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The surviving rate of digraphs[★]

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ABSTRACT

Let \overrightarrow{G} be a connected digraph with $n \ge 2$ vertices. Suppose that a fire breaks out at a vertex v of \overrightarrow{G} . A firefighter starts to protect vertices. At each time interval, the firefighter protects k vertices not yet on fire. Afterward, the fire spreads to all unprotected neighbors that are heads of some arcs starting from the vertices on fire. Let $\operatorname{sn}_k(v)$ denote the maximum number of vertices in \overrightarrow{G} that the firefighter can save when a fire breaks out at vertex v. The k-surviving rate $\rho_k(\overrightarrow{G})$ of \overrightarrow{G} is defined as $\sum_{v \in V(\overrightarrow{G})} \operatorname{sn}_k(v)/n^2$. In this paper, we consider the k-surviving rate of digraphs. Main results are as follows:

In this paper, we consider the *k*-surviving rate of digraphs. Main results are as follows: (1) if \overrightarrow{G} is a *k*-degenerate digraph, then $\rho_k(\overrightarrow{G}) \ge \frac{1}{k+1}$; (2) if \overrightarrow{G} is a planar digraph, then $\rho_2(\overrightarrow{G}) > \frac{1}{40}$; (3) if \overrightarrow{G} is a planar digraph without 4-cycles, then $\rho_1(\overrightarrow{G}) > \frac{1}{51}$. © 2014 Elsevier B.V. All rights reserved.

1. Introduction

In 1995, Hartnell [7] introduced the firefighter problem on a finite graph *G*. Assume that a fire breaks out at a vertex v of *G*. A firefighter (or defender) chooses a vertex not yet on fire to protect. Then the firefighter and the fire alternately move on the graph. Once a vertex has been chosen by the firefighter, it is considered protected or safe from any further moves of the fire. After the firefighter's move, the fire makes its move by spreading to all vertices which are adjacent to the vertices on fire, except for those that are protected. The process ends when the fire can no longer spread.

Let sn(v) denote the maximum number of vertices in *G* that the firefighter can save when a fire breaks out at vertex *v*. Determining for a graph *G*, vertex $v \in V(G)$ and an integer *l*, whether $sn(v) \ge l$ is NP-complete, even when *G* is restricted to bipartite graphs [11], cubic graphs [8] and trees with maximum degree three [4]. For a survey of related results the reader is referred to [5].

The surviving rate $\rho(G)$ of a graph *G* with *n* vertices was introduced by Cai and Wang [2] and is defined to be the average proportion of vertices that can be saved when a fire breaks out at one vertex of the graph. More generally, for an integer $k \ge 1$, the *k*-firefighter problem is the same as the firefighter problem, except that at each move, the firefighter protects *k* vertices. We use $\operatorname{sn}_k(v)$ to denote the maximum number of vertices in *G* that the firefighter can save when a fire breaks out at vertex *v*. The *k*-surviving rate $\rho_k(G)$ of a graph *G* with *n* vertices is defined by

$$\rho_k(G) = \frac{\sum_{v \in V(G)} \operatorname{sn}_k(v)}{n^2}.$$

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In particular, $\rho_1(G) = \rho(G)$. By the definition, it is evident that for any integer $k \ge 1$ and a graph G on n vertices, $0 \le \rho_k(G) < 1$, and $\rho_k(G) = 0$ if and only if n = 1. Thus, we always assume that $n \ge 2$ in the following arguments.

Wang et al. [13] proved that for any $k \ge 1$, the *k*-surviving rate of almost all graphs is arbitrarily close to zero and therefore they began studying classes of special graphs, e.g., planar graphs, with the *k*-surviving rate bounded away from zero. In [3], Esperet et al. defined the firefighter number for a class of graph C. Formally, the firefighter number for a class of graph C, denoted by ff(C), is the minimum integer *k* such that there exists $\epsilon > 0$ and an integer *N* so that every $G \in C$ with at least *N* vertices has $\rho_k(G) > \epsilon$. The graph $K_{2,n}$ shows that for \mathcal{P} , the class of planar graphs, $ff(\mathcal{P}) \ge 2$. The firefighter number of the class of planar graphs with girth at least seven is one [14]. The firefighter number of planar graphs with girth five and six remains open. Two independent proofs have shown that $ff(\mathcal{P}) \le 4$ [3,9], and this was recently improved as $ff(\mathcal{P}) \le 3$ [6,10] and it was conjectured that $ff(\mathcal{P}) = 2$ [3]. This conjecture was confirmed for planar graphs without 3-cycles [3], without 4-cycles [15], or without 6-cycles [12]. For other results on the surviving rate of graphs readers are referred to [1,16].

In this paper, we consider the firefighter problem on a digraph *D*. Suppose that a fire breaks out at a vertex *v* of *D*. A firefighter chooses a vertex not yet on fire to protect. Once a vertex has been chosen by the firefighter, it is considered protected or safe from any further moves of the fire. After the firefighter's move, the fire spreads to all unprotected neighbors that are heads of some arcs starting from the vertices on fire. The process ends when the fire can no longer spread. Similarly, we use $sn_k(v)$ to denote the maximum number of vertices in *D* that the firefighter can save when a fire breaks out at vertex *v*. The *k*-surviving rate $\rho_k(D)$ of a digraph *D* with *n* vertices is defined by

$$\rho_k(D) = \frac{\sum_{v \in V(D)} \operatorname{sn}_k(v)}{n^2}.$$

We first consider the *k*-surviving rate on a digraph *D* by showing that $\rho_k(D) \ge \frac{1}{k+1}$ for a *k*-degenerate digraph *D*. Then we consider a planar digraph *D* and show the following results: (1) $\rho_2(D) > \frac{1}{40}$; (2) $\rho(D) > \frac{1}{51}$ if *D* has no 4-cycles.

2. Notation

A plane graph is a particular drawing in the Euclidean plane of a certain planar graph. For a plane graph *G*, we denote its vertex set, edge set, and face set by V(G), E(G), andF(G), respectively. Let n = |V(G)|. For a face $f \in F(G)$, we use b(f) to denote the boundary walk of f and write $f = [u_1u_2 \dots u_m]$ if u_1, u_2, \dots, u_m are the vertices of b(f) in the clockwise order. Repeated occurrences of a vertex are allowed. The degree of a face is the number of edge-steps in its boundary walk. Note that each cut-edge is counted twice. For $x \in V(G) \cup F(G)$, let $d_G(x)$, or simply d(x), denote the degree of x in G. A face of degree k, at least k, or at most k is called a k-face, k^+ -face, or k^- -face, respectively.

A digraph D is an order pair (V, A) consisting of a set V of vertices and a set A, disjoint from V, of arcs, together with an incidence function ψ_D that associates with each arc of D an ordered pair of vertices of D. If a is an arc and $\psi_D(a) = (u, v)$, then a is said to join u to v; we also say that u dominates v. The vertex u is called the *tail* of a, and the vertex v its *head*; they are the two ends of a. The vertices which dominate a vertex v are its in-neighbors, those which are dominated by the vertex its out-neighbors. These sets are denoted by $N_D^-(v)$ and $N_D^+(v)$, respectively.

Given a graph *G*, we may obtain a digraph by replacing each edge by just one of the two possible arcs with the same ends. Such a digraph is called an *orientation* of *G*. We often use the symbol \overrightarrow{G} to express an orientation of *G*. An orientation of a simple graph is referred to as an *oriented graph*. The degree of a vertex v in a digraph *D* is simply the degree of v in *G*, the underlying graph of *D*. The *indegree* $d_D^-(v)$ of a vertex v in *D* is the number of arcs with v as head, and the *outdegree* $d_D^+(v)$ of v is the number of arcs with v as a tail.

Let $k \ge 1$ be an integer. A class of graphs, \mathcal{G} , is said to be *k*-good if the *k*-surviving rate of any graph $G \in \mathcal{G}$ is greater than or equal to a positive constant *c*.

3. Degenerate digraphs

In this section, we consider the *k*-surviving rate on digraphs.

Theorem 1. Let $0 < \varepsilon \le 1$ be a real number and $k \ge 1$ be an integer. If *D* is a digraph with $n(\ge k+1)$ vertices and *m* arcs such that $m \le (k+1-\varepsilon)n$, then $\rho_k(D) \ge \frac{\varepsilon}{k+1}$.

Proof. Let V_{out}^* denote the set of vertices with outdegree at most k and $n^* = |V_{out}^*|$. Clearly, $\operatorname{sn}_k(v) = n - 1$ for any vertex $v \in V_{out}^*$. As $m \le (k + 1 - \varepsilon)n$ and $m = \sum_{v \in V(D)} d^+(v)$, we have

$$(k+1-\varepsilon)n \geq \sum_{v \in V(D)} d^+(v) = \sum_{v \in V_{out}^*} d^+(v) + \sum_{v \in V(D) \setminus V_{out}^*} d^+(v) \geq (k+1)(n-n^*).$$

Thus, $n^* \geq \frac{\varepsilon n}{k+1}$ and

$$\sum_{v \in V(D)} \operatorname{sn}_{k}(v) = \sum_{v \in V_{out}^{*}} \operatorname{sn}_{k}(v) + \sum_{v \in V(D) \setminus V_{out}^{*}} \operatorname{sn}_{k}(v)$$

$$\geq n^{*}(n-1) + k(n-n^{*})$$

$$= n^{*}(n-1-k) + kn$$

$$\geq \frac{\varepsilon n(n-1-k)}{k+1} + kn$$

$$= \frac{\varepsilon n^{2}}{k+1} + (k-\varepsilon)n$$

$$\geq \frac{\varepsilon n^{2}}{k+1}.$$

Therefore,

$$\rho_k(D) = \frac{\sum_{v \in V(D)} \operatorname{sn}_k(v)}{n^2} \ge \frac{\varepsilon}{k+1}. \quad \Box$$

Let $k \ge 1$ be an integer. A graph *G* is called *k*-degenerate if every induced subgraph *H* of *G* contains a vertex of degree at most *k* in *H*. Obviously, 1-degenerate graphs are forests, and 2-degenerate graphs include outerplanar graphs, K_4 -minor-free graphs, planar graphs of girth at least six, etc. It is easy to see that $|E(G)| \le k|V(G)|$ if *G* is a *k*-degenerate graph.

Corollary 2. Let G be a k-degenerate graph and \overrightarrow{G} be an orientation of G. Then $\rho_k(\overrightarrow{G}) \geq \frac{1}{k+1}$.

Since planar graphs are 5-degenerate, Corollary 2 shows that planar orientated graphs are 5-good. In fact, this result can be further improved. Recall that a planar graph *G* with *n* vertices and *m* edges satisfies $m \le 3n - 6 < 3n$. This fact together with Theorem 1 gives immediately the following consequence, which implies that planar orientated graphs are 3-good.

Corollary 3. Let G be a planar graph and \overrightarrow{G} be an orientation of G. Then $\rho_3(\overrightarrow{G}) \geq \frac{1}{4}$.

The girth, denoted by g(G), of a graph G is the length of a shortest cycle in G. Let G be a planar graph embedded in the plane. For a face $f \in F(G)$, it is obvious that $d(f) \ge g(G)$. By using Euler's formula |V(G)| - |E(G)| + |F(G)| = 2,

$$2m = \sum_{f \in F(G)} d(f) \ge g(G)|F(G)| = g(G)(m+2-n)$$

Consequently, when $g(G) \ge 5$,

$$m \le \frac{g(G)}{g(G)-2}(n-2) \le \frac{5}{3}(n-2) < \frac{5}{3}n.$$

By Theorem 1, we obtain the following corollary.

Corollary 4. Let G be a planar graph and \overrightarrow{G} be an orientation of G. If $g(G) \ge 5$, then $\rho(\overrightarrow{G}) \ge \frac{1}{6}$.

4. Planar digraphs

Kong et al. [9] and Esperet et al. [3], independently, showed that planar graphs are 4-good. More recently, Gordinowicz [6] and independently Kong et al. [10] further improved this result by showing that planar graphs are 3-good, and it was conjectured in [3] that planar graphs are 2-good. In this section, we shall prove that planar digraphs are 2-good. Before proving the main result, we first establish the following useful lemma.

Before proving the main result, we first establish the following useful lemma.

Lemma 5. Assume that there is a weight function w on the vertices of \overrightarrow{G} such that the total weight is negative and let V^g be the set of vertices v such that $sn_k(v) \ge n - l$ $(n \ge l + k)$, for some integers k, l. Assume further that for some constants $\beta > 0$ and $\alpha < \beta$ such that $-\alpha k \ge \beta l$, we have $w(v) \ge \alpha$ if $v \in V^g$ and $w(v) \ge \beta$ otherwise. Then $\rho_k(\overrightarrow{G}) > \frac{\beta}{\beta - \alpha}$.

Proof. According to the assumption, it follows that

$$0 > \sum_{v \in V(\overrightarrow{G})} w(v) = \sum_{v \in V^g} w(v) + \sum_{v \in V(\overrightarrow{G}) \setminus V^g} w(v)$$

$$\geq \alpha n^g + \beta (n - n^g).$$

This gives $n^g \geq \frac{\beta}{\beta-\alpha}n$.

It is easy to see that when a fire breaks out at a vertex $v \in V(\overrightarrow{G}) \setminus V^g$, the firefighter can save at least k vertices. Thus,

$$\sum_{v \in V(\vec{G})} \operatorname{sn}_{k}(v) = \sum_{v \in V^{g}} \operatorname{sn}_{k}(v) + \sum_{v \in V(\vec{G}) \setminus V^{g}} \operatorname{sn}_{k}(v)$$

$$\geq (n-l)n^{g} + k(n-n^{g})$$

$$= (n-l-k)n^{g} + kn$$

$$\geq \frac{n(n-l-k)\beta}{\beta-\alpha} + kn$$

$$= \frac{\beta}{\beta-\alpha}n^{2} + \frac{-\alpha k - \beta l}{\beta-\alpha}n$$

$$> \frac{\beta}{\beta-\alpha}n^{2}.$$

Therefore,

$$\rho_k(\overrightarrow{G}) = \frac{\sum\limits_{v \in V(\overrightarrow{G})} \operatorname{sn}_k(v)}{n^2} > \frac{\beta}{\beta - \alpha}. \quad \Box$$

A planar graph G is maximal if no new edge can be added without violating the planarity of G. Let G be a maximal plane graph with *n* vertices and *m* edges. Obviously, if $n \le 4$, then *G* is the complete graph K_n . If $n \ge 5$, then $3 \le \delta(G) \le 5$ and m = 3n - 6, where $\delta(G)$ denotes the minimum degree of *G*.

Given a planar digraph \overrightarrow{H} with *n* vertices, there exists a maximal planar digraph \overrightarrow{G} with *n* vertices such that \overrightarrow{H} is a spanning subgraph of \overrightarrow{G} . It is easy to observe that $\rho_k(\overrightarrow{H}) \ge \rho_k(\overrightarrow{G})$. Hence, we consider the maximal planar digraphs in Section 4

Lemma 6. Suppose that G is a maximal plane graph and \overrightarrow{G} is an orientation of G. Let V^g be the set of vertices with $\operatorname{sn}_2(v) \ge n-3$, and let $V^b = V(\overrightarrow{G}) \setminus V^g$. Then there exists a weight function w(v) for a vertex $v \in V(G)$ such that

(1) $\sum_{v \in V(\vec{G})} w(v) < 0.$ (2) If $v \in V^g$, then $w(v) \ge -3.$ (3) If $v \in V^b$, then $w(v) \ge \frac{1}{13}.$

Proof. First, we give an initial weight function $w_0(x) = d^+(x) - 3$ for a vertex $x \in V(\overrightarrow{G})$. Then we define the following discharging rules (R1)–(R3) for a vertex $v \in V^b$ with $d^+(v) = 3$ as follows.

- (R1) If a vertex $u \in N_D^-(v)$ has $d^+(u) \ge 4$, then u sends $\frac{1}{13}$ to v.
- (R2) If a 3-face f = [vxy] has $(v, x), (x, y) \in A(\overrightarrow{G})$ and $d^+(x) \ge 4$, then x sends $\frac{1}{13}$ to v along the arc (x, y) and through the face f (see Fig. 1). (R3) If $u \in N_D^-(v) \cap V^g$, then u sends $\frac{1}{13}$ to v.

Once these rules are carried out on \overrightarrow{G} , we obtain a new weight function w(v) for $v \in V(\overrightarrow{G})$. Now we prove that w(v)satisfies the required properties (1)-(3).

(1) Since \overrightarrow{G} is an orientation of a maximal plane graph *G*, we have $|A(\overrightarrow{G})| = 3n - 6$. By the Handshake lemma, we get:

$$\sum_{v \in V(\overrightarrow{G})} d^+(v) = \sum_{v \in V(\overrightarrow{G})} d^-(v) = 3n - 6.$$

Thus,

$$\sum_{v \in V(\overrightarrow{G})} w(v) = \sum_{v \in V(\overrightarrow{G})} w_0(v) = \sum_{v \in V(\overrightarrow{G})} (d^+(v) - 3) = -6 < 0.$$

(2) Let $v \in V^g$. There are at most $d^+(v)$ vertices in $N_D^+(v) \cap V^b$. By (R1)–(R3),

$$w(v) \ge w_0(v) - 4 \cdot \frac{1}{13} \cdot d^+(v)$$

= $d^+(v) - 3 - \frac{4}{13}d^+(v)$
= $\frac{9}{13}d^+(v) - 3$
> -3 .



Fig. 1. Rule 2.

(3) Let $v \in V^b$. Then $d^+(v) \ge 3$. If $d^+(v) \ge 4$, then v sends at most $d^+(v) \cdot \frac{1}{13}$ to the vertices in $N_D^+(v) \cap V^b$ with outdegree 3 and $2 \cdot d^+(v) \cdot \frac{1}{13}$ to the other vertices in $V^b \setminus N_D^+(v)$ by (R2) as an edge belongs to at most two triangles. Thus,

$$w(v) \ge w_0(v) - \frac{d^+(v)}{13} - \frac{2d^+(v)}{13}$$
$$= d^+(v) - 3 - \frac{3d^+(v)}{13}$$
$$\ge \frac{10d^+(v) - 39}{13}$$
$$\ge \frac{1}{13}.$$

Therefore, we may assume that $d^+(v) = 3$.

Let $N_D^+(v) = \{x, y, z\}$. It is easy to see that $d^+(x), d^+(y), d^+(z) \ge 3$. Otherwise, v is in V^g . If $N_D^-(v) = \emptyset$, then x, y, z are mutually adjacent. We consider the following two cases, depending on the orientations of the three edges (up to symmetry).

• $(x, y), (y, z), (z, x) \in A(\overrightarrow{G}).$

It is easy to see that x, y, z have outdegree at least 4 as v is a bad vertex. By (R2), v receives $\frac{1}{13}$ from each of x, y, z. Hence, $w(v) \ge w_0(v) + 3 \times \frac{1}{13} = \frac{3}{13}$.

• $(x, y), (y, z), (x, z) \in A(\overrightarrow{G})$. Similarly, we can see that both x and y have outdegree at least 4. Thus, $w(v) \ge w_0(v) + 3 \times \frac{1}{13} = \frac{3}{13}$ by (R2).

Suppose that $N_D^-(v) \neq \emptyset$. If there is a vertex in $N_D^-(v)$ with outdegree at least 4, then we have $w(v) \ge w_0(v) + \frac{1}{13} = \frac{1}{13}$ by (R1). Otherwise, all the vertices in $N_D^-(v)$ have outdegree at most 3. If there is a vertex u in $N_D^-(v)$ with outdegree at most 2, then $u \in V^g$. Hence, we have $w(v) \ge w_0(v) + \frac{1}{13} = \frac{1}{13}$ by (R3). Thus, we assume that all the vertices in $N_D^-(v)$ have outdegree 3.

Since \overrightarrow{G} is maximal, there exists a vertex w in $N_D^-(v)$ adjacent to some of x, y, z, say z. Assume that $(w, z) \in A(\overrightarrow{G})$. Let $N_D^+(w) = \{v, z, s\}$. It follows that $w \in V^g$ since, when a fire breaks out at w, we can define a strategy for the firefighter by first defending $\{z, s\}$ and then $\{x, y\}$. Hence, $w(v) \ge w_0(v) + \frac{1}{13} = \frac{1}{13}$ by (R3). Assume that $(z, w) \in A(\overrightarrow{G})$. If z has outdegree at least 4, then by (R1), it is not difficult to see that $w(v) \ge w_0(v) + \frac{1}{13} = \frac{1}{13}$. Otherwise, let $N_D^+(z) = \{w, z_1, z_2\}$ and $N_D^+(w) = \{v, w_1, w_2\}$. $w \in V^g$ since the firefighter can first protect $\{w_1, w_2\}$, then $\{x, y\}$ and finally $\{z_1, z_2\}$ when a fire breaks out at w. By (R3), w sends $\frac{1}{13}$ to v along (z, w) through the face [vzw]. Hence, $w(v) \ge w_0(v) + \frac{1}{13} = \frac{1}{13}$. \Box

Combining Lemmas 5 and 6, we have the following consequence by choosing k = 2, l = 3, $\alpha = -3$ and $\beta = \frac{1}{13}$.

Corollary 7. If \overrightarrow{G} is an oriented planar graph, then $\rho_2(\overrightarrow{G}) > \frac{1}{40}$.

5. Planar digraphs without 4-cycles

Wang et al. [15] showed that planar graphs without 4-cycles are 2-good. In this section, we prove that oriented planar graphs without 4-cycles are 1-good. Suppose that \overrightarrow{G} is an oriented planar graph without 4-cycles. Then \overrightarrow{G} contains neither 4-faces nor adjacent 3-faces.

Lemma 8. Suppose that G is a plane graph without 4-cycles and \overrightarrow{G} is an orientation of G. Let V^g be the set of vertices with $sn(v) \ge n-3$, and $V^b = V(\overrightarrow{G}) \setminus V^g$. Then there exists a weight function w(x) for $x \in V(\overrightarrow{G}) \cup F(\overrightarrow{G})$ such that

(1) $\sum_{x \in V(\vec{G}) \cup F(\vec{G})} w(x) < 0.$ (2) If $v \in V^g$, then $w(v) \ge -4.$ (3) If $v \in V^b$, then $w(v) \ge \frac{2}{25}.$

- (4) If $f \in F(\overrightarrow{G})$, then $w(f) \ge 0$.

Proof. For convenience, we say that vertices are good if they are in V^g and all other vertices bad vertices. We first define an initial weight function $w_0(v) = 2d^+(v) - 4$ for $v \in V(\overrightarrow{G})$ and $w_0(f) = d(f) - 4$ for $f \in F(\overrightarrow{G})$. Then we define the following discharging rules:

- (R1) Every 5⁺-face f gives $\frac{3}{25}$ to each adjacent 3-face. (R2) Every 5⁺-face f gives $\frac{2}{25}$ to each incident bad vertex with outdegree 2.
- (R3) Let f = [uvw] be a 3-face with $(v, u) \in A(\overrightarrow{G})$. If v is a good vertex or has outdegree at least 3, then v gives $\frac{16}{25}$ to f along the arc (v, u).

Once these rules are carried out on \overrightarrow{G} , we obtain a new weight function $w(x), x \in V(\overrightarrow{G}) \cup F(\overrightarrow{G})$. Let us prove that the weight function w(v) satisfies the required properties (1)–(4).

(1) By Euler's formula |V(G)| - |E(G)| + |F(G)| = 2 and the relations

$$\sum_{v \in V(\overrightarrow{G})} d^+(v) = |E(\overrightarrow{G})|, \qquad \sum_{f \in F(\overrightarrow{G})} d(f) = 2|E(\overrightarrow{G})|,$$

we derive the following identity:

$$\sum_{v \in V(\overrightarrow{G})} w(v) + \sum_{f \in F(\overrightarrow{G})} w(f) = \sum_{v \in V(\overrightarrow{G})} w_0(v) + \sum_{f \in F(\overrightarrow{G})} w_0(f)$$
$$= \sum_{v \in V(\overrightarrow{G})} (2d^+(v) - 4) + \sum_{f \in F(\overrightarrow{G})} (d(f) - 4)$$
$$= 2|E(\overrightarrow{G})| - 4|V(\overrightarrow{G})| + 2|E(\overrightarrow{G})| - 4|F(\overrightarrow{G})|$$
$$= -8 < 0.$$

(2) Let $v \in V^g$. Then v only lose charge in (R3). Thus, $w(v) \ge w_0(v) - \frac{16}{25}d^+(v) = 2d^+(v) - 4 - \frac{16}{25}d^+(v) = \frac{34}{25}d^+(v) - 4 \ge -4$. (3) Assume that $v \in V^b$. Then $d^+(v) \ge 2$. If $d^+(v) = 2$, then it is easy to derive that v is incident to at least one 5⁺-face as \overrightarrow{G} is an oriented planar graph without 4-cycles. Thus, by rule (R2), $w(v) \ge w_0(v) + \frac{2}{25} = \frac{2}{25}$. If $d^+(v) \ge 3$, then v only loses charge in (R3). It follows that v sends at most $\frac{16}{25}d^+(v)$ to incident 3-faces as \overrightarrow{G} contains no adjacent 3-faces. Therefore, $w(v) \ge w_0(v) - \frac{16}{25}d^+(v) = 2d^+(v) - 4 - \frac{16}{25}d^+(v) = \frac{34}{25}d^+(v) - 4 \ge \frac{2}{25}$. (4) Let $f \in F(G)$. Then $d(f) \ne 4$.

Case 1 d(f) = 3.

Then $\tilde{w}_0(f) = -1$. Without loss of generality, let f = [uvw], and then f is adjacent to three 5⁺-faces. By (R1), f receives $3 \times \frac{3}{25} = \frac{9}{25}$ from its adjacent 5⁺-faces. We need to consider the following two cases (up to symmetry).

(1.1) $(v, u), (u, w), (w, v) \in A(\overrightarrow{G}).$

If one of u, v, w is a vertex with outdegree 1, say u, then u gives $\frac{16}{25}$ to f along the arc (u, w) by (R3), so $w(f) \ge 1$ $-1 + \frac{9}{25} + \frac{16}{25} = 0$. Similarly, if one of u, v, w is a vertex with outdegree at least 3, say u, then u gives $\frac{16}{25}$ to f along the arc (u, w) by (R3). Thus, $w(f) \ge -1 + \frac{9}{25} + \frac{16}{25} = 0$. Hence, we assume that all of u, v, w are vertices with outdegree 2. Let $N_D^+(v) = \{u, v_1\}, N_D^+(u) = \{w, u_1\}, N_D^+(w) = \{v, w_1\}. u \in V^g$ since the firefighter can protect u_1, v_1, w_1 successively when a fire breaks out at *u*. Hence, $w(f) \ge -1 + \frac{9}{25} + \frac{16}{25} = 0$ by (R3).

(1.2) $(v, u), (u, w), (v, w) \in A(\overrightarrow{G}).$

If one of v, u is a vertex with outdegree 1 or at least 3, say v, then v gives $\frac{16}{25}$ to f along the arc (v, u) by (R3). Thus, $w(f) \ge -1 + \frac{9}{25} + \frac{16}{25} = 0$. Otherwise, v and u are vertices with outdegree 2. Without loss of generality, say, $N_D^+(u) = \{s, w\}$. $v \in V^g$ since the firefighter first protect w, and then protect s when a fire breaks out at v. Hence, $\omega(f) \ge -1 + \frac{9}{25} + \frac{16}{25} = 0$ by (R3).

Case 2 d(f) > 5.

It is easy to see that f gives at most $\frac{3}{25}d(f)$ to adjacent 3-faces and $\frac{2}{25}d(f)$ to incident bad vertices with outdegree 2, by (R1) and (R2). Therefore, $\omega(f) \ge \omega_0(f) - \frac{3}{25}d(f) - \frac{2}{25}d(f) = d(f) - 4 - \frac{1}{5}d(f) = \frac{4}{5}d(f) - 4 \ge 0.$

By Lemma 8(1) and (4), we have the following

$$0 > \sum_{x \in V(\vec{G}) \cup F(\vec{G})} w(x)$$

=
$$\sum_{v \in V(\vec{G})} w(v) + \sum_{f \in F(\vec{G})} w(f)$$

\ge
$$\sum_{v \in V(\vec{G})} w(v).$$

That is, there is a weight function w on the vertices of \overrightarrow{G} such that the total weight is negative. Thus, the following holds obviously from Lemma 5 by choosing k = 1, l = 3, $\alpha = -4$ and $\beta = \frac{2}{25}$.

Corollary 9. If \overrightarrow{G} is an oriented planar graph without 4-cycles, then $\rho(\overrightarrow{G}) > \frac{1}{51}$.

6. Concluding remarks

Gordinowicz [6] and Kong et al. [10] showed that planar graphs are 3-good, independently, and Esperet et al. [3] conjectured that planar graphs are 2-good. In Section 4, we show that oriented planar graphs are 2-good. Thus, the following is a natural problem.

Question 1. Are oriented planar graphs 1-good?

Corollary 9 in Section 5 asserts that oriented planar graphs without 4-cycles are 1-good, which gives a partial solution of Question 1. Wang et al. [15] showed that planar graphs without 4-cycles are 2-good.

Based on the above results, we like to put forward the following problem.

Question 2. If *C* is a 2k-good graph, then is \overrightarrow{C} *k*-good?

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