), N(x) \cap N(z), N(y) \cap N(z)$, there is at most one vertex in $V(G) \setminus N$ with a neighbor in the corresponding set, since G is $\{K_4, fork\}$-free. If two vertices belong to one of the sets $N'(x), N'(y), N'(z)$ (see the notation in the proof of the previous lemma) and have distinct neighbors in $V(G) \setminus N$, then they are adjacent, since G is *fork*-free. Hence, $|V(G) \setminus N| \le 9$ (otherwise, $V(G) \setminus N$ contains vertices a, b, c and one of the sets $N'(x), N'(y), N'(z)$ contains vertices a', b', c' with $aa' \in E(G)$, $bb' \in E(G)$, $cc' \in E(G)$, $ab' \notin E(G)$, $ac' \notin E(G)$, $ba' \notin E(G)$, $bc' \notin E(G)$, $ca' \notin E(G)$, $cb' \notin E(G)$; the vertices a', b', c' must be pairwise adjacent, and G is not K_4-free). ∎

Lemma 9. *If $G' \in \{C_3, C_3+K_1, C_3+O_2, C_3+K_2, paw, paw+K_1, hammer_2\}$, then the 3-colorability problem for $Free(\{fork, G'\})$ can be polynomially reduced to the same problem for $\{fork, paw\}$-free graphs.*

Proof. Let G be a connected $\{fork, G', K_4\}$-free graph having an induced copy of a *paw* with triangle (x, y, z), and let H be the graph from Lemma 7. The graph G is 3-colorable if and only if there is an independent set IS containing $N(x) \cap N(y)$ or $N(x) \cap N(z)$ or $N(y) \cap N(z)$ plus at most 15 vertices of H, such that $G \setminus IS$ is bipartite (see Lemmas 7 and 8). The existence of such a set can be checked in polynomial time. Thus, we have a polynomial-time reduction to the *paw*-free part of $Free(\{fork, G'\})$. ■

Lemma 10. *Any connected graph $G \in Free(\{fork, butterfly, K_4\})$ is either $K_{1,3}$-free or $10K_2$-free.*

Proof. Let G contain a vertex x and its pairwise nonadjacent neighbors y_1, y_2, y_3. We will show that if $v \in N_2(x)$ has a neighbor $u \in N_3(x)$, then v is adjacent to at least two elements of $Y = \{y_1, y_2, y_3\}$. The fact is clear, when v has a neighbor in the set Y, as G contains a *fork* as an induced subgraph otherwise. If $vy_1 \notin E(G)$, $vy_2 \notin E(G)$, $vy_3 \notin E(G)$, then $N(x) \setminus Y$ contains a neighbor y of v. To avoid an induced copy of a *fork*, the vertex y must be adjacent to at least two vertices in Y. But, the vertex y, two of its neighbors in Y, v, and u induce a *fork*. This completes the proof of the claim. It follows that $N_4(x) = \emptyset$, since G is *fork*-free

We will also show that if $u, v \in N_2(x)$ are adjacent, then v or u has a neighbor among y_1, y_2, y_3. Suppose the opposite. The vertex v has a neighbor $y \in N(x) \setminus Y$. The vertex y is adjacent to at least two elements of Y, since G is *fork*-free. As G is *butterfly*-free, $yu \notin E(G)$. Hence, v, u, y, and two neighbors of y in Y induce a *fork*. This is a contradiction, completing the proof of the claim.

Suppose that $N_3(x)$ has an independent set IS with four vertices. No two elements of IS have a common neighbor in $N_2(x)$, since G is *fork*-free. Hence, $N_2(x)$ has four distinct vertices, each of which is a neighbor of some vertex in IS. Since G is K_4-free, there are $a_1, b_1 \in IS$ and $a_2, b_2 \in N_2(x)$ with $a_1a_2 \in E(G)$, $b_1b_2 \in E(G)$, $a_2b_2 \notin E(G)$. The vertices a_2 and b_2 have a common neighbor in Y. Hence, this neighbor, x and a_1, a_2, b_2 induce a *fork*. Thus, the assumption was false, so the maximum size of an independent set in $N_3(x)$ is at most three.

Let E' be a maximum induced matching of G, m_i $(i \in \{1, 2, 3\})$ be the number of its edges with both ends in $N_i(x)$, and $m_{i,i+1}$ $(i \in \{0, 1, 2\})$ be the number of edges incident to a vertex in $N_i(x)$ and to a vertex in $N_{i+1}(x)$. Clearly, $m_{0,1} + m_1 \leq 1$ (since G is *butterfly*-free), $m_{1,2} \leq 2$ (since G is *fork*-free), and $m_3 + m_{2,3} \leq 3$. If $m_2 \geq 4$, by the pigeonhole principle, for some edges $a'b', a''b'' \in E'$ (where $a', a'', b', b'' \in N_2(x)$) and a vertex $y^* \in Y$, we have $a'y^* \in E(G)$ and $a''y^* \in E(G)$. As G is *fork*-free, y^*b' and y^*b'' are edges of G. Hence, a', b', a'', b'', and y^* induce a *butterfly*. Thus, $m_2 \leq 3$ and $|E'| = m_1 + m_2 + m_3 + m_{0,1} + m_{1,2} + m_{2,3} \leq 9$, so G is $10K_2$-free. ■

5.4. On some polynomial-time reductions

A graph G is said to be *locally k-eliminable* if there is a vertex x and an independent set $IS \subseteq N(x)$, such that $|N(x) \setminus IS| \leq k$ and $G \setminus IS$ is bipartite. For each fixed k, this property can be verified in polynomial time.

Lemma 11. *If G is a 3-colorable connected graph in $Free(\{K_{1,3} + K_1\})$ containing a vertex x, then $V(G) \setminus N(x)$ is independent or G is locally 10-eliminable or $deg(x) \leq 15$.*

Proof. Let $deg(x) > 15$. Each vertex in $V(G) \setminus N[x]$ has at least $|N(x)| - 5$ neighbors in $N(x)$, since any graph with six vertices contains either a K_3 or an O_3, and G is a 3-colorable $K_{1,3} + K_1$-free graph. Any two (three) vertices in $V(G) \setminus N[x]$ have at least $|N(x)| - 10$ $(|N(x)| - 15)$ common neighbors in $N(x)$. Hence, if $V(G) \setminus N[x]$ has adjacent vertices, then for each 3-coloring of G this set contains elements of exactly two colors, and one of them is the color of x. Hence, there are at least $|N(x)| - 10$ vertices in $N(x)$ with the third color, and G is locally 10-eliminable. ■

The following lemma is easy to prove.

Lemma 12. *Any $H + K_1$-free graph G is H-free or it contains at most $|V(H)|(\Delta(G) + 1)$ vertices.*

Lemma 13. *The 3-colorability problem for any hereditary subclass of $Free(\{K_{1,3} + K_1\})$ can be reduced in polynomial time to the same problem for $K_{1,3}$-free graphs in this class.*

Proof. Let $G \in Free(\{K_{1,3} + K_1\})$ be a connected graph. If G is locally 10-eliminable, then G is 3-colorable. If $\Delta(G) \leq 15$, then G is $K_{1,3}$-free or $|V(G)| \leq 64$ by Lemma 12. If G is not 10-eliminable and $|V(G)| > 64$, then a necessary condition for 3-colorability of G is to have a vertex x with an independent set $V(G) \setminus N(x)$ by Lemma 11. The existence of such a vertex can be verified in polynomial time. If it exists, then G is 3-colorable if and only if $N(x)$ induces a bipartite subgraph. Hence, we have a polynomial-time reduction. ■

6. Main result

Recall that $\mathcal{F}$ is the class of forests, $\mathcal{T}'$ is the set of line graphs of forests with degrees at most three.

Theorem 2. *Let G_1 and G_2 be graphs with at most five vertices. The 3-colorability problem is polynomial-time solvable for $\mathcal{X} = Free(\{G_1, G_2\})$ if at least one of the graphs G_1, G_2 is a forest, at least one of them belongs to $\mathcal{T}'$, and $\{G_1, G_2\} \neq \{K_{1,4}, bull\}$ or $\{K_{1,4}, butterfly\}$. It is NP-complete for all remaining cases.*

Proof. The sets $\mathcal{F}$ and $\mathcal{T}'$ are limit classes for the 3-colorability problem. Hence, if $\mathcal{X}$ includes either $\mathcal{F}$ or $\mathcal{T}'$, then the problem is NP-complete for the class (by Theorem 1). By Lemma 1, the 3-colorability problem is NP-complete for $Free(\{K_{1,4}, bull\})$ and $Free(\{K_{1,4}, butterfly\})$.

If H is a graph with at most five vertices and $H \in \mathcal{T}' \setminus (\mathcal{F} \cup \{bull, butterfly\})$, then H is $hammer_4$-free. The 3-colorability problem for $\{K_{1,4}, H\}$-free graphs can therefore be polynomially reduced to $\{K_{1,4}, K_4, hammer_4\}$-free graphs, i.e. graphs having degrees of all vertices at most eight, as every graph with nine vertices contains an induced K_3 or O_4. There is a finite number of connected graphs having diameter at most 10 and degrees of all vertices at most eight. Hence, by Lemma 2, we have a polynomial-time reduction to $\{K_{1,4}, C_3\}$-free graphs. By Brook's theorem [5], all $\{K_{1,4}, C_3\}$-free graphs are 3-colorable. So, the problem is tractable for $Free(\{K_{1,4}, H\})$.

If H' is a *linear forest* (the disjoint union of simple paths) with $|V(H')| \leq 6$, then the 3-colorability problem can be solved in polynomial time for H'-free graphs [4]. Since $\mathcal{F} \cap \mathcal{T}'$ is the set of all linear forests, one need only consider the case, when $G_1 \in \mathcal{F} \setminus (\mathcal{T}' \cup \{K_{1,4}\})$ (i.e. $G_1 \in \{K_{1,3}, K_{1,3} + K_1, fork\}$) and $G_2 \in \mathcal{T}' \setminus \mathcal{F}$ (i.e. $G_2 \in \{C_3, C_3 + K_1, C_3 + O_2, C_3 + K_2, paw, paw + K_1, bull, hammer_2, butterfly\}$). The classes of perfect, $\{fork, paw\}$-free, and $\{K_{1,3}, butterfly\}$-free graphs are easy for the 3-colorability problem [10,8,22]. For each fixed s, the 3-colorability problem is polynomial-time solvable in the class of sK_2-free graphs [7]. Hence, by Lemmas 6, 9, 10 and 13, the problem is polynomial-time solvable for the graph classes defined by pairs of forbidden induced subgraphs of the form mentioned above. ■

Acknowledgments

The research is partially supported by LATNA laboratory, National Research University Higher School of Economics, RF government grant, ag. 11.G34.31.00357; by RF President grant MK-1148.2013.1. This study was carried out within "The National Research University Higher School of Economics' Academic Fund Program" in 2013–2014, research grant No. 12-01-0035.

The author would like to thank anonymous reviewers for their interest in the paper and valuable comments.

References

[1] V.E. Alekseev, On easy and hard hereditary classes of graphs with respect to the independent set problem, Discrete Appl. Math. 132 (2004) 17–26.
[2] V.E. Alekseev, R. Boliac, D.V. Korobitsyn, V.V. Lozin, NP-hard graph problems and boundary classes of graphs, Theoret. Comput. Sci. 389 (2007) 219–236.
[3] V.E. Alekseev, D.V. Korobitsyn, V.V. Lozin, Boundary classes of graphs for the dominating set problem, Discrete Math. 285 (2004) 1–6.
[4] H.J. Broersma, P.A. Golovach, D. Paulusma, J. Song, Updating the complexity status of coloring graphs without a fixed induced linear forest, Theoret. Comput. Sci. 414 (2012) 9–19.
[5] R.L. Brooks, On colouring the nodes of a network, Proceedings of the Cambridge philosophical society, Math. Phys. Sci. 37 (1941) 194–197.
[6] M. Chudnovsky, N. Robertson, P.D. Seymour, R. Thomas, The strong perfect graph theorem, Ann. of Math. 164 (2006) 51–229.
[7] K.K. Dabrowski, V.V. Lozin, R. Raman, B. Ries, Colouring vertices of triangle-free graphs without forests, Discrete Math. 312 (2012) 1372–1385.
[8] P. Golovach, D. Paulusma, List coloring in the absence of two subgraphs, Discrete Appl. Math. 166 (2014) 123–130.
[9] P. Golovach, D. Paulusma, J. Song, 4-coloring H-free graphs when H is small, Discrete Appl. Math. 161 (2013) 140–150.
[10] M. Grötschel, L. Lovász, A. Schrijver, The ellipsoid method and its consequences in combinatorial optimization, Combinatorica 1 (1981) 169–197.
[11] M. Kochol, V.V. Lozin, B. Randerath, The 3-colorability problem on graphs with maximum degree four, SIAM J. Comput. 32 (2003) 1128–1139.
[12] N. Korpelainen, V.V. Lozin, D. Malyshev, A. Tiskin, Boundary properties of graphs for algorithmic graph problems, Theoret. Comput. Sci. 412 (2011) 3544–3554.
[13] D. Kral', J. Kratochvil, Z. Tuza, G. Woeginger, Complexity of coloring graphs without forbidden induced subgraphs, Lecture Notes in Comput. Sci. 2204 (2001) 254–262.
[14] V.V. Lozin, M. Kaminski, Coloring edges and vertices of graphs without short or long cycles, Contrib. Discrete Math. 2 (2007).
[15] V.V. Lozin, D.S. Malyshev, Vertex coloring of graphs with few obstructions, Discrete Appl. Math. (2015) http://dx.doi.org/10.1016/j.dam.2015.02.015.
[16] F. Maffray, M. Preissmann, On the NP-completeness of the k-colorability problem for triangle-free graphs, Discrete Math. 162 (1996) 313–317.
[17] D.S. Malyshev, The coloring problem for classes with two small obstructions, Optimiz. Lett. 8 (2014) 2261–2270.
[18] D.S. Malyshev, Two cases of polynomial-time solvability for the coloring problem, J. Comb. Optim. (2015) http://dx.doi.org/10.1007/s10878-014-9792-3.
[19] D.S. Malyshev, Continued sets of boundary classes of graphs for colorability problems, Diskretnyi analiz i issledivanie operatsii 16 (2009) 41–51 (in Russian).
[20] D.S. Malyshev, On intersection and symmetric difference of families of boundary classes in the problems of colouring and on the chromatic number, Diskretnaya Matematika 24 (2012) 75–78 (in Russian); English translation in Discrete Math. Appl. 21 (2011) 645–649.
[21] D.S. Malyshev, O.O. Lobanova, The coloring problem for $\{P_5, \overline{P_5}\}$-free graphs and $\{P_5, K_p - e\}$-free graphs is polynomial, arXiv:1503.02550.
[22] B. Randerath, I. Schiermeyer, M. Tewes, Three-colourability and forbidden subgraphs. II: polynomial algorithms, Discrete Math. 251 (2002) 137–153.