## Note

# Total chromatic number of generalized Mycielski graphs ${ }^{\text {T}}$ 

<br>a School of Applied Mathematics, Xiamen University of Technology, Xiamen, Fujian 361024, China<br>${ }^{\mathrm{b}}$ School of Mathematical Sciences, Xiamen University, Xiamen, Fujian 361005, China<br>${ }^{\text {c }}$ Laboratoire de Recherche en Informatique, UMR 8623, C.N.R.S.-Université de Paris-sud, 91405-Orsay Cedex, France

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#### Abstract

A total coloring of a simple graph $G$ is a coloring of both the edges and the vertices. A total coloring is proper if no two adjacent or incident elements receive the same color. The minimum number of colors required for a proper total coloring of $G$ is called the total chromatic number of $G$ and denoted by $\chi_{t}(G)$. The Total Coloring Conjecture (TCC) states that for every simple graph $G, \Delta(G)+1 \leq \chi_{t}(G) \leq \Delta(G)+2$. $G$ is called Type 1 (resp. Type 2) if $\chi_{t}(G)=\Delta(G)+1$ (resp. $\left.\chi_{t}(G)=\Delta(G)+2\right)$. In this paper, we prove that the generalized Mycielski graphs satisfy TCC. Furthermore, we get that if $\Delta(G) \leq \frac{|V(G)|-1}{2}$, then the generalized Mycielski graph $\mu_{m}(G)$ is Type 1.


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## 1. Introduction

All graphs considered in this paper are finite, simple, connected and undirected. Terminology and notation not defined here are followed [3]. Let $G$ be a graph. We use $V(G), E(G)$ and $\Delta(G)$ (or simply $V, E$ and $\Delta$ ) to denote the vertex set, the edge set and the maximum degree of $G$, respectively.

A $k$-total coloring $h: V \cup E \rightarrow\{1,2, \ldots, k\}$ of a graph $G=(V, E)$ is an assignment of $k$ colors to both the edges and the vertices of $G$. The total coloring $h$ is called a proper $k$-total coloring if no incident or adjacent elements (vertices or edges) receive the same color. The total chromatic number of $G, \chi_{t}(G)$, is the least integer $k$ for which $G$ admits a proper $k$-total coloring. Behzad [1] and Vizing [20] proposed independently the following famous conjecture, which is known as the Total Coloring Conjecture (TCC).

Conjecture 1. For any graph $G, \Delta(G)+1 \leq \chi_{t}(G) \leq \Delta(G)+2$.
The lower bound of this conjecture is obvious, the upper bound remains to be proved. Sanchez-Arroyo proved that the problem of determining the total chromatic number is NP-hard [17]. Later, McDiarmid and Sanchez-Arroyo proved that the problem of deciding the total chromatic number of regular bipartite graphs is also NP-hard [15]. By extending a given vertex coloring of a graph to a total coloring, McDiarmid and Sanchez-Arroyo [14] proved that $\frac{7}{5} \Delta+3$ is an upper bound for the total chromatic number of graphs. Using probabilistic methods, Molloy and Reed (1998) showed that the total chromatic

[^0]number of a simple graph $G$ is at most $\Delta(G)+10^{26}$, provided that $\Delta(G)$ is sufficiently large. Hilton and Hind [9] proved that TCC is correct for those graphs $G$ having $\Delta(G) \geq \frac{3}{4}|V(G)|$.

Let $G$ be a graph with vertex set $V^{0}=\left\{v_{1}^{0}, v_{2}^{0}, \ldots, v_{n}^{0}\right\}$ and edge set $E^{0}$. Given an integer $m \geq 1$, the $m$-Mycielskian of $G$, denoted by $\mu_{m}(G)$, is the graph with vertex set $V^{0} \cup V^{1} \cup V^{2} \cup \cdots \cup V^{m} \cup\{u\}$, where $V^{i}=\left\{v_{j}^{i}: v_{j}^{0} \in V^{0}\right\}$ is the ith distinct copy of $V^{0}$ for $i=1,2, \ldots, m$, and edge set $E^{0} \cup\left(\bigcup_{i=0}^{m-1}\left\{v_{j}^{i} v_{j^{\prime}}^{i+1}: v_{j}^{0} v_{j^{\prime}}^{0} \in E^{0}\right\}\right) \cup\left\{v_{j}^{m} u: v_{j}^{m} \in V^{m}\right\}$. The Mycielskian of $G$, $\mu_{1}(G)$ will be denoted simply by $\mu(G)$. Note that the $m$-Mycielskian graph $\mu_{m}(G)$ of $G$ contains $G$ itself as a subgraph.

In order to build a graph with a high chromatic number and a small clique number, Mycielski [16] introduced $\mu(G)$. It is not hard to prove that $\chi(\mu(G))=\chi(G)+1$. Thus $\mu^{k}\left(K_{2}\right)=\mu\left(\mu^{k-1}\left(K_{2}\right)\right)$ has chromatic number $k+2$ and clique number 2. In recent years, there have been results on Mycielski graphs in relation to several coloring problems [4,5,10,12,13,18]. In [11], the authors generalized the Mycielskian of $G$ to the $m$-Mycielskian of $G$, where $m \geq 1$. In this paper, we consider the total chromatic number of the generalized Mycielski graphs and prove that TCC is true for the generalized Mycielski graphs. Before presenting the main result, we would like to recall some useful definitions and theorems.

A proper $k$-edge-coloring of a graph $G$ is an assignment of colors from a color set $C$ to each edge of $G$ such that every two adjacent edges receive different colors, where $C$ is a set of $k$ colors. The edge-chromatic number of a graph $G$, denoted by $\chi^{\prime}(G)$, is the minimum $k$ for which $G$ has a proper $k$-edge-coloring. In 1964, Vizing [19] showed that every simple graph $G$ has edge-chromatic number either $\Delta(G)$ (known as a Class I graph) or $\Delta(G)+1$ (known as a Class II graph).

Theorem 2 ([19]). For any graph $G, \Delta(G) \leq \chi^{\prime}(G) \leq \Delta(G)+1$.
We will need the following strengthening of this theorem.
Theorem 3 ([6]). For any graph $G$, if the subgraph of $G$ induced by the vertices of maximum degree is a forest, then $G$ is Class I.
In list edge-coloring, each edge $e$ of $G$ has a set $L(e)$ of colors, called the list of $e$. Then, a proper edge-coloring $f$ of $G$ is called an $L$-edge-coloring of $G$ if $f(e) \in L(e)$ for each edge $e$, where $f(e)$ denotes the color assigned to $e$ by $f$. If $G$ admits an $L$-edge-coloring, then it is $L$-edge-colorable. For $k \in \mathbb{N}$, the graph is $k$-edge-choosable if it is $L$-edge-colorable for every list assignment $L$ with $|L(e)| \geq k$ for each $e \in E(G)$. Galvin established that every bipartite multigraph $G$ is $\Delta(G)$-edge-choosable.

Theorem 4 ([7]). Every bipartite multigraph $G$ is $\Delta(G)$-edge-choosable.

## 2. Main result

In this section, we consider the total chromatic number of the generalized Mycielski graphs. We begin with some definitions and a lemma which will be used in the proof of our main result.

Consider $\mu_{m}(G)$ and let $G_{i}$ be the bipartite subgraph on the vertex set $V^{i-1} \cup V^{i}$ and the edges with one end in $V^{i-1}$ and the other in $V^{i}$, where $1 \leq i \leq m$. By the definition of $\mu_{m}(G), G_{i}$ is isomorphic to $G_{j}$ for any $j \neq i, 1 \leq j \leq m$. If $v_{j}^{0} v_{j^{\prime}}^{0} \in E^{0}$, then both $v_{j}^{i-1} v_{j^{\prime}}^{i} \in E\left(G_{i}\right)$ and $v_{j^{\prime}}^{i-1} v_{j}^{i} \in E\left(G_{i}\right)$. Clearly, $\Delta\left(G_{i}\right)=\Delta(G)=\Delta$. In accordance, we also denote $G$ by $G_{0}$. Let $X^{0}$ and $X^{1}$ be the set of vertices in $V^{0}$ and $V^{1}$ (respectively) of degree $\Delta$ in $G_{1}$. First, we would like to show that there exists a matching in $G_{1}$ which saturates the vertices of both $X^{0}$ and $X^{1}$.

Lemma 5. There exists a matching $M^{1}$ in $G_{1}$ which saturates the vertices of both $X^{0}$ and $X^{1}$.
Proof. Assume $\emptyset \neq S \subseteq X^{0}$. Let $N(S)$ be the set of all neighbors of $S$ in $V^{1}$ and let $E_{1}$ and $E_{2}$ be the sets of edges of $G_{1}$ incident with $S$ and $N(S)$, respectively. By definition of $N(S)$, we have $E_{1} \subseteq E_{2}$. Therefore, $\Delta|N(S)| \geq\left|E_{2}\right| \geq\left|E_{1}\right|=\Delta|S|$. We can obviously assume $\Delta \geq 1$, it then follows that $|N(S)| \geq|S|$ and hence, by Hall's theorem [8], there exists a matching from $X^{0}$ to $V^{1}$; denote it by $M$. Let $M^{\prime}$ be the symmetric matching of $M$, that is, $v_{i}^{0} v_{j}^{1} \in M^{\prime}$ if and only if $v_{j}^{0} v_{i}^{1} \in M$. Thus, $M^{\prime}$ is a matching from $X^{1}$ to $V^{0} . M \cup M^{\prime}$ together induce a graph in which each connected component is either an even cycle or a path. There may exist multiple edges which we view them as 2-cycles. In the following, we will select some edges from $M \cup M^{\prime}$ to form a matching which saturates the vertices of both $X^{0}$ and $X^{1}$.

For a component of $M \cup M^{\prime}$, if the component is an even cycle $C_{2 k}$, let $e_{1}, e_{2}, \ldots, e_{2 k}$ be the edge sequence of $C_{2 k}$, we pick $e_{1}, e_{3}, \ldots, e_{2 k-1}$. Then $e_{1}, e_{3}, \ldots, e_{2 k-1}$ saturates all the vertices of $C_{2 k}$. If the component is an odd path $P_{2 k}$, let $e_{1}, e_{2}, \ldots, e_{2 k-1}$ be the edge sequence of $P_{2 k}$, we pick $e_{1}, e_{3}, \ldots, e_{2 k-1}$. Then $e_{1}, e_{3}, \ldots, e_{2 k-1}$ saturates all the vertices of $P_{2 k}$. If the component is an even path $P_{2 k+1}$, let $e_{1}, e_{2}, \ldots, e_{2 k}$ be the edge sequence of $P_{2 k+1}$. By symmetry, without loss of generality, assume that the vertex sequence of $P_{2 k+1}$ is $v_{i_{1}}^{1}, v_{j_{1}}^{0}, v_{i_{2}}^{1}, v_{j_{2}}^{0}, \ldots, v_{i_{k}}^{1}, v_{j_{k}}^{0}, v_{i_{k+1}}^{1}$. Then we conclude that it cannot happen that both $v_{i_{1}}^{1}$ and $v_{i_{k+1}}^{1}$ belong to $X^{1}$. Otherwise, suppose that both $v_{i_{1}}^{1}$ and $v_{i_{k+1}}^{1}$ are maximum degree vertices. Since $P_{2 k+1}$ is an even path, either $e_{1}$ or $e_{2 k}$ belongs to $M$, assume $e_{1} \in M$; then $v_{i_{1}}^{1}$ is also saturated by $M^{\prime}$ by the fact that $v_{i_{1}}^{1} \in X^{1}$, so $P_{2 k+1}$ can be enlarged, a contradiction. Suppose $d_{G_{1}}\left(v_{i_{k+1}}^{1}\right)<\Delta(G)$; then we pick $e_{1}, e_{3}, \ldots, e_{2 k-1}$. They saturate the vertices of $P_{2 k+1}$ with maximum degree in $G_{1}$.

Therefore, the edges we select from every component of $M \cup M^{\prime}$ form a matching which saturates the maximum degree vertices in $G_{1}$. We denote this set of edges by $M^{1}$.

We are now ready to prove the main result of this paper.
Theorem 6. For each integer $m \geq 1$, the $m$-Mycielskian $\mu_{m}(G)$ of a graph $G$ satisfies $\chi_{t}\left(\mu_{m}(G)\right) \leq \Delta\left(\mu_{m}(G)\right)+2$.
Proof. Let $G$ be a graph with $n$ vertices, note that $\Delta\left(\mu_{m}(G)\right)=\max \{2 \Delta(G), n\}$. Depending on whether $\Delta\left(\mu_{m}(G)\right)=2 \Delta(G)$ or $\Delta\left(\mu_{m}(G)\right)=n$, we will consider two cases. In both cases we will use an edge-coloring of the following graph. Let $G$ be a graph obtained from $G_{0}$ by adding a new vertex $w$ joined to all vertices of $G_{0}$.

Case 1. $n \geq 2 \Delta(G)+\underset{\sim}{1}$. In the case we have $\Delta\left(\mu_{m}(G)\right)=n$. Thus $\left.\Delta \widetilde{G}\right)=n$ and $d_{\widetilde{G}}\left(v_{j}^{0}\right) \leq \Delta(G)+1<n$ for every $j$. Hence, the subgraph of $\widetilde{G}$ induced by the vertices with maximum degree is just $K_{\mathcal{F}}$. By Theorem 3, there exists a proper n-edge coloring $\widetilde{f}: E(\widetilde{G}) \rightarrow\left\{c_{1}, c_{2}, \ldots, c_{n}\right\}$ for $\widetilde{G}$. We will modify and extend $\widetilde{f}$ to be a proper $(n+1)$-total coloring $f: V\left(\mu_{m}(G)\right) \cup E\left(\mu_{m}(G)\right) \rightarrow\left\{c_{1}, \ldots, c_{n}, c_{n+1}\right\}$ for $\mu_{m}(G)$.

For any $v_{j}^{0} v_{l}^{0} \in E\left(G_{0}\right)$, let $f\left(v_{j}^{0} v_{l}^{0}\right)=\widetilde{f}\left(v_{j}^{0} v_{l}^{0}\right)$, and for any $j \in\{1, \ldots, n\}$, let $f\left(v_{j}^{0}\right)=\widetilde{f}\left(w v_{j}^{0}\right), f\left(v_{j}^{m} u\right)=c_{j}$. Suppose $M^{m} \subseteq E\left(G_{m}\right)$ is a matching which is corresponding to the matching $M^{1}$ in $G_{1}$. Let $f\left(M^{m}\right)=f(u)=c_{n+1}$. For the other edges and vertices of $\mu_{m}(G)$ that are not colored yet, we will color them by the order $E\left(G_{1}\right), V^{1}, E\left(G_{2}\right), V^{2}, \ldots, E\left(G_{m}\right) \backslash M^{m}, V^{m}$.

For $1 \leq i \leq m-1$, and each edge $v_{j}^{i-1} v_{l}^{i} \in E\left(G_{i}\right)$, let $L\left(v_{j}^{i-1} v_{l}^{i}\right)=\left\{c_{1}, c_{2}, \ldots, c_{n}\right\} \backslash\left\{f\left(v_{j}^{i-1}\right), f\left(v_{j}^{i-1} x\right) \mid v_{j}^{i-1} x \in E\left(G_{i-1}\right)\right\}$ be a list of colors for edge $v_{j}^{i-1} v_{l}^{i}$. Since $n \geq 2 \Delta+1$ and at most $\Delta+1$ colors are removed, we have $\left|L\left(v_{j}^{i-1} v_{l}^{i}\right)\right| \geq \Delta(G)$. Since $\Delta\left(G_{i}\right)=\Delta(G)$, by Theorem 4 , we can color the edges in $E\left(G_{i}\right)$ properly by the set $\left\{c_{1}, c_{2}, \ldots, c_{n}\right\}$. For $v_{j}^{i} \in V^{i}, j=1,2, \ldots, n$, define $\bar{F}\left(v_{j}^{i}\right)=\left\{c_{1}, c_{2}, \ldots, c_{n}\right\} \backslash\left\{f\left(v_{j}^{i} x\right), f(x) \mid v_{j}^{i} x \in E\left(G_{i}\right)\right\}$. Clearly, we have $\left|\bar{F}\left(v_{j}^{i}\right)\right| \geq 1$. Choose one color from the set $\bar{F}\left(v_{j}^{i}\right)$ to color the vertex $v_{j}^{i}$.

For any $v_{j}^{m-1} v_{l}^{m} \in E\left(G_{m}\right) \backslash M^{m}$, let $L\left(v_{j}^{m-1} v_{l}^{m}\right)=\left\{c_{1}, c_{2}, \ldots, c_{n}\right\} \backslash\left\{f\left(v_{j}^{m-1}\right), f\left(v_{l}^{m} u\right), f\left(v_{j}^{m-1} x\right) \mid v_{j}^{m-1} x \in E\left(G_{m-1}\right)\right\}$; then $\left|L\left(v_{j}^{m-1} v_{l}^{m}\right)\right| \geq \Delta-1$. Since $\Delta\left(G_{m} \backslash M^{m}\right)=\Delta-1$, by Theorem 4 , we can color the edges in $E\left(G_{m}\right) \backslash M^{m}$ properly by the set $\left\{c_{1}, c_{2}, \ldots, c_{n}\right\}$. For $v_{j}^{m} \in V^{m}, j=1,2, \ldots, n$, define $\bar{F}\left(v_{j}^{m}\right)=\left\{c_{1}, \ldots, c_{n}, c_{n+1}\right\} \backslash\left\{f\left(v_{i}^{m-1} v_{j}^{m}\right), f\left(v_{i}^{m-1}\right), f\left(v_{j}^{m} u\right)\right.$, $\left.f(u) \mid v_{i}^{m-1} v_{j}^{m} \in E\left(G_{m}\right)\right\}$. Note that there are at most $\Delta$ colors for $f\left(v_{i}^{m-1} v_{j}^{m}\right)$ and at most $\Delta$ colors for $f\left(v_{i}^{m-1}\right)$ and one color for each of $f\left(v_{j}^{m} u\right)$ and $f(u)$. Furthermore, if there are exactly $\Delta$ colors for $f\left(v_{i}^{m-1} v_{j}^{m}\right)^{\prime}$ 's, then the color for $f(u)$ is among them. Thus the set $\left\{f\left(v_{i}^{m-1} v_{j}^{m}\right), f\left(v_{i}^{m-1}\right), f\left(v_{j}^{m} u\right), f(u) \mid v_{i}^{m-1} v_{j}^{m} \in E\left(G_{m}\right)\right\}$ has at most $2 \Delta+1$ elements. Since $n+1 \geq 2 \Delta+2$, the set $\bar{F}\left(v_{j}^{m}\right)$ is not empty, which implies that $\left|\bar{F}\left(v_{j}^{m}\right)\right| \geq 1$. Choose one color from the set $\bar{F}\left(v_{j}^{m}\right)$ to color the vertex $v_{j}^{m}$. This forms a proper $(n+1)$-total coloring for $\mu_{m}(G)$. So we get that $\chi_{t}\left(\mu_{m}(G)\right) \leq n+1=\Delta\left(\mu_{m}(G)\right)+1$. On the other hand, $\chi_{t}\left(\mu_{m}(G)\right) \geq \Delta\left(\mu_{m}(G)\right)+1=n+1$. Hence, $\chi_{t}\left(\mu_{m}(G)\right)=\Delta\left(\mu_{m}(G)\right)+1 \sim n+1$ in this case.

Case 2. $n \leq 2 \Delta(G)$. In the case we have $\underset{\sim}{\Delta}\left(\mu_{m}(G)\right)=2 \Delta(G)$. Since $\Delta(G)=n \lesssim 2 \Delta(G)$, Theorem 2 implies that there exists a proper $(2 \Delta(G)+1)$-edge coloring $\widetilde{f}: E(\widetilde{G}) \rightarrow\left\{c_{1}, c_{2}, \ldots, c_{2 \Delta(G)+1}\right\}$ for $\widetilde{G}$. We will modify and extend $\widetilde{f}$ to be a proper $(2 \Delta(G)+2)$-total coloring $f: V\left(\mu_{m}(G)\right) \cup \underset{\sim}{v} E\left(\mu_{m}(G)\right) \rightarrow\left\{c_{1}, \ldots, c_{2 \Delta(G)+1}, c_{2 \Delta(G)+2}\right\}$ for $\mu_{m}(G)$.

For each edge $v_{j}^{0} v_{l}^{0} \in E\left(G_{0}\right)$, let $f\left(v_{j}^{0} v_{l}^{0}\right)=\widetilde{f}\left(v_{j}^{0} v_{l}^{0}\right)$, and for $j=1,2, \ldots, n$, let $f\left(v_{j}^{0}\right)=\widetilde{f}\left(w v_{j}^{0}\right)$ and $f\left(v_{j}^{m} u\right)=c_{j}$. Let $f\left(M^{m}\right)=f(u)=c_{2 \Delta(G)+2}$. For the other edges and vertices that are not colored yet, we will color them by the order $E\left(G_{1}\right), V^{1}, E\left(G_{2}\right), V^{2}, \ldots, E\left(G_{m}\right) \backslash M^{m}, V^{m}$.

For $1 \leq i \leq m-1$, let $v_{j}^{i-1} v_{l}^{i} \in E\left(G_{i}\right)$, and $L\left(v_{j}^{i-1} v_{l}^{i}\right)=\left\{c_{1}, \ldots, c_{2 \Delta+1}\right\} \backslash\left\{f\left(v_{j}^{i-1}\right), f\left(v_{j}^{i-1} x\right) \mid v_{j}^{i-1} x \in E\left(G_{i-1}\right)\right\}$ be a list of colors for edge $v_{j}^{i-1} v_{l}^{i}$; then we have $\left|L\left(v_{j}^{i-1} v_{l}^{i}\right)\right| \geq \Delta(G)$. Since $\Delta\left(G_{i}\right)=\Delta(G)$, by Theorem 4 , we can color the edges in $E\left(G_{i}\right)$ properly by the set $\left\{c_{1}, \ldots, c_{2 \Delta(G)+1}\right\}$. For the vertex $v_{j}^{i} \in V^{i}$, define $\bar{F}\left(v_{j}^{i}\right)=\left\{c_{1}, \ldots, c_{2 \Delta(G)+1}\right\} \backslash\left\{f\left(v_{j}^{i} x\right), f(x) \mid v_{j}^{i} x \in E\left(G_{i}\right)\right\}$. Clearly, we have $\left|\bar{F}\left(v_{j}^{i}\right)\right| \geq 1$. Choose one color from the set $\bar{F}\left(v_{j}^{i}\right)$ to color the vertex $v_{j}^{i}$.

For each edge $v_{j}^{m-1} v_{l}^{m} \in E\left(G_{m}\right) \backslash M^{m}$, define the color set $L\left(v_{j}^{m-1} v_{l}^{m}\right)=\left\{c_{1}, \ldots, c_{2 \Delta(G)+1}\right\} \backslash\left\{f\left(v_{j}^{m-1}\right), f\left(v_{l}^{m} u\right), f\left(v_{j}^{m-1}\right.\right.$ $\left.x) \mid v_{j}^{m-1} x \in E\left(G_{m-1}\right)\right\}$; then $\left|L\left(v_{j}^{m-1} v_{l}^{m}\right)\right| \geq \Delta(G)-1$. Since $\Delta\left(G_{m} \backslash M^{m}\right)=\Delta(G)-1$, by Theorem 4 , we can color the edges in $E\left(G_{m}\right) \backslash M^{m}$ properly by the set $\left\{c_{1}, \ldots, c_{2 \Delta(G)+1}\right\}$. For $v_{j}^{m} \in V^{m}, j=1,2, \ldots, n$, let $\bar{F}\left(v_{j}^{m}\right)=\left\{c_{1}, \ldots, c_{2 \Delta(G)+1}, c_{2 \Delta(G)+2}\right\} \backslash$ $\left\{f\left(v_{i}^{m-1} v_{j}^{m}\right), f\left(v_{i}^{m-1}\right), f\left(v_{j}^{m} u\right), f(u) \mid v_{i}^{m-1} v_{j}^{m} \in E\left(G_{m}\right)\right\}$. Similar to the previous case, we have $\left|\bar{F}\left(v_{j}^{m}\right)\right| \geq 1$. Choose one color from the set $\bar{F}\left(v_{j}^{m}\right)$ to color the vertex $v_{j}^{m}$. This forms a proper $(2 \Delta(G)+2)$-total coloring for $\mu_{m}(G)$. Hence $\chi_{t}\left(\mu_{m}(G)\right) \leq$ $2 \Delta(G)+2=\Delta\left(\mu_{m}(G)\right)+2$. On the other hand, $\chi_{t}\left(\mu_{m}(G)\right) \geq \Delta\left(\mu_{m}(G)\right)+1$. Therefore, $\Delta\left(\mu_{m}(G)\right)+1 \leq \chi_{t}\left(\mu_{m}(G)\right) \leq$ $\Delta\left(\mu_{m}(G)\right)+2$ in this case. This completes the proof of the theorem.

By case 1 of Theorem 6, we can get the following corollary immediately.
Corollary 7. If $\Delta(G) \leq \frac{|V(G)|-1}{2}$, then $\mu_{m}(G)$ is Type 1 for any integer $m \geq 1$.
Moreover, we would like to show that the bounds in case 2 of Theorem 6 are reachable.
Lemma 8. For integer $m \geq 1$, if $m \equiv 0 \bmod 3$, then $\mu_{m}\left(K_{2}\right)$ is Type 1 . Otherwise, it is Type 2.
Proof. Since $\mu_{m}\left(K_{2}\right) \cong C_{2 m+3}$, , the lemma holds by $\chi_{t}\left(C_{2 r+1}\right)= \begin{cases}3, & \text { if } 2 r+1 \equiv 0 \bmod 3 ; \\ 4, & \text { otherwise } .\end{cases}$

To conclude, we also show that the Mycielski of complete graph is always Type 1.
Theorem 9. For any integer $n \geq 3, \mu\left(K_{n}\right)$ is Type 1.
Proof. It is known that $\chi_{t}\left(K_{n}\right)=n$ if $n$ is odd and $\chi_{t}\left(K_{n}\right)=n+1$ if $n$ is even [2].
Note that $\Delta\left(\mu\left(K_{n}\right)\right)=\max \{2(n-1), n\}=2(n-1)$. If $n$ is odd, since $\chi_{t}\left(K_{n}\right)=n$, let $f: V\left(K_{n}\right) \cup E\left(K_{n}\right) \rightarrow\{1,2, \ldots, n\}$ be a proper $n$-total coloring of $K_{n}$. It is obvious that every vertex of $K_{n}$ should be colored differently, without loss of generality, suppose $f\left(v_{i}^{0}\right)=i, i=1,2, \ldots, n$. Then we can color properly the edges of $G_{1}$ by $n-1$ colors $\{n+1, n+2, \ldots, 2 n-1\}$. Let $f\left(v_{i}^{1}\right)=i ; f\left(v_{i}^{1} u\right)=i+1$ for $1 \leq i \leq n-1$ and $f\left(v_{n}^{1} u\right)=1, f(u)=n+1$. This gives a proper $(2 n-1)$-total coloring of $\mu\left(K_{n}\right)$.

If $n$ is even, consider $K_{n}$ as a subgraph of $K_{n+1}$ where $w$ is the new vertex. Since $n+1$ is odd, let $f: V\left(K_{n+1}\right) \cup E\left(K_{n+1}\right) \rightarrow$ $\{1,2, \ldots, n+1\}$ be a proper total coloring of $K_{n+1}$. Let $M_{0}, M_{1}$ be two disjoint perfect matchings of $G_{1}$. Assume $v_{i}^{1} v_{j}^{0} \in M_{0}$, $v_{i}^{1} v_{l}^{0} \in M_{1}$, and let $f\left(v_{i}^{1} v_{j}^{0}\right)=f\left(w v_{j}^{0}\right)$ and $f\left(v_{i}^{1} u\right)=f\left(w v_{l}^{0}\right)$. Then we can color properly the edges of $G_{1} \backslash M_{0}$ by $n-2$ colors $\{n+2, n+3, \ldots, 2 n-1\}$. Let $f\left(v_{i}^{1}\right)=f(w), f(u)=n+2$. This gives a proper $(2 n-1)$-total coloring of $\mu\left(K_{n}\right)$.

## 3. Remark

Motivated by Case 1 of Theorem 6, we propose the following problem.
Problem. If a graph $G$ has only one vertex of maximum degree and there are at least $\Delta(G)+2$ vertices in $G$, then it satisfies TCC.

A proof of this special case of TCC would provide a strong support for TCC. Indeed if this special case is proven, then we can prove that for any graph $H$, the total chromatic number of $H$ is at most $\Delta(H)+4$. To see this we first add a new vertex $w$ and connect it with $\Delta(H)+2$ vertices of $H$. Denote the resulting graph by $H^{\prime}$. Thus $H^{\prime}$ is a graph with maximum degree $\Delta(H)+2$ and $w$ is the only vertex attaining the maximum degree. Hence, $\chi_{t}\left(H^{\prime}\right) \leq \Delta\left(H^{\prime}\right)+2=\Delta(H)+4$. Therefore, we can color the vertices and edges of $H$ properly with at most $\Delta(H)+4$ colors.

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    * Corresponding author.

    E-mail address: li@lri.fr (H. Li).

