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Effect of plastic volumetric strains on shakedown of hardening granular soils

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ABSTRACT

The granular soils are pressure dependent materials and substantial volume change occurs during loading. Some granular soils contract (volume decrease) when sheared and some dilate (volume increase), depending on their initial state. Owing to the plastic volume change that accompanied plastic deformation, the lack of normality rule arises and the granular soil exhibits a non-associated flow rule. As known the non normality has destabilizing effect on the soil behavior in the hardening regime (loss of positive definiteness of the second order work). The unstable behavior usually develops in association with dilation. A contracting soil displaying hardening and stable material behavior under drained conditions, may succumb to unstable flow type when its behavior becomes dilating. This paper deals with the extension of the static shakedown theorem to dilative granular soils in drained conditions, exhibiting pre-failure instability in the hardening regime, in the framework of non associated plasticity.

1. Introduction

The soils in the subgrade underneath railways, road pavements or offshore structures are subjected to a large number of loads, traffic or propagation of water waves, during the life span of their service. This type of loading is characterized by the number of repetitions being formidably large and therefore, even though the intensity is trivial, its accumulated effects could be of engineering significance. A number of models have been constructed specifically to describe the behavior of soils under cyclic loading. However the nature of these models requires that in actual application, for traffic or wave loading analysis, a stress-strain calculation performed for each individual cycle. This makes the procedure cumbersome and computationally expensive when a large numbers of cycles are involved.

Shakedown theory is an alternative for the study of the long-time behavior of soils, subjected to a set of loads fluctuating arbitrarily within given bounds. It gives a safe criterion against failure caused by the unlimited accumulation of plastic strains during loading, leading to either incremental collapse or alternating plasticity. If, on the contrary, plastic strains cease to develop further after some time and the soil responds purely elastically for subsequent load cycles, one says that the soil shakes down.

The static and kinematic shakedown theorems were derived by [1,2] for materials obeying to the concept of normality which have an associated flow rule. Since these theorems are extended to cover various class of material behaviors and applications [3–12]. Among them the soils and frictional materials which have non-associated flow rule [3–6]. In the same spirit numerical procedures in conjunction with finite element technique have been developed for the prediction of shakedown or lack of it [9–12].

The lack of normality owing the plastic volume change has a destabilizing effect on the soil behavior in the hardening regime, well before failure conditions are reached [13,21,22]. Since non-associativeness implies non-symmetry of elastoplastic stiffness matrix. In turn this implies that the loss of positive definiteness of elastoplastic stiffness matrix occurs when its determinant is still positive [14]. In other terms the loss of its positive definiteness occurs in the hardening regime and coincides with the loss of the positive definiteness of the second order work [14]. The unstable behavior of the granular soil in drained conditions develops usually in association with dilation (volume increase). A contracting sand displaying hardening and stable material behavior, may succumb to unstable flow type when its behavior becomes dilating [15,16]. Except if the hardening modulus $H > H_{cr}$, where H_{cr} being a threshold which marks the limits between stability and instability.

The hardening considered herein is assumed to be isotropic depending on the plastic volumetric strains. It is introduced solely to compensate for the destabilizing effect of the non normality.

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However, to cover both alternating plasticity and incremental collapse a limited kinematic hardening is more appropriate [8].

This paper deals with the extension of the static shakedown theorem to dilative granular soils in drained conditions, exhibiting pre-failure instability in the hardening regime, in the framework of non associated plasticity.

2. Basic equations

Let consider a solid occupying volume V bounded by surface S. The solid is subjected to surface tractions and body forces, which vary so slowly with time that inertia effects may be disregarded. The surface tractions T_i are prescribed on the part S_T of the surface S, while the displacement U_i are prescribed on the remainder part S_u of this surface. The body forces F_i are prescribed throughout V. The precise manner in which T_i and F_i vary with time need not to be known, but at any point of S_T a range of variation must be specified for each component of T_i, and at any point of V a range of variation must be assigned to each component of F_i.

Soils are compressible materials which display plastic volumetric strains under loading. To include this feature the plasticity criterion is that of Drucker-Prager with the yield function given by [20]

$$F(\sigma_{ij}, R) = -\alpha I_1 + \sqrt{J_2} - k(R) \le 0 \tag{1}$$

in which α is material parameter, I_1 and J_2 denote the first invariant of the stress tensor σ_{ij} and the second invariant of the deviatoric stress tensor S_{ij}. The positive increasing function $k(R)$ of the hardening variable R represents a measure of the current size of the yield surface. The changes of the yield surface ensue from the development of the plastic volumetric deformations. A suitable plastic potential can be derived from the Drucker-Prager criterion such that

$$g(\sigma_{ij}) = -\beta I_1 + \sqrt{J_2} \tag{2}$$

where β denotes the dilatancy factor. The plastic flow occurs with the plastic strain rate $\dot{\varepsilon}^p_{ij}$ directed along the gradient vector of the plastic potential $g(\sigma_{ij})$:

$$\dot{\varepsilon}^p_{ij} = \dot{\lambda}\frac{\partial g}{\partial \sigma_{ij}} = \dot{\lambda}\left[-\beta\,\delta_{ij} + \frac{S_{ij}}{2\sqrt{J_2}}\right] \tag{3}$$

while $\dot{\lambda}$ is a positive scalar which is non zero only when plastic deformation occur, and δ_{ij} is the Kronecker symbol. If the function F and g coincide, the flow rule (3) is called associated.

For an elastoplastic isotropic strain hardening material with non associated flow rule, the evolution law of the hardening variable R (an increasing function of time) is given by

$$\dot{R} = \left[\frac{2}{3}\dot{\varepsilon}^p_{ij}\dot{\varepsilon}^p_{ij}\right]^{1/2} = \dot{\lambda}\left[2\,\beta^2 + \frac{1}{3}\right]^{1/2} = \Re\,\dot{\lambda} \tag{4}$$

During plastic flow, the stress state must remain on the yield surface, and the consistency condition

$$\frac{\partial F}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial F}{\partial R}\dot{R} = \frac{\partial F}{\partial \sigma_{ij}}\dot{\sigma}_{ij} - H\cdot\Re\cdot\dot{\lambda} = 0 \tag{5}$$

must be satisfied. The yield surface evolves as the plastic flow continues. This is reflected by the term $-H\,\Re\,\dot{\lambda}$ in Eq. (5). Note that H is the strain hardening modulus which is positive, zero, or negative for strain hardening, perfect, and strain softening plasticity, respectively.

By substituting Eqs. (3), (4) and the elastic stress-strain law whose terms are defined below by Eqs. (13), (20) and (24)

$$\dot{\sigma}_{ij} = \dot{\sigma}^e_{ij} + \dot{\rho}_{ij} = E^e_{ijkl}\left(\dot{\varepsilon}^e_{kl} + \dot{\varepsilon}^{er}_{kl} = \dot{\varepsilon}_{kl} - \dot{\varepsilon}^p_{kl}\right) \tag{6}$$

into the consistency condition Eq. (5) one gets the value of the plastic multiplier $\dot{\lambda}$. Replacing this value of $\dot{\lambda}$ into Eqs. (3), (6) and using again Eq. (5), one obtains the elastoplastic stiffness tensor [23]

$$E^{ep}_{ijkl} = E^e_{ijkl} - \frac{E^e_{ijmn}(\partial F/\partial\sigma_{mn})(\partial g/\partial\sigma_{rs})E^e_{rskl}}{H\,\Re + (\partial F/\partial\sigma_{mn})\,E^e_{mnrs}(\partial g/\partial\sigma_{rs})} \tag{7}$$

Herein E^e_{ijkl} denotes the isotropic linear elastic stiffness tensor given by

$$E^e_{ijkl} = \left(K - \frac{2}{3}G\right)\delta_{ij}\delta_{kl} + G\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \tag{8}$$

where K is the Bulk modulus and G the shear modulus for the elastic material.

The gradient of the yield function is obtained simply by replacing the dilatancy coefficient β by the friction coefficient α in Eq. (3). The value of $\Re$ is given by Eq. (4). Because $\delta_{ij}\ \delta_{ij} = 3$, $S_{ij}\ S_{ij} = 2\ J_2$ and $\delta_{ij}\ S_{ij} = 0$. Substituting into the general formula (7) and using the relation

$$E^e_{ijkl}\ \delta_{ij} = 3K\ \delta_{ij} \tag{9}$$

$$E^e_{ijkl}\ S_{ij} = 2\ G\ S_{ij} \tag{10}$$

one obtains the explicit expression

$$E^{ep}_{ijkl} = E^{ep}_{ijkl} - \frac{9K^2\alpha\beta\,\delta_{ij}\otimes\delta_{kl} + 3KG/\sqrt{J_2}\left[\alpha\ S_{ij}\otimes\delta_{kl} + \beta\,\delta_{ij}\otimes S_{kl}\right] + \left[G^2/J_2\right]\ S_{ij}\otimes S_{kl}}{9K\alpha\beta + G + H\left[2\beta^2 + \frac{1}{3}\right]^{1/2}} \tag{11}$$

The elastoplastic compliance tensor C^{ep}_{ijkl}, the inverse of E^{ep}_{ijkl}, is given by

$$C^{ep}_{ijkl} = \left[1 - \frac{9K\alpha\beta + G}{H\left[2\beta^2 + \frac{1}{3}\right]^{1/2}}\right]C^e_{ijkl} \tag{12}$$

where C^{ep}_{ijkl} is the isotropic linear elastic compliance tensor.

In an elastic-plastic body under given loads T_i, F_i, the stress field σ_{ij} can be written as

$$\sigma_{ij} = \sigma^e_{ij} + \rho_{ij} \tag{13}$$

where σ^e_{ij} is the elastic stress field corresponding to the given loads, and ρ_{ij} is the residual stress field. Such a field is in equilibrium with zero body forces and zero prescribed surface tractions

$$\rho_{ij,j} = 0 \quad \text{in V} \tag{14}$$

$$\rho_{ij}\ n_j = 0 \quad \text{on } S_T \tag{15}$$

where n is the unit vector along the outer normal of S, a comma denotes partial derivative with respect to the space variable x_i. In turn the elastic stress field must satisfy the following equations

$$\sigma^e_{ij,j} + F_i = 0 \quad \text{in } V \tag{16}$$

$$\sigma^e_{ij}\ n_j = T_i \quad \text{on } S_T \tag{17}$$

Accordingly the strain field can be written as

$$\varepsilon_{ij} = \varepsilon^e_{ij} + \varepsilon^r_{ij} \tag{18}$$

$$\varepsilon^r_{ij} = \varepsilon^{er}_{ij} + \varepsilon^p_{ij} \tag{19}$$

The term ε^e_{ij} represents the elastic strain field in the hypothetical elastic solid under the prescribed loads, whereas ε^{er}_{ij} denotes the residual elastic strain field generated by the non compatible plastic strain distribution ε^p_{ij}.

The total strain field ε_{ij} is compatible with the displacement filed $u_i = u^e_i + u^r_i$. Thus the elastic strain field ε^e_{ij} is derivable from the elastic displacement field u^e_i,

$$\varepsilon^e_{ij} = (1/2)\left(u^e_{i,j} + u^e_{j,i}\right) \quad \text{in } V \tag{20}$$

$$\varepsilon_{ij}^{e} = C_{ijkl}^{e}\,\sigma_{kl}^{e} \tag{21}$$

$$u_i^e = U_i \quad \text{on } S_u \tag{22}$$

and the residual strain field ε_{ij}^{r} is derivable from the residual displacement field u_i^r,

$$\varepsilon_{ij}^{r} = (1/2)\left(u_{i,j}^{r} + u_{j,i}^{r}\right) \quad \text{in } V \tag{23}$$

$$\varepsilon_{ij}^{er} = C_{ijkl}^{e}\,\rho_{kl} \tag{24}$$

$$u_i^r = 0 \quad \text{on } S_u \tag{25}$$

3. Material stability

According to Hill's stability criterion [17], a sufficient condition for stability is the positive definiteness of the second order work:

$$2\ d^2W = \dot{\sigma}_{ij}\ \dot{\varepsilon}_{ij} = \dot{\sigma}_{ij}\ \dot{\varepsilon}_{ij}^{e} + \dot{\sigma}_{ij}\ \dot{\varepsilon}_{ij}^{p} > 0,\ \forall \dot{\varepsilon}_{ij} \tag{26}$$

where $\dot{\sigma}_{ij}$ represents the stress rate tensor and $\dot{\varepsilon}_{ij}$ the corresponding strain rate tensor. On the basis of this stability criterion it was shown in [13] that a material obeying to non associated flow is stable if and only if $H \geq H_{cr}$, $H_{cr} \geq 0$ represents a critical value of the hardening modulus defined as [13]:

$$H_{cr} = \frac{1}{2}\left[\frac{\partial g}{\partial \sigma_{ij}} E_{ijkl}^{e} \frac{\partial g}{\partial \sigma_{kl}}\right]^{1/2}\left[\frac{\partial F}{\partial \sigma_{rs}} E_{rsmn}^{e} \frac{\partial F}{\partial \sigma_{mn}}\right]^{1/2} - \frac{1}{2}\left[\frac{\partial g}{\partial \sigma_{ij}} E_{ijkl}^{e} \frac{\partial F}{\partial \sigma_{kl}}\right] > 0 \tag{27}$$

Due to the positive definiteness of E_{ijkl}^{e}, one has $\dot{\sigma}_{ij}\ \dot{\varepsilon}_{ij}^{e} > 0$. It follows that $\dot{\sigma}_{ij}\ \dot{\varepsilon}_{ij}^{p} > 0$ for $H > H_{cr}$, referred as the stability in the small [18]. In the large it is equivalent to the postulate of maximum plastic dissipation [19]

$$\left(\sigma_{ij} - \bar{\sigma}_{ij}\right)\ \dot{\varepsilon}_{ij}^{p}\ >\ 0 \tag{28}$$

for any stress state σ_{ij} generating a nonzero plastic strain rate $\dot{\varepsilon}_{ij}^{p}$, and for any stress state $\bar{\sigma}_{ij}$ such that $F(\bar{\sigma}_{ij}, R) < 0$. Note that the stress state $\bar{\sigma}_{ij}$ may be inside or outside the convex plastic potential surface $g(\sigma_{ij}) = 0$ which is strictly included inside the yield surface $F(\sigma_{ij}, R) = 0$. Since the only requirement to be met for the fulfillment of the stability condition (28) is that $H > H_{cr}$.

By substituting Eqs. (1), (2) and (8) into Eq. (27), one gets the explicit expression of the critical hardening modulus

$$H_{cr} = \frac{1}{2}\left[9K\alpha^2 + G\right]^{1/2}\left[9K\beta^2 + G\right]^{1/2} - \frac{1}{2}\left[9K\alpha\beta + G\right] > 0 \tag{29}$$

For $\alpha = \beta$ one has an associated flow, for $\beta < 0$ a non associated contractive flow, and for $\beta > 0$ a non associated dilative flow. The inspection of Eq. (29) reveals that the granular soil is completely stable ($H > H_{cr}$) when its behavior is contractive ($\beta < 0$). Since the critical value H_{cr} of the hardening modulus corresponds to dilative behavior ($\beta > 0$). It is worth noting that the plastic volumetric strain rate $\dot{\varepsilon}_{v}^{p}$ ($\dot{\varepsilon}_{v}^{p} = -\ 3\ \dot{\lambda}\ \beta$) is negative for dilation (volume increase) and is positive for contraction (volume decrease), following the soil mechanics sign convention.

On other hand non-associativeness implies non-symmetry of the stiffness and the compliance matrices. The loss of the positive definiteness of matrix E^{eP} (and C^{eP}) occurs when its determinant is still positive [14]. In other terms the loss of its positive definiteness occurs in the hardening regime and coincides with the loss of the positive definiteness of the second order work. At the onset of loss of positive definiteness one has [14]

$$Det\ E^{eP} \geq Det\ E_s^{eP} = Det\ C_s^{eP} = 0 \tag{30}$$

where E_s^{eP} and C_s^{eP} are the symmetric part of E^{eP} and C^{eP} respectively. Alternatively the positive definiteness of the second order work $2\ d^2W = \dot{\tilde{\varepsilon}}E^{eP}\dot{\varepsilon} = \dot{\tilde{\varepsilon}}E_s^{eP}\dot{\varepsilon} > 0$ entails the positive definiteness of the stiffness matrix E^{eP}. The tilde is used to indicate transpose in matrix notation. The compliance matrix C^{eP} is also positive definite. This implies that the value in the square bracket of Eq. (12) is positive regardless of the type of the plastic volumetric strains displayed by the granular soil which can be either contractive or dilative. This result is essential for the proof of the extended shakedown theorem.

4. The extended shakedown theorem

Shakedown will occur if any time-independent distribution of residual stress $\bar{\rho}_{ij}$ and the hardening variable $\bar{R}$ can be found so that the sum of these residual stresses and the elastic stresses σ_{ij}^{e} (stress based on purely elastic behavior) is a safe state of stress, i.e.

$$F\left(\bar{\sigma}_{ij} = \sigma_{ij}^{e} + \bar{\rho}_{ij}, \bar{R}\right) < 0 \tag{31}$$

for all possible load combination within the prescribed range of the loads.

To prove the theorem, let us consider the non negative quantity

$$\mathrm{W(t)} = \frac{1}{2}\int_{V} C_{ijkl}^{ep}\left(\rho_{ij} - \bar{\rho}_{ij}\right)\left(\rho_{kl} - \bar{\rho}_{kl}\right)\mathrm{dV} + \int_{V} Q\left(R, \bar{R}\right)\mathrm{dV} \tag{32}$$

where ρ_{ij} denotes the actual time dependent residual stresses, and $Q(R, \bar{R})$ represents a positive scalar function defined as

$$Q\left(R,\ \bar{R}\right) = \int_{\bar{R}}^{R} M\left(\vartheta,\ \bar{R}\right)\, d\vartheta \tag{33}$$

The time derivative of $Q(R, \bar{R})$ is equal to $M(R, \bar{R})\ \dot{R}$. In turn the time-derivative of Eq. (32) gives

$$\dot{W}(t) = \int_{V} C_{ijkl}^{ep}\ \left(\rho_{ij} - \bar{\rho}_{ij}\right)\dot{\rho}_{kl}\ dV + \int_{V} M\ \left(R,\ \bar{R}\right)\ \dot{R}\ dV \tag{34}$$

which making use of Eqs. (12) and (24) becomes

$$\dot{W}(t) = \left[1 - \frac{9K\alpha\beta + G}{H\left[2\beta^2 + \frac{1}{3}\right]^{1/2}}\right]\int_{V}\left(\rho_{ij} - \bar{\rho}_{ij}\right)\dot{\varepsilon}_{ij}^{er}\ dV + \int_{V} M\left(R, \bar{R}\right)\dot{R}\ dV \tag{35}$$

The distribution of residual stresses $(\rho_{ij} - \bar{\rho}_{ij})$ is self-equilibrating, and the residual strain rate field ($\dot{\varepsilon}_{ij}^{r} = \dot{\varepsilon}_{ij}^{er} + \dot{\varepsilon}_{ij}^{p}$) forms a compatible strain field. Hence Eq. (35) may be simplified by means of the virtual work equation into

$$\dot{\mathrm{W}}(t) = -\left[1 - \frac{9K\alpha\beta + G}{H\left[2\beta^2 + \frac{1}{3}\right]^{1/2}}\right]\int_{V}\left(\rho_{ij} - \bar{\rho}_{ij}\right)\dot{\varepsilon}_{ij}^{p}\ \mathrm{dV} + \int_{V} M\left(R, \bar{R}\right)\dot{R}\ \mathrm{dV} \tag{36}$$

From the definition above $\sigma_{ij} = \sigma_{ij}^{e} + \rho_{ij}$ and $\bar{\sigma}_{ij} = \sigma_{ij}^{e} + \bar{\rho}_{ij}$, Eq. (36) can be rewritten as

$$\dot{W}(t) = -\left[1 - \frac{9K\alpha\beta + G}{H\left[2\beta^2 + \frac{1}{3}\right]^{1/2}}\right]\int_{v}\left(\sigma_{ij} - \bar{\sigma}_{ij}\right)\dot{\varepsilon}_{ij}^{p}\ dV + \int_{v} M\left(R, \bar{R}\right)\ \dot{R}\, dV \tag{37}$$

The function $M(R, \bar{R}) = k(R) - k(\bar{R}) < 0$ since it has the sign of $(R - \bar{R})$. Therefore $Q(R, \bar{R})$ is a positive decreasing function of R

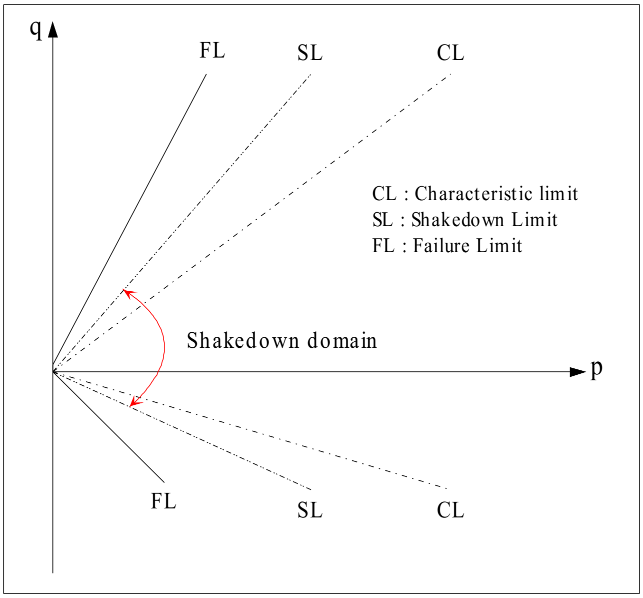

Fig. 1. Shakedown criterion.

(positive for the value of R distinct of $\bar{R}$). By considering the above definition of $M(R, \bar{R})$, Eq. (37) becomes

$$\dot{W}(t) = -\left[1 - \frac{9K\alpha\beta + G}{H\left[2\beta^2 + \frac{1}{3}\right]^{1/2}}\right] \int_v \left(\sigma_{ij} - \bar{\sigma}_{ij}\right) \dot{\varepsilon}_{ij}^p \, dV + \left[2\beta^2 + \frac{1}{3}\right] \int_v \left[k(R) - k(\bar{R})\right] \dot{\lambda} \, dV \tag{38}$$

In drained conditions, the onset of instability is flagged by the transition of the volumetric strain from contraction to dilation [15]. However, if under repeated cyclic loads the behavior of the granular soil still always contractive ($\beta < 0$), the hardening modulus H still always greater than H_{cr} and the material is stable. This implies that $\dot{W}(t) \leq 0$, where the equality holds only in the absence of plastic flow ($\dot{\varepsilon}_{ij}^p = 0$). As $W(t) > 0$ by definition, the condition $\dot{W}(t) = 0$ must eventually be reached, and this condition corresponds to shakedown. Alternatively if the granular soil exhibits a dilative behavior ($\beta > 0$), shakedown will occur only and only if the hardening modulus $H > H_{cr}$. Otherwise the granular soil succumbs to unstable flow type. In other terms dilative volumetric strains displayed by the granular soils have destabilizing effect. The condition of stability (28) can be violated even in the hardening regime.

Therefore the stress domain is separated into two perfectly distinct regions:

- The shakedown domain characterized by $H > H_{cr}$, inside which one have stabilization of the plastic strains. This domain is delimited by the shakedown lines (SL) in compression and in tension as shown in Fig. 1. These lines are located below the failure lines (FL) and above the characteristic lines (CL), which mark the boundary between the contractive behavior and the dilative behavior.
- The non-shakedown domain over the range of positive hardening moduli $0 < H < H_{cr}$. This domain is located between the shakedown lines (SL) and the failure lines (FL).
- For an associated flow rule ($\alpha = \beta$) the limit of the hardening modulus degenerates to $H_{cr} = 0$. The unstable behavior occurs in the post peak softening regime ($H < 0$). Unlike non associated flow rule, dilation has no effect upon the shakedown process in the hardening regime.
- For a granular soil which exhibits a perfectly plastic behavior ($H = 0$), shakedown will not occur. Since the lack of normality generates material instability.

5. Conclusions

In this paper an extension of the static shakedown theorem is proposed to granular soils in drained conditions, exhibiting pre-failure instability in the hardening regime, in the frame work of non associated plasticity. It is shown that shakedown is predicted to always occur if the flow is non associated contractive. However if the flow is non associated dilative shakedown will occur if the hardening can compensate for the destabilizing effect of dilation i.e. if $H > H_{cr}$. Note that $H_{cr} \geq 0$ being a threshold of the hardening modulus which marks the limits between stability and instability. This peculiarity of the non associated flow law of elastoplasticity disappears an associated flow law in which the critical hardening modulus reduces to $H_{cr} = 0$.

Declaration of Competing Interest

None.

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