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The least eigenvalue of signless Laplacian of non-bipartite graphs with given domination number[☆]

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ABSTRACT

Let G be a connected non-bipartite graph on n vertices with domination number $\gamma \le \frac{n+1}{3}$. We present a lower bound for the least eigenvalue of the signless Laplacian of G in terms of the domination number.

1. Introduction

Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The *adjacency matrix* of G is defined to be the $(0, 1)$-matrix $A(G) = [a_{ij}]$, where $a_{ij} = 1$ if v_i is adjacent to v_j, and $a_{ij} = 0$ otherwise. The *degree matrix* of G is defined by $D(G) = \mathrm{diag}(d_G(v_1), d_G(v_2), \ldots, d_G(v_n))$, where $d_G(v)$ or simply $d(v)$ is the degree of a vertex v in G. The matrix $Q(G) = D(G) + A(G)$ is called the *signless Laplacian matrix* (or *Q-matrix*) of G. It is known that $Q(G)$ is nonnegative, symmetric and positive semidefinite, so its eigenvalues are real and can be arranged as: $q_1(G) \ge q_2(G) \ge \cdots \ge q_n(G) \ge 0$. We call the eigenvalues of $Q(G)$ the *Q-eigenvalues* of G and refer the readers to [3–7] for the survey on this topic. The least Q-eigenvalue $q_n(G)$ is denoted by $q_{\min}(G)$, and the eigenvectors corresponding to $q_{\min}(G)$ are called the *first Q-eigenvectors* of G.

If G is connected, then $q_{\min}(G) = 0$ if and only if G is bipartite. So, here we are concerned with the least eigenvalue of connected non-bipartite graph. Desai and Rao [9] use the least Q-eigenvalue to characterize the bipartiteness of graphs. As a consequence of this work, Shaun and Fan [10] establish the relationship between the least Q-eigenvalue and some parameters such as vertex or edge bipartiteness. In [11] they present upper bounds for the least Q-eigenvalue in terms of the edge bipartiteness and lower bounds for the signless Laplacian spread. Cardoso et al. [2] give a lower bound for the least Q-eigenvalue of non-bipartite graphs. Liu et al. [17] investigate the minimum least Q-eigenvalue of non-bipartite unicyclic graphs with fixed number of pendant vertices. Lima et al. [8] survey the known results and present some new ones for the least Q-eigenvalues of graphs. Our research group investigate how the least Q-eigenvalue changes when relocating bipartite

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branches [22], which provides an easier way to get some known results on this topic, and characterize the unique graph whose least Q-eigenvalue attains the minimum among all non-bipartite graphs with fixed number of pendant vertices [13]. In a more general setting, the least eigenvalue of the Laplacian of mixed graphs has been discussed in [12,21].

Recall that a set $S \subset V(G)$ of a graph G is called a *dominating set* if every vertex of $V(G)\backslash S$ is adjacent to at least one vertex of S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum of the cardinalities of all domination sets of G. Surely, if G has no isolated vertices then $\gamma(G) \leq \frac{|V(G)|}{2}$. With respect to the adjacency matrix, Stevanović et al. [20] determine the unique graph with maximal spectral radius among all graphs with no isolated vertices and fixed domination number; Zhu [23] characterize the unique graph whose least eigenvalue achieves the minimum among all graphs with fixed domination number. With respect to the Laplacian matrix, Brand and Seifter [1] give an upper bound for the spectral radius in terms of the domination number. Lu et al. [18], Nikiforov et al. [19] and Feng [15] give some bounds for the second least eigenvalue and the spectral radius of graphs, respectively. In addition, Feng et al. [14] characterize the minimum Laplacian spectral radius of trees with given domination number.

But few work appeared on the relation between the signless Laplacian eigenvalue and the domination number, except that He and Zhou [16] use the domination number to give an upper bound for the least signless Laplacian eigenvalue. In this paper, we will investigate the lower bound for the least Q-eigenvalue of a non-bipartite graph in terms of the domination number. For convenience, a graph is called *minimizing* in a certain non-bipartite graph class if its least Q-eigenvalue attains the minimum among all graphs in the class. Denote by $\mathscr{G}_n^\gamma$ the set of all connected non-bipartite graphs of order n with the domination number γ, and by $\mathscr{G}_n^\gamma(g)$ $(g < n)$ the set of graphs in $\mathscr{G}_n^\gamma$ for which the minimum length of odd cycles is g. When $\gamma \leq \frac{n+1}{3}$, we characterize the unique minimizing graph among all graphs in $\mathscr{G}_n^\gamma$, and hence provide a lower bound for the least Q-eigenvalue in terms of the domination number.

2. Preliminaries

Let C_n, P_n and $S_{1,n-1}$ denote a cycle, a path and a star, all on n vertices, respectively. A graph G is called *trivial* if it contains only one vertex; otherwise, it is called *nontrivial*. A graph G is called *unicyclic* if it is connected and contains exactly one cycle. The minimum length of all cycles in G is called the *girth* of G. A *pendant vertex* of G is a vertex of degree 1 and a *quasi-pendant vertex* is one adjacent to a pendant vertex.

Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Denote $|x| = (|x_1|, |x_2|, \ldots, |x_n|)$. Let G be a graph on vertices $v_1, v_2, \ldots, v_n$. The vector x can be considered as a function defined on $V(G)$, which maps each vertex v_i of G to the value x_i, i.e. $x(v_i) = x_i$. If x is an eigenvector of $Q(G)$, then it defines on G naturally, i.e. $x(v)$ denotes the entry of x corresponding to v. Also, the quadratic form $x^TQ(G)x$ can be written as

$$x^TQ(G)x = \sum_{uv\in E(G)} [x(u) + x(v)]^2. \tag{2.1}$$

The eigenvector equation $Q(G)x = \lambda x$ can be interpreted as

$$[\lambda - d(v)]x(v) = \sum_{u\in N_G(v)} x(u) \quad \text{for each } v \in V(G), \tag{2.2}$$

where $N_G(v)$ denotes the neighborhood of v in G. In addition, for an arbitrary unit vector $x \in \mathbb{R}^n$,

$$q_{\min}(G) \leq x^TQ(G)x, \tag{2.3}$$

with equality if and only if x is a first Q-eigenvector of G.

Let G_1, G_2 be two vertex-disjoint graphs, and let $v \in V(G_1), u \in V(G_2)$. The *coalescence* of G_1 and G_2 with respect to v and u, denoted by $G_1(v) \diamond G_2(u)$, is obtained from G_1 and G_2 by identifying v with u, thus giving rise to a new vertex p, which is also denoted as $G_1(p) \diamond G_2(p)$. If a connected graph G can be expressed as $G = G_1(p) \diamond G_2(p)$, where G_1 and G_2 are nontrivial subgraphs of G both containing p, then G_1 (or G_2) is called a *branch* of G with root p. Given a vector x defined on G, a branch G_1 of G is called a *zero branch* with respect to x if $x(v) = 0$ for all $v \in V(G_1)$. Let $G = G_1(v_2) \diamond G_2(u)$ and $G^* = G_1(v_1) \diamond G_2(u)$, where v_1 and v_2 are two distinct vertices of G_1 and u is a vertex of G_2. We say that G^* is obtained from G by *relocating* G_2 *from* v_2 *to* v_1.

Lemma 2.1 (*[22]*). *Let H be a bipartite branch of a connected graph G with root u. Let x be a first Q-eigenvector of G.*

(1) *If $x(u) = 0$, then H is a zero branch of G with respect to x.*

(2) *If $x(u) \neq 0$, then $x(p) \neq 0$ for every vertex p of H. Furthermore, for every vertex p of H, $x(p)x(u)$ is positive or negative, depending on whether p is or is not in the same part of bipartite graph H as u; consequently, $x(p)x(q) < 0$ for each edge $pq \in E(H)$.*

The following result is a supplement of [22, Lemma 2.5]. For completeness we will restate some discussion as used in its proof.

Lemma 2.2. *Let G_1 be a connected graph containing at least two vertices v_1, v_2, and let G_2 be a connected bipartite graph containing a vertex u. Let $G = G_1(v_2) \diamond G_2(u)$ and $G^* = G_1(v_1) \diamond G_2(u)$. If there exists a first Q-eigenvector x of G such that $|x(v_1)| \geq |x(v_2)|$, then $q_{\min}(G^*) \leq q_{\min}(G)$, with equality only if $|x(v_1)| = |x(v_2)|$, $d_{G_2(u)}x(u) = -\Sigma_{v\in N_{G_2}(u)}x(v)$, and G^* has a first Q-eigenvector $\widetilde{x}$ such that $|\widetilde{x}| = |x|$.*

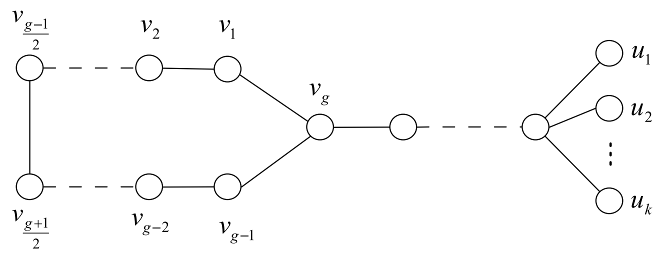

Fig. 3.1. The graph $U_n^k(g)$ with $g < n$.

Proof. Assume that x is a unit vector with $x(v_1) \geq 0$. Let U, W be the two parts of the bipartite graph G_2, where $u \in U$. Let $\widetilde{x}$ be the vector defined on G^* given by $\widetilde{x}(v) = x(v)$ if $v \in V(G_1)$, $\widetilde{x}(v) = |x(v)| + [x(v_1) - |x(v_2)|]$ if $v \in U \backslash \{u\}$, and $\widetilde{x}(v) = -|x(v)| - [x(v_1) - |x(v_2)|]$ if $v \in W$. By (2.1) and Lemma 2.1, one can verify that $\widetilde{x}^T Q(G^*)x = x^T Q(G)x$, and since $x(v_1) = |x(v_1)| \geq |x(v_2)|$,

$$\|\widetilde{x}\|^2 = \sum_{v \in V(G_1)} x(v)^2 + \sum_{v \in V(G_2)\backslash\{u\}} [|x(v)| + x(v_1) - |x(v_2)|]^2 \geq \sum_{v \in V(G)} x(v)^2 = \|x\|^2 = 1.$$

So

$$q_{\min}(G^*) \leq \|\widetilde{x}\|^{-2} \widetilde{x}^T Q(G^*)x \leq \widetilde{x}^T Q(G^*)x = x^T Q(G)x = q_{\min}(G).$$

If $q_{\min}(G^*) = q_{\min}(G)$, then $\|\widetilde{x}\|^2 = \|x\|^2$, and $\widetilde{x}$ is also a first Q-eigenvector of G^*. The former condition implies that $x(v_1) = |x(v_1)| = |x(v_2)|$ and hence $|\widetilde{x}| = |x|$, while the latter implies that $d_{G_2(u)} x(u) = -\sum_{v \in N_{G_2}(u)} x(v)$ by the eigenvector equations of x and $\widetilde{x}$ both at v_2. ■

Lemma 2.3 ([22]). *Let G be a connected non-bipartite graph, and let x be a first Q-eigenvector of G. Let T be a tree, which is a nonzero branch of G with respect to x and with root u. Then $|x(q)| < |x(p)|$ whenever p, q are vertices of T such that q lies on the unique path from u to p.*

3. Minimizing the least Q-eigenvalue among all graphs in $\mathscr{G}_n^\gamma$

Denote by $U_n^k(g)$ the unicyclic graph of order n, which is obtained from an odd cycle C_g ($g < n$) and a star $S_{1,k}$ by adding a path P_l connecting (or identifying) one vertex of the cycle and the center of the star (if $l = 1$), where $l = n + 1 - g - k$; see Fig. 3.1. Surely, if $k \geq 2$, then

$$\gamma(U_n^k(g)) \leq \gamma(U_n^{k-1}(g)) \leq \cdots \leq \gamma(U_n^1(g)) =: \gamma_{n,g}.$$

Fixed n and odd $g \in [3, n-1]$, for each $\gamma \in [\lceil \frac{g}{3} \rceil, \gamma_{n,g}]$, there exists one or more graphs $U_n^k(g)$ with domination number γ; the unique one with minimum k among those graphs is denoted by $V_n^\gamma(g)$.

Lemma 3.1 ([13]). *Let $U_n^k(g)$ be the graph with some vertices labeled as in Fig. 3.1, where $v_1, v_2, \ldots, v_g$ are the vertices of the unique cycle C_g labeled in an anticlockwise way. Let x be a first Q-eigenvector of $U_n^k(g)$. Then*

(1) $x(v_i) = x(v_{g-i})$ *for* $i = 1, 2, \ldots, \frac{g-1}{2}$;
(2) $x(v_{\frac{g-1}{2}})x(v_{\frac{g+1}{2}}) > 0$, *and* $x(v)x(w) < 0$ *for other edges* vw *of* $U_n^k(g)$ *except* $v_{\frac{g-1}{2}} v_{\frac{g+1}{2}}$;
(3) $|x(v_g)| > |x(v_1)| > |x(v_2)| > \cdots > |x(v_{\frac{g-1}{2}})| > 0$.

Denote by $\mathscr{U}_n^k(g)$ the set of unicyclic graphs of order n with odd girth g and $k \geq 1$ pendant vertices.

Lemma 3.2 ([13]). *Among all graphs in $\mathscr{U}_n^k(g)$, $U_n^k(g)$ is the unique minimizing graph.*

Lemma 3.3 ([13]). *The least Q-eigenvalue of $U_n^k(g)$ is strictly increasing with respect to $k \geq 1$ and odd $g \geq 3$, respectively.*

Corollary 3.4. *The least Q-eigenvalue of $V_n^\gamma(g)$ is strictly decreasing with respect to γ.*

Proof. Suppose that $\gamma \geq \lceil \frac{g}{3} \rceil + 1$, $V_n^\gamma(g) =: U_n^k(g)$, $V_n^{\gamma-1}(g) =: U_n^{k'}(g)$. Clearly, $k < k'$, and the result follows by Lemma 3.3. ■

Corollary 3.5. *Let $\gamma(U_n^k(g)) = \gamma$. If $U_n^k(g) \neq V_n^\gamma(g)$, then $q_{\min}(U_n^k(g)) > q_{\min}(V_n^\gamma(g))$.*

Proof. Suppose that $V_n^\gamma(g) = U_n^{k'}(g)$. If $U_n^k(g) \neq V_n^\gamma(g)$, then $k > k'$ by the choice of $V_n^\gamma(g)$. The result follows by Lemma 3.3. ■

Corollary 3.6. *The least Q-eigenvalue of $V_n^\gamma(g)$ is strictly increasing with respect to odd g.*

Proof. Suppose that $g \geq 5$. For the graph $V_n^\gamma(g) =: U_n^k(g)$ in Fig. 3.1, replacing the edge $v_{g-2}v_{g-1}$ by $v_{g-2}v_1$, we obtain a graph $G' \in \mathscr{U}_n^{k+1}(g-2)$. Let x be a unit first Q-eigenvector of $V_n^\gamma(g)$. Then

$$q_{\min}(V_n^\gamma(g)) = x^T Q(V_n^\gamma(g))x = x^T Q(G')x \geq q_{\min}(G') \geq q_{\min}(U_n^{k+1}(g-2)),$$

where the second equality holds as $x(v_1) = x(v_{g-1})$ by Lemma 3.1, and the last inequality holds by Lemma 3.2. Furthermore, $q_{\min}(V_n^\gamma(g)) > q_{\min}(G')$; otherwise, x is also a first Q-eigenvector of G', and by considering the eigenvector equations of $V_n^\gamma(g)$ and G' on the vertex v_{g-1} both associated with x, we will have $x(v_{g-1}) = -x(v_{g-2})$; a contradiction to Lemma 3.1.

Note that $\gamma(U_n^{k+1}(g-2)) =: \gamma' \leq \gamma(U_n^k(g)) = \gamma$. So, by Corollaries 3.5 and 3.4,

$$q_{\min}(V_n^\gamma(g)) > q_{\min}(G') \geq q_{\min}(U_n^{k+1}(g-2)) \geq q_{\min}(V_n^{\gamma'}(g-2)) \geq q_{\min}(V_n^\gamma(g-2)). \quad ■$$

Lemma 3.7. *Let $G \in \mathscr{G}_n^\gamma$. Then G contains a non-bipartite spanning unicyclic subgraph with domination number γ.*

Proof. If $\gamma = 1$, the result is easily verified. So we assume that $\gamma \geq 2$. Let $U = \{u_1, u_2, \ldots, u_\gamma\}$ be a dominating set of G of size γ, let $W = V(G)\backslash U$. Let B be a bipartite spanning subgraph of G, which is obtained by deleting all edges within U or W.

Case 1: Suppose that B is connected. Thus, there exist two vertices in U, say u_1 and u_2, such that $N_B(u_1) \cap N_B(u_2) \neq \emptyset$. Assume that $w_1 \in N_B(u_1) \cap N_B(u_2)$. Deleting all edges between u_2 and the vertices of $(N_B(u_1) \cap N_B(u_2))\backslash\{w_1\}$ (if it is nonempty), we get a subgraph B_1 of B such that u_2 shares exactly one neighbor with u_1. If $U\backslash\{u_1, u_2\} \neq \emptyset$, noting that B_1 is also connected, there exists one vertex $w_2 \in N_B(u_1) \cup N_B(u_2)$ such that w_2 is adjacent to one vertex, say u_3 in $U\backslash\{u_1, u_2\}$. Deleting all edges between u_3 and the vertices of $(N_{B_1}(u_3)\backslash\{w_2\}) \cap (N_B(u_1) \cup N_B(u_2))$, we get a subgraph B_2 of B_1 such that u_3 shares exactly one neighbor with exactly one of u_1 and u_2. Repeating the above process, we will arrive at a subgraph $B_{\gamma-1}$ of B such that for each $i = 2, 3, \ldots, \gamma$, u_i shares exactly one neighbor with exactly one of $u_1, u_2, \ldots, u_{i-1}$. So $B_{\gamma-1}$ is a tree with domination number γ. Since G is non-bipartite, there exists at least one edge e within U or W. Adding the edge e to $B_{\gamma-1}$, the resulting graph is as desired.

Case 2: Suppose that B is not connected. Let $B_1, B_2, \ldots, B_k$ be the components of B with bipartitions $(U_1, W_1), (U_2, W_2), \ldots, (U_k, W_k)$ respectively. Since G is connected, there exists a spanning tree B_1 of G obtained from B by adding $k-1$ edges between sets U_i and U_j or sets W_i and W_j. As G is non-bipartite, adding the edges of $E(G)\backslash E(B_1)$ to B_1 such that the first odd cycle C appears, we arrive at a graph B_2. If B_2 contains only one cycle (i.e. the cycle C), the result follows. Otherwise, B_2 contains some even cycles, each of which must contain an edge with endpoints both within U_is or W_js. Deleting one of such edges will break an even cycle and preserve an odd cycle despite the deleted edge shared by C or not. Repeating the process of breaking even cycles, we finally arrive at a connected subgraph containing exactly one odd cycle (not necessarily being C), as desired. ■

Denote by $\mathscr{V}_n^\gamma(g)$ the set of unicyclic graphs of order n with odd girth $g < n$ and domination number γ.

Theorem 3.8. *Among all graphs in $\mathscr{V}_n^\gamma(g)$, where $\lceil \frac{g}{3} \rceil \leq \gamma \leq \gamma_{n,g}$, $V_n^\gamma(g)$ is the unique minimizing graph.*

Proof. Let G be a minimizing graph in $\mathscr{V}_n^\gamma(g)$, and let x be a unit first Q-eigenvector of G. In order to obtain the result, it suffices to prove that G contains exactly one *pendant star* (i.e. the star centered at a quasi-pendant vertex with maximum possible size). Then $G = U_n^k(g)$ for some k and $U_n^k(g) = V_n^\gamma(g)$ by Corollary 3.5.

Suppose that G contains at least two pendant stars. Among all pendant stars of G, choose two pendant stars attached at p and q respectively such that one of them has maximum size. Without loss of generality, assume that $|x(p)| \geq |x(q)|$. Relocating the pendant star attached at q to p, we get a graph G_1 such that $\varrho(G_1) > \varrho(G)$ and $\gamma(G) \geq \gamma(G_1)$, where $\varrho(G)$ denotes the maximum size of the pendant stars of G. By Lemma 2.2, we also have $q_{\min}(G_1) \leq q_{\min}(G)$.

If the graph G_1 has more than one pendant stars, repeating the above process of relocating pendant stars, we will get a sequence of graphs $G = G_0, G_1, \ldots, G_m$ such that from G_{i-1} to G_i the pendant star at q_{i-1} is relocated to p_{i-1} for $i = 1, 2, \ldots, m$, where $p_0 := p$ and $q_0 := q$, and

$$\gamma = \gamma(G) \geq \gamma(G_1) \geq \cdots \geq \gamma(G_m) =: \gamma', \qquad q_{\min}(G) \geq q_{\min}(G_1) \geq \cdots \geq q_{\min}(G_m),$$

where G_m contains exactly one pendant star, i.e. $G_m = U_n^k(g)$ for some k. This can be done as $\varrho(G_i)$ is strictly increasing and is bounded by a finite number.

Now by the above discussion and Corollaries 3.4 and 3.5, we have

$$q_{\min}(G) \geq q_{\min}(G_{m-1}) \geq q_{\min}(U_n^k(g)) \geq q_{\min}(V_n^{\gamma'}(g)) \geq q_{\min}(V_n^\gamma(g)). \tag{3.1}$$

Since G is the minimizing graph in $\mathscr{V}_n^\gamma(g)$, all the inequalities in (3.1) become equalities. So by Corollary 3.4 $\gamma = \gamma'$, and by Corollary 3.5 $U_n^k(g) = V_n^\gamma(g)$. Let y be the first Q-eigenvector of G_{m-1}. Then by the equality $q_{\min}(G_{m-1}) = q_{\min}(U_n^k(g))$ and

Lemma 2.2, $|y(p_{m-1})| = |y(q_{m-1})|$, and $U_n^k(g)$ has a first Q-eigenvector z such that $|z| = |y|$. So, $|z(p_{m-1})| = |z(q_{m-1})|$. Note that p_{m-1} is exactly the quasi-pendant vertex of $U_n^k(g)$, and $|z(p_{m-1})|$ is not equal to the modulus of the z-value of any other vertex by Lemmas 2.3 and 3.1; a contradiction. ■

Theorem 3.9. *Among all graphs in $\mathscr{G}_n^\gamma(g)$, where $\lceil \frac{g}{3} \rceil \le \gamma \le \gamma_{n,g}$, $V_n^\gamma(g)$ is the unique minimizing graph.*

Proof. Let G be a minimizing graph in $\mathscr{G}_n^\gamma(g)$. By Lemma 3.7, G contains a non-bipartite spanning unicyclic subgraph with domination number γ, which we denoted by $\widetilde{G}$. If $\widetilde{G} = C_n$ and since $G \ne C_n$, there exists an edge $uv \in E(G)\backslash E(\widetilde{G})$ joining two vertices of C_n. Then C_n is split into two cycles C^1, C^2 by the edge uv, where C^1 is odd and C^2 is even. Noting that

$$\gamma = \gamma(G) \ge \gamma(C_n + uv) \ge \gamma(C_n) = \gamma(\widetilde{G}) = \gamma,$$

so $\gamma(C_n+uv) = \gamma(C_n) = \gamma$. Let $w_1, w_2 \in V(C_2)\backslash V(C_1)$ such that w_1 is adjacent to u and w_2 is adjacent to w_1 both in C_n+uv. Without loss of generality, we may assume that C_n has a dominating set S of size γ such that $u \in S$ and $w_1, w_2 \notin S$. Therefore, S is still a dominating set of $C_n + uv - w_1w_2$, and hence $\gamma(C_n + uv - w_1w_2) \le \gamma$. Furthermore, $\gamma(C_n + uv - w_1w_2) = \gamma$, since $\gamma(C_n + uv - w_1w_2) \ge \gamma(C_n - w_1w_2) = \gamma(P_n) = \gamma(C_n) = \gamma$.

So we assume that $\widetilde{G}$ contains pendant vertices and has girth $\widetilde{g}$, where $g \le \widetilde{g} < n$. By Theorem 3.8 and Corollary 3.6,

$$q_{\min}(G) \ge q_{\min}(\widetilde{G}) \ge q_{\min}(V_n^\gamma(\widetilde{g})) \ge q_{\min}(V_n^\gamma(g)).$$

Since G is the minimizing graph in $\mathscr{G}_n^\gamma(g)$, we have $q_{\min}(G) = q_{\min}(V_n^\gamma(g))$, which implies that $\widetilde{g} = g$ by Corollary 3.6, and $\widetilde{G} = V_n^\gamma(\widetilde{g}) = V_n^\gamma(g)$ by Theorem 3.8. We now consider the original graph G, which is obtained from $\widetilde{G} = V_n^\gamma(g)$ possibly by adding some edges. Assume that $E(G)\backslash E(V_n^\gamma(g)) \ne \emptyset$. Let x be a unit first Q-eigenvector of G. Then

$$\begin{aligned} q_{\min}(G) &= \sum_{uv\in E(G)} [x(u)+x(v)]^2 \\ &= \sum_{uv\in E(V_n^\gamma(g))} [x(u)+x(v)]^2 + \sum_{uv\in E(G)\backslash E(V_n^\gamma(g))} [x(u)+x(v)]^2 \\ &\ge \sum_{uv\in E(V_n^\gamma(g))} [x(u)+x(v)]^2 \ge q_{\min}(V_n^\gamma(g)). \end{aligned}$$

Since $q_{\min}(G) = q_{\min}(V_n^\gamma(g))$, x is also the first Q-eigenvector of $V_n^\gamma(g)$, and $x(u) + x(v) = 0$ for each edge $uv \in E(G)\backslash E(V_n^\gamma(g))$, which cannot hold by Lemmas 3.1 and 2.3. So, $G = V_n^\gamma(g)$ is the unique minimizing graph. ■

Corollary 3.10. *Among all graphs in $\mathscr{G}_n^\gamma$, where $1 \le \gamma \le \frac{n+1}{3}$, $V_n^\gamma(3)$ is the unique minimizing graph.*

Proof. Let G be a minimizing graph in $\mathscr{G}_n^\gamma$. Assume that the minimum length of odd cycles of G is g. If $g = n$, then $G = C_n$. Note that $q_{\min}(C_n) = 2(1 - \cos\frac{\pi}{n})$; and by a simple verification using (2.2), the following vector x is a corresponding eigenvector which is defined by

$$x(v_i) = (-1)^i \cos\frac{j\pi}{n}, \quad \text{for } i = 1, 2, \ldots, n,$$

where $v_1, v_2, \ldots, v_n$ are the vertices of C_n labeled in an anticlockwise way. Replacing the edge $v_{n-2}v_{n-1}$ by $v_{n-2}v_1$, we get the graph $U_n^1(n-2)$, which has the same domination number as C_n. As $x(v_1) = x(v_{n-1})$, $x^TQ(C_n)x = x^TQ(U_n^1(n-2))x$ and then $q_{\min}(C_n) \ge q_{\min}(U_n^1(n-2))$. In fact, $q_{\min}(C_n) > q_{\min}(U_n^1(n-2))$; otherwise x is also a first Q-eigenvector of $U_n^1(n-2)$, which yields a contradiction by a similar discussion as in the proof of Corollary 3.6. So, we may assume that $g < n$. By Theorem 3.9 and Corollary 3.6,

$$q_{\min}(G) \ge q_{\min}(V_n^\gamma(g)) \ge q_{\min}(V_n^\gamma(3)).$$

Since G is minimizing, all the inequalities become equalities, so $G = V_n^\gamma(g)$ by Theorem 3.9 and $g = 3$ by Corollary 3.6, which implies that $G = V_n^\gamma(3)$. ■

If $n \ge 3\gamma + 1$, then $V_n^\gamma(3) = U_n^{n-3\gamma}(3)$. If $n = 3\gamma - 1$ or $n = 3\gamma$, then $V_n^\gamma(3) = U_n^1(3)$, i.e. $V_n^\gamma(3)$ is obtained from C_3 by appending a path P_{n-2}.

We finally present the main result of this paper.

Corollary 3.11. *Let G be a connected non-bipartite graph of order n with domination number $\gamma \le \frac{n+1}{3}$. Then $q_{\min}(G) \ge q_{\min}(V_n^\gamma(3))$, with equality if and only if $G = V_n^\gamma(3)$.*

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