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Efficient calculation of the response statistics of two-dimensional fractional diffusive systems

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ABSTRACT

Various natural phenomena are described by anomalous diffusion processes. Notable examples relate to the study of diffusion of tracers in particles in turbulent flows, or in the propagation of acoustic waves. In this context, the governing equations involve a fractional Laplacian operator, which replaces the classical Laplacian, and may involve nonlinear terms. This leads to problems described by fractional nonlinear diffusion equations. In general, no analytical solutions are available for determining the response of these systems. Thus, the development of approximate approaches circumventing the use of computationally demanding numerical techniques is desirable. This paper proposes a statistical linearization based approach, which allows calculating approximately, albeit iteratively, the response statistics. The method is developed using a recently proposed representation of the fractional Laplacian in conjunction with a mode expansion of the system response. It is implemented by introducing non-orthogonal eigenfunctions of the fractional Laplacian of the response, which are obtained from the linear modes of the classical diffusion equation. Such a representation allows deriving a system of nonlinear ordinary differential equations, which is linearized in a stochastic mean square sense. Then, the response statistics and power spectral density are determined by an iterative procedure. Numerical results pertaining to a system with white noise excitation demonstrate the efficiency of the proposed approximate approach. Further, comparisons with data from relevant Monte Carlo results assess the reliability of the estimated response.

1. Introduction

Anomalous diffusion is a phenomenon observed in several theoretical and experimental studies. Examples can be found in conduction [1–3], or in the propagation of waves in scattered media [4–6] where, for instance, it has been observed in the propagation of acoustic fields [7–10]. Most experimental evidence of anomalous diffusion has been reported in case of random or disordered media. For example, Barthelemy et al. [11] showed that light waves perform Lévy flights in adequately designed optical materials; Asatryan et al. [12] reported an example of anomalous diffusion of electromagnetic waves in disordered two-dimensional photonic crystals; and Burresi et al. [13] reported experimental observations of interference effects in transport based on Lévy statistics in super-diffusive materials. Nevertheless, recently, Buonocore et al. [14] demonstrated, theoretically and numerically, that anomalous diffusion occurs also in periodic media with no disorder. Their analysis pertains to acoustic wave fields, but the results can be generalized in other domains.

From a modelling perspective, the critical feature of this phenomenon is that the associated field variables have a power-law distribution. This fact leads to models generalizing the classical diffusion equation by a fractional diffusion equation. That is, equations involving the use of a fractional Laplacian operator generalizing the classical Laplace operator. Notable applications of this concept were proposed by Benson et al. [15,16] and Mainardi [17].

The fractional Laplacian operator is an integro-differential operator used in nonlocal models, such as nonlocal wave equations [18] and phase transitions [19], as well as for modelling the anomalous diffusion, as mentioned previously. Since the hyper-singular integral definition [20] of the fractional Laplacian was introduced, a number of representations were proposed [21] which share the common feature of generalizing the classical Laplace operator. However, limited results are available relating to solutions of the equations involving such an operator. In this regard, Huang and Oberman [22] derived a finite difference method for solving equations involving the fractional Laplacian in one dimension. Varlamov [23] investigated the existence and uniqueness of a global-in-time solution in a nonlinear heat equation.

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Chen and Pang [21] introduced an implicit definition of the fractional Laplacian, and applied the singular boundary method to a fractional Laplacian equation.

The problem of estimating the response of systems governed by equations involving a fractional Laplacian is quite challenging especially when the governing equations include also nonlinear terms. Indeed, in these context, no analytical solutions are available. Thus, it is desirable to develop approximate analytical solutions which may be used as an alternative to time consuming numerical methods. The open literature addresses the use of approximate approaches for systems endowed with fractional elements, especially in situations where the fractional derivative operates in the time domain. For instance, approximate response statistics were derived by Spanos and Evangelatos [24] by statistical linearization for nonlinear oscillators with fractional derivative elements. In this context, other techniques were proposed by Huang and Jin [25], by stochastic averaging, Cottone et al. [26], by the Fokker–Plank–Kolmogorov equation, and by Di Matteo et al. [27] by Wiener Path Integrals. Further contributions on this subject were made by Kougioumtzoglou and Spanos [28] who proposed a harmonic wavelets based approximate analytical technique for calculating the response of both linear and non-linear time-variant oscillators; and by Spanos et al. [29], who used a Hilbert transform based stochastic averaging technique. Approximate approaches were used for estimating the response of nonlinear continua, as well. In this context, Li [30] and Li et al. [31] investigated the behaviour of Timoshenko beams with fractional derivative constitutive equation by the Galerkin method. The response statistics of nonlinear beams and plates were determined approximately by statistical linearization by Spanos and Malara [32] and Malara and Spanos [33], respectively. Analytical solutions for the vibration problem of a linear beam involving fractional derivatives were derived by Agrawal [34], by a Laplace transform technique; by Di Lorenzo et al. [35] in the frequency domain; and by Liaskos et al. [36] in an implicit form.

This paper considers the problem of determining the response statistics of a system governed by a nonlinear fractional diffusion equation. This subject was analysed by Jiao [37], that developed a numerical approach for estimating the system response in the time domain. Herein, an approximate approach based on the statistical linearization technique is developed. The approach is implemented by introducing non-orthogonal eigenfunctions of the fractional Laplacian of the response, which are derived from the linear modes of a classical linear diffusion equation. In this manner, the response can be represented by a modal expansion, which is used for deriving a system of nonlinear ordinary differential equations determining the mode amplitudes. It is shown that the approach can be implemented in an iterative manner, and comparisons with relevant Monte Carlo data assess the reliability of the method.

2. Preliminary concepts on fractional Laplacian

This section establishes the fundamental concepts leading to the formulation of a fractional Laplacian operator, and describes general results that will be applied in the development of the statistical linearization based approach. A detailed mathematical background is provided in the monograph by Samko et al. [20].

A natural framework for the definition of the fractional Laplacian is the Fourier transform theory. In this context, denoting by $(-\Delta)^{\alpha/2}u(\boldsymbol{x})$ the fractional Laplacian of order α of a scalar function $u(\boldsymbol{x})$, with the vector $\boldsymbol{x}\in\mathbb{R}^d$, this operator is defined by the equation,

$$(-\Delta)^{\alpha/2}\,u(\boldsymbol{x})=\mathcal{F}^{-1}\,|\boldsymbol{x}|^{\alpha}\,\mathcal{F}u(\boldsymbol{x}), \tag{1}$$

Where the vector $\boldsymbol{\omega}\in\mathbb{R}^d$ is the Fourier variable vector; the Fourier transform is given by the equation

$$\mathcal{F}\{u(\boldsymbol{x})\}=\int_{\mathbb{R}^d}u(\boldsymbol{x})e^{i\boldsymbol{\omega}\cdot\boldsymbol{x}}d\boldsymbol{x}, \tag{2}$$

and $\mathcal{F}^{-1}\{\cdot\}$ renders the inverse Fourier transform by the equation

$$\mathcal{F}^{-1}\{g(\boldsymbol{\omega})\}=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}g(\boldsymbol{\omega})e^{-i\boldsymbol{\omega}\cdot\boldsymbol{x}}d\boldsymbol{\omega}. \tag{3}$$

While constructing such an operator, it is seen that the negative powers lead to the so-called Riesz potentials,

$$I^{\alpha}u(\boldsymbol{x})=\frac{1}{\gamma_d(\alpha)}\int_{\mathbb{R}^d}\frac{u(\boldsymbol{y})\,d\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^{d-\alpha}},\alpha\neq d,d+2,d+4,\ldots \tag{4}$$

in which $\gamma_d(\alpha)$ is a normalizing constant. Instead, the positive powers lead to convolution integrals having an order of singularity higher than the dimension of the space $\mathbb{R}^d$. That is, they are hyper-singular integrals. In this context, to guarantee the convergence of the convolution, this operation is introduced by utilizing finite differences. Thus, if finite differences of the function $u(\boldsymbol{x})$ are denoted by the symbol $\Delta_h u(\boldsymbol{x})$, the operation $(-\Delta)^{\alpha/2}$, for $\alpha>0$, is given by the equation

$$D^{\alpha}u(\boldsymbol{x})=\frac{1}{d_n(\alpha)}\int_{\mathbb{R}^d}\frac{\Delta_y u(\boldsymbol{x})}{|\boldsymbol{y}|^{d+\alpha}}d\boldsymbol{y}, \tag{5}$$

where $d_n(\alpha)$ is a normalizing constant. Eq. (5) is indeed the Riesz derivative, however, it is necessary to select an appropriate value of the normalizing constant for making this equation consistent with the fundamental relation (1). Thus, the operation leading to a generalization of the Laplace operator is

$$(-\Delta)^{\frac{\alpha}{2}}\,u(\boldsymbol{x})=\mathcal{F}^{-1}\,|\boldsymbol{x}|^{\alpha}\,\mathcal{F}u(\boldsymbol{x})=\begin{cases}I^{-\alpha}u(\boldsymbol{x}),\alpha<0\\ D^{\alpha}u(\boldsymbol{x}),\alpha>0.\end{cases} \tag{6}$$

Various other representations of the fractional Laplacian have been proposed in the open literature. The representation adopted in the following section was described by Chen and Holm [38]. Specifically, they proposed the Caputo-type representation of the fractional Laplacian of a bounded function $u(\mathbf{x})$,

$$(-\Delta)^{\frac{\alpha}{2}}\,u(\boldsymbol{x})=I_d^{2-\alpha}\left[-\Delta u(\boldsymbol{x})\right],\ \text{for}\ 1<\alpha<2, \tag{7}$$

where $\Delta u(x)$ is the classical Laplacian operator. In this case, the Riesz potential on a bounded convex domain Ω in $\mathbb{R}^d$ is defined as

$$I_d^{2-\alpha}\varphi(\boldsymbol{x})=c(\alpha)\int_{\Omega}\frac{\varphi(\boldsymbol{\xi})}{|\boldsymbol{x}-\boldsymbol{\xi}|^{d+\alpha-2}}d\boldsymbol{\xi}, \tag{8}$$

where $\boldsymbol{\xi}$ is the coordinate vector in the domain Ω and

$$c(\alpha)=\frac{\Gamma[\frac{d-2+\alpha}{2}]}{\pi^{d/2}2^{2-\alpha}\Gamma[\frac{2-\alpha}{2}]}, \tag{9}$$

with Γ being the Gamma function [39]. Since such an operator naturally includes the boundary conditions and reduces the singularity of the integrand, it is commonly used in bounded systems. It can be proved that, under sufficiently mild conditions on the function $u(\mathbf{x})$, the limit of such an expression [Eq. (7)], when the fractional order tends to 2, is the classical Laplace operator. That is,

$$\lim_{\alpha\to 2^-}(-\Delta)^{\frac{\alpha}{2}}\,u(\boldsymbol{x})=-\Delta u(\boldsymbol{x}). \tag{10}$$

Even in this case, the representation is consistent with Eq. (1). That is,

$$\mathcal{F}\{I_d^{2-\alpha}[-\Delta u(\boldsymbol{x})]\}=|\boldsymbol{\omega}|^{\alpha-2}\,\mathcal{F}\{-\Delta u(\boldsymbol{x})\}=|\boldsymbol{\omega}|^{\alpha}\,\mathcal{F}\{u(\boldsymbol{x})\}. \tag{11}$$

3. Statistical linearization method

In this section the development of the statistical linearization approach is pursued by considering a system governed by a nonlinear fractional partial differential equation on a rectangular domain [37]. In the following, the rectangular domain is defined over the region

$$\Omega=\{(x,y):\ -a\le x\le a,-b\le y\le b\}, \tag{12}$$

its boundary is denoted as $\partial\Omega$, and the problem is two-dimensional ($d=2$). Specifically, the governing equation is

$$\dot{u}(x,y,t)+(-\Delta)^{\frac{\alpha}{2}}\,u(x,y,t)+ku^3(x,y,t)=q(x,y,t),1<\alpha<2, \tag{13}$$

in which t denotes the time variable, the over-dot denotes differentiation with respect to time, and k is a nonlinear parameter accounting for the relevance of the nonlinear term. This equation describes the classical diffusion when $\alpha = 2$, while the case $1 < \alpha < 2$ corresponds to the anomalous diffusion. A known physical mechanism described by this equation is the heat propagation, and for this reason it is also known as fractional heat equation [40]. The source term $q = q(x, y, t)$ is assumed of a separable form. Thus,

$$q(x, y, t) = p(x, y) f(t), \tag{14}$$

where $p(x, y)$ is a deterministic function and $f(t)$ is a zero mean stationary Gaussian random process with given power spectral density function. The boundary condition associated with Eq. (13) is

$$\beta_1 u + \beta_2 \frac{\partial u}{\partial n} = \beta_3, \text{ on } \partial\Omega, \tag{15}$$

and the initial condition is

$$u(x, y, 0) = u_0(x, y), \tag{16}$$

where $\beta_1, \beta_2, \beta_3$ are known functions defined on the boundary, and $u_0(x, y)$ is a known space dependent function.

To solve Eq. (13) approximately, the system response is represented by the series expansion

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{mn}(t) v_{mn}(x, y), \tag{17}$$

in which $v_{mn}(x, y)$ are the eigenfunctions of the linear diffusion equation associated with the boundary condition (15), and $w_{mn}(t)$ are time-dependent amplitudes. The eigenfunctions are selected by utilizing the equation

$$-\Delta v_{mn}(x, y) = \lambda_{mn} v_{mn}(x, y), \tag{18}$$

where λ_{mn} are constants, and the orthogonality condition holds. That is,

$$\int_{-b}^{b}\int_{-a}^{a} v_{sl} v_{mn} dx dy = C_{mn}\delta_{ms}\delta_{nl}, \tag{19}$$

with δ_{ms} being the Kronecker delta, and C_{mn} being a constant. Similar expansions have been used for solving other problems involving fractional operators, such as beam and plate vibration problems [32,33], but in those problems the fractional operator is operating in the time domain. Instead, the fractional operator in Eq. (13) is defined in the space domain. This fact poses the problem of estimating appropriately the fractional Laplacian of the response represented by Eq. (17). For this purpose and to adopt the series expansion (17), the Riesz potential associated with the eigenfunctions v_{mn}, denoted as $z_{mn}(x, y)$, is derived. Specifically, if $P = (x_P, y_P)$ and $Q = (x_Q, y_Q)$ are two points in the domain Ω, the Riez potential is given by the equation

$$z_{mn}(P) = I_d^{2-\alpha}(v_{mn}) = c(\alpha)\int_{\Omega} \frac{v_{mn}(Q)}{|P - Q|^{\alpha}} d\Omega(Q), \tag{20}$$

where the distance between the points is given by the equation

$$|P - Q| = \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2}. \tag{21}$$

In this manner, the fractional Laplacian of the eigenfunctions is derived from Eq. (18). That is,

$$(-\Delta)^{\frac{\alpha}{2}} v_{mn}(x, y) = \lambda_{mn} z_{mn}(x, y). \tag{22}$$

Thus, the fractional Laplacian of the system response is given by the equation

$$(-\Delta)^{\alpha/2} u = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \lambda_{mn} w_{mn}(t) z_{mn}(x, y). \tag{23}$$

Eq. (23) is used in conjunction with Eq. (17) into Eq. (13). Then, the resulting residual is projected on the space of modes v_{mn}. By exploiting the orthogonality condition (19), the following system of nonlinear ordinary differential equations governing the evolution of the mode amplitudes is obtained:

$$C_{mn}\dot{w}_{mn} + \sum_{s=1}^{\infty}\sum_{l=1}^{\infty} \beta_{mn,sl} w_{sl} + k\int_{\Omega} v_{mn}\left(\sum_{s=1}^{\infty}\sum_{l=1}^{\infty} w_{sl} v_{sl}\right)^3 d\Omega = P_{mn} f(t),$$
$$\text{for } m, n = 1, 2, \ldots, \infty. \tag{24}$$

The constants in Eq. (24) are given by the equations,

$$\beta_{mn,sl} = \lambda_{sl}\int_{\Omega} v_{mn} z_{sl} d\Omega, \tag{25}$$

and

$$P_{mn} = \int_{\Omega} v_{mn}(x, y) p(x, y) d\Omega. \tag{26}$$

The formulation captured in Eq. (24) points out the critical difference of the case considered herein from the classical heat equation. Indeed, in the classical heat equation the orthogonality condition requires determining only the natural frequencies, $\beta_{mn,mn}$. Instead, herein the influence of the fractional Laplacian relates to the presence of off-diagonal elements $\beta_{mn,sl}$ associated with the linear terms w_{mn}. This relates to the fact that the functions $\{z_{mn}\}$ are not orthogonal, in general.

Eq. (24) constitute a system of nonlinear ordinary stochastic differential equations, which can be recast in a matrix form as

$$\mathbf{C}\dot{\mathbf{w}} + \mathbf{K}\mathbf{w} + \mathbf{g}(\mathbf{w}) = \mathbf{q}, \tag{27}$$

where $\mathbf{w}$ is the vector of the mode amplitudes; $\mathbf{C}$ is a diagonal matrix with elements C_{mn}; and $\mathbf{K}$ is a fully populated matrix with elements

$$K_{ij} = \beta_{mn,sl}, \text{ for } i = (m, n) \text{ and } j = (s, l). \tag{28}$$

Further, the load vector has elements

$$q_i = P_{mn} f(t), \tag{29}$$

and $\mathbf{g}(\mathbf{w})$ is a vector function capturing all of the associated nonlinearities.

Next, an approximate solution of Eq. (27) is sought by the statistical linearization scheme. Specifically, the original nonlinear system of equations is replaced by the surrogate linear system

$$\mathbf{C}\dot{\mathbf{w}} + \left(\mathbf{K} + \mathbf{K}_{\mathbf{eq}}\right)\mathbf{w} = \mathbf{q}. \tag{30}$$

In this equation the matrix $\mathbf{K}_{\mathbf{eq}}$ is an equivalent coefficient matrix, whose elements are determined by minimizing the error $\boldsymbol{\varepsilon}$ between the linear and the nonlinear system in a mean square sense. Specifically, seek to satisfy the criterion

$$\left\langle \boldsymbol{\varepsilon}^T \boldsymbol{\varepsilon} \right\rangle = \text{minimum}, \tag{31}$$

where $\langle\cdot\rangle$ denotes the averaging operator and the error $\boldsymbol{\varepsilon}$ defined

$$\boldsymbol{\varepsilon} = \boldsymbol{g}(\boldsymbol{w}) - \boldsymbol{K}_{\boldsymbol{eq}}\boldsymbol{w}. \tag{32}$$

The necessary conditions for solving this minimization problem are

$$\frac{\partial}{\partial \boldsymbol{K}_{ij}^e}\left\langle \boldsymbol{\varepsilon}^T \boldsymbol{\varepsilon} \right\rangle = 0, \tag{33}$$

where K_{ij}^e is the (i, j) element of the matrix $\mathbf{K}_{\mathbf{eq}}$. Note that the excitation $\mathbf{q}$ is a Gaussian random vector. Thus, the system response calculated by the equivalent linear system is Gaussian, as well. In this context, it can be argued [41] that the equivalent coefficients are calculated directly by the equation

$$\boldsymbol{K}_{ij}^e = \left\langle \frac{\partial g_i}{\partial w j} \right\rangle. \tag{34}$$

Thus, $\mathbf{K}_{\mathbf{eq}}$ can be determined by the equations

$$K_{ij}^e = 3k I_{iijj}\left\langle w_j^2 \right\rangle + 6k\sum_{l\neq j} I_{ijjl}\left\langle w_j w_l \right\rangle + 3k\sum_{l_1\neq j}\sum_{l_2\neq j} I_{ijl_1l_2}\left\langle w_{l_1} w_{l_2} \right\rangle,$$
$$\text{for } i, j = 1, \ldots, \infty, \tag{35}$$

where

$$I_{il_1l_2l_3} = \int_{\Omega} v_i v_{l_1} v_{l_2} v_{l_3} d\Omega. \tag{36}$$

Since $f(t)$ is a stationary Gaussian process with given power spectral density function matrix $\boldsymbol{S_q}(\omega)$, the spectral density matrix of the response $\mathbf{w}$ in Eq. (30) is calculated by the classical stochastic input–output relationship

$$\boldsymbol{S_w}(\omega) = \boldsymbol{H}(i\omega)\boldsymbol{S_q}(\omega)\boldsymbol{H}^{T*}(i\omega), \tag{37}$$

where ^{T*} denotes the conjugate transpose operation, and $\mathbf{H}(i\omega)$ is the frequency response matrix

$$\mathbf{H}(i\omega) = [i\omega\mathbf{C} + (\mathbf{K} + \mathbf{K_{eq}})]^{-1}. \tag{38}$$

In this context, the average values involved in Eq. (35) can be readily calculated once the response power spectral density matrix is known by the equation

$$K^e_{ij} = 3kI_{ijjj}S_{w_jw_j} + 6k\sum_{l\neq j} I_{ijjl}S_{w_jw_l} + 3k\sum_{l_1\neq j}\sum_{l_2\neq j} I_{ijl_1l_2}S_{w_{l1}w_{l2}},$$

$$\text{for } i,j = 1,\dots,\infty, \tag{39}$$

where the quantities $S_{w_iw_j}$ are estimated directly from the elements of the response power spectral density as

$$S_{w_iw_j} = \int_{-\infty}^{+\infty} S_{w_iw_j}(\omega)\, d\omega. \tag{40}$$

It is seen that the calculation of the equivalent coefficients requires the *a priori* knowledge of the response statistics. This fact implies that the calculation is pursued by an iterative procedure. Specifically, the algorithm is initiated by assuming an equivalent matrix $\mathbf{K_{eq}}$ populated by zeros. Then, the response statistical moments are calculated by utilizing the stochastic input–output relation (37), and the result is used for updating the values of the equivalent coefficients by Eq. (39). This procedure usually converges in a few iterations, relating to the criterion that the variation in consecutive estimated equivalent stiffness coefficients becoming lower than a reasonable pre-assigned threshold.

Finally, the resulting equivalent linear system is used for calculating the response statistics. Specifically, the variance of the response is estimated by the equation

$$\sigma^2_u(x,y) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} v_i v_j S_{w_iw_j}, \tag{41}$$

and the power spectral density function of the response at a certain point is

$$S_u(x,y,\omega) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} v_i v_j S_{w_iw_j}(\omega), \tag{42}$$

$S_{w_iw_j}(\omega)$ being the elements of the response power spectral density function $\boldsymbol{S_w}(\omega)$.

4. Numerical results

This section involves a numerical application of the statistical linearization approach. For assessing the reliability of the method, the statistical linearization results are compared against relevant Monte Carlo data. In this regard, the algorithm described by Jiao [37] is employed for synthesizing a spectrum compatible excitation and solving numerically the fractional partial differential equation (13) by an approach based on the Boundary Element Method. So that, the response statistics are determined directly from the simulated response time histories.

The implementation of the iterative algorithm described in the previous section requires the calculation of the coefficients $\beta_{mn,sl}$ given in Eq. (25). In general, the integral involved in this calculation cannot be determined analytically. Therefore, its estimation is conducted by numerical integration. Specifically, the domain Ω is discretized in small

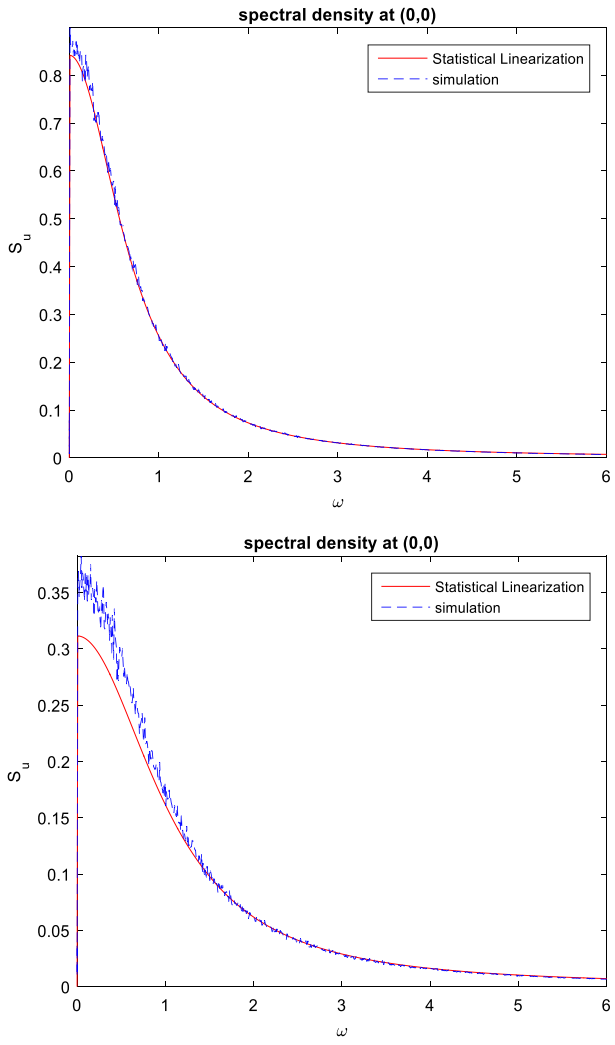

Fig. 1. Power spectral density function of the response u at point (0, 0) calculated by numerical simulation and statistical linearization in case of fractional derivative order $\alpha = 1.9$. Nonlinear parameter: $k = 0.1$ (left panel); and $k = 0.5$ (right panel).

panels so that v_{mn} can be assumed constant over each panel, and the elements $\{z_{mn}\}$ of the vector $\mathbf{z_{mn}}$ involved in the integral are calculated by the equation

$$\mathbf{z_{mn}} = \mathbf{M}\cdot\mathbf{v_{mn}}. \tag{43}$$

The elements of the matrix $\mathbf{M}$ are derived directly from Eq. (20). Specifically,

$$M_{ij} = c(\alpha)\int_{\Omega_j} \frac{1}{|P_i - Q|^{\alpha}} d\Omega. \tag{44}$$

where P_i is kept constant, and Q varies over the jth element of the domain. The numerical computation of Eq. (44) requires a note of caution in the computation of the diagonal elements of the matrix $\mathbf{M}$, because the integral is singular. In this situation, the numerical method described by Chen and Pang [21] is utilized.

The numerical application concerns a rectangular plate with sides $a = 5$ and $b = 2.5$ (note that all variables and values are dimensionless). A Dirichlet boundary condition is posed. That is,

$$u = 0 \text{ on } \Gamma. \tag{45}$$

The time-dependent part $f(t)$ of the excitation is a Gaussian random process compatible with a white noise spectrum having spectral level $S(\omega) = 0.5$, while the space dependent part is constant, $p(x,y) = 1$.

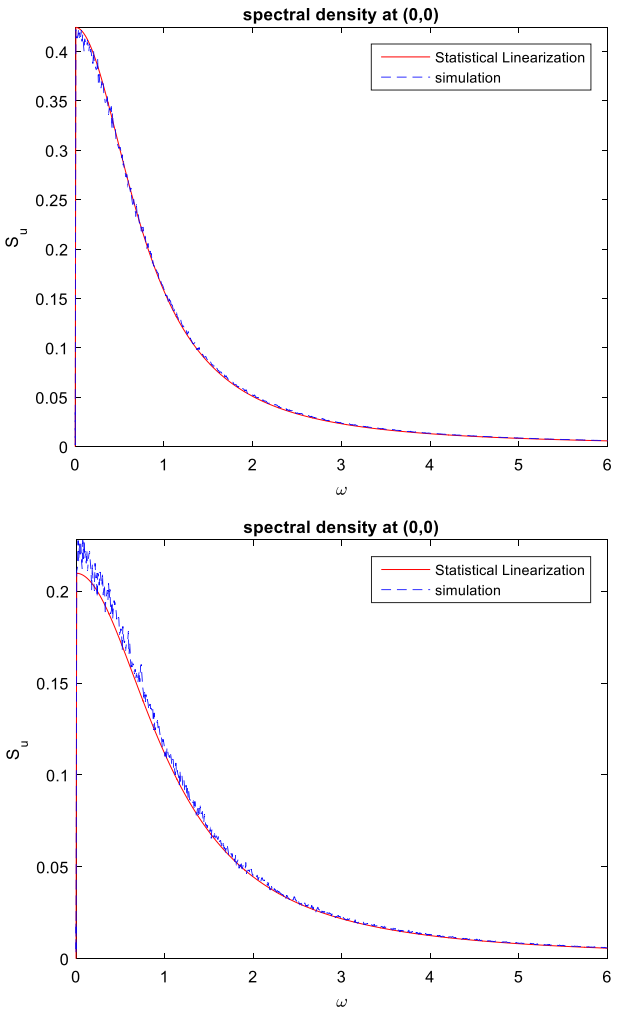

Fig. 2. Power spectral density function of the response u at point (0, 0) calculated by numerical simulation and statistical linearization in case of fractional derivative order $\alpha = 1.5$. Nonlinear parameter: $k = 0.1$ (left panel); and $k = 0.5$ (right panel).

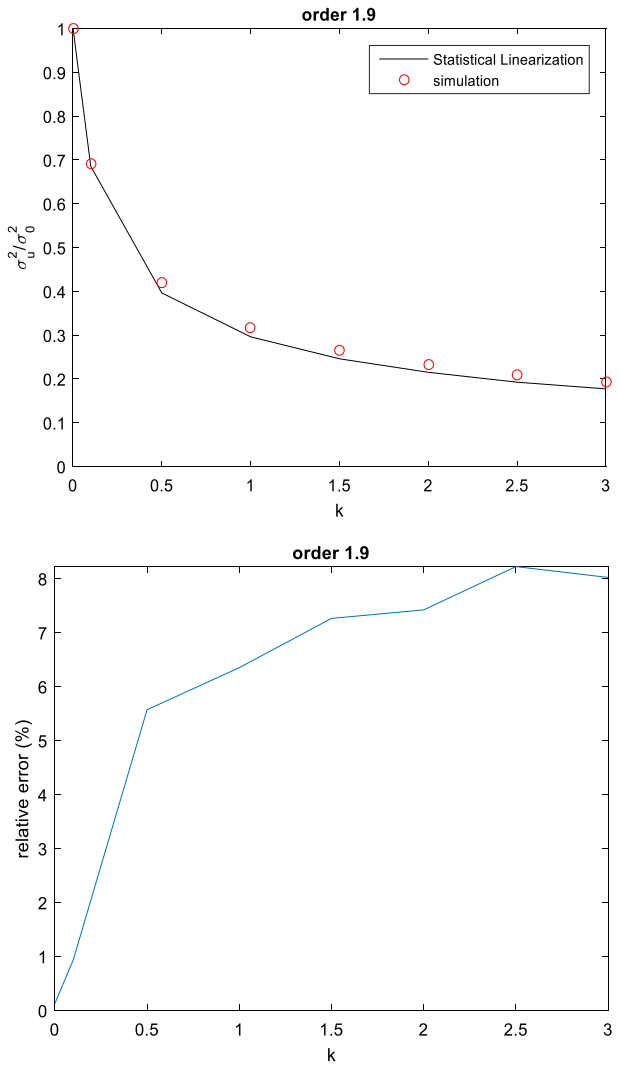

Fig. 3. Variance of the response u at point (0, 0) calculated by numerical simulation and by statistical linearization (left panel) and relative error (right panel) under the assumption of fractional derivative order $\alpha = 1.9$.

In the simulation, the spectral method [42] is utilized to generate spectrum-compatible time histories of the excitations, with $S(\omega) = 0.5$ for $0 < \omega \leq 20\pi$ and $S(\omega = 0) = 0$.

The modes implemented in this specific numerical example are given by the equation

$$v_{mn}(x, y) = \sin\left[\frac{m\pi(x + a)}{2a}\right] \sin\left[\frac{n\pi(y + b)}{2b}\right]. \tag{46}$$

The numerical results pertain to different fractional derivative orders, and levels of the nonlinear parameter k. The comparisons are conducted with the purpose of capturing the impact of these parameters on the reliability of the statistical linearization. The first comparison is shown in Figs. 1 and 2. The figures show the power spectral density functions of the response calculated at the geometrical centre of the domain (0, 0). Each figure shows the response spectrum estimated by statistical linearization (red line), and the one determined via Monte Carlo data (blue dotted line). In the application of the statistical linearization approach a variable number of mode shapes is used. Specifically, 81 modes have been employed for deriving Fig. 1, while 121 modes have been employed for Fig. 2. Including more modes did not yield further variations in the estimated power spectrum. It is seen that there is an excellent agreement between the approximate solution, and the Monte Carlo data in the case of small nonlinearities (left panels). Indeed, the approximation is able to estimate the power spectrum behaviour over the whole frequency domain. The reliability of the approach deteriorates slightly for higher values of the nonlinear parameter. Indeed, in this context (right panels), there are some "under –estimations" in the lower frequency band. However, it is worth-mentioning that the high frequency tail is predicted quite well.

The reliability of the approach for estimating the response statistics is studied in Figs. 3 and 4. The figures show the variance of the response calculated at the centre of the domain. Specifically, they show the influence of the nonlinear parameter k on the response variance. In this regard, note that the figures show variance values normalized by σ_0^2, which is the response variance associated with the linear system obtained by neglecting the nonlinear contribution in Eq. (13) (i.e., by setting $k = 0$). The figures demonstrate that the approximate approach estimates quite reliably the response statistics even in case of large nonlinearities. Clearly, the agreement between the approximate solution and the numerical one is affected by the strength of the nonlinear term. Indeed, larger discrepancies are observed for larger values of the nonlinear parameter k. However, the relative error study (right panels) points out that such an error does not exceed 15%. An interesting feature of this approximate approach is that it is not affected by the fractional derivative order. Indeed, it has provided reliable response estimates both in the case of nearly classical diffusion (Fig. 3), and in the case of anomalous diffusion (Fig. 4).

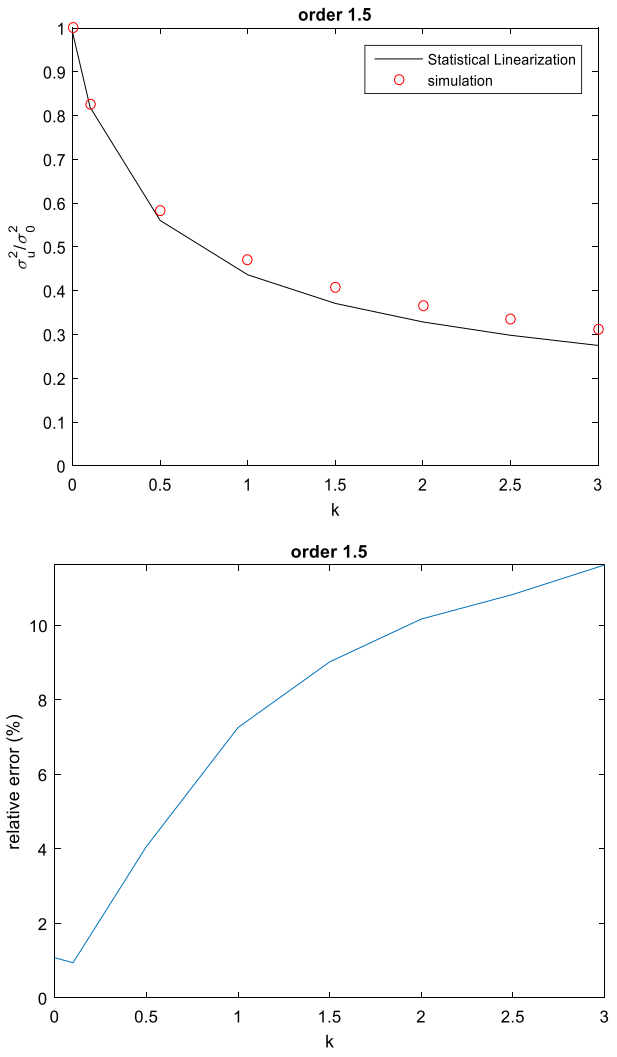

Fig. 4. Variance of the response u at point (0, 0) calculated by numerical simulation and by statistical linearization (left) and relative error (right) under the assumption of fractional derivative order $\alpha = 1.5$.

5. Concluding remarks

This paper has dealt with the problem of estimating the response of a system governed by a nonlinear partial differential equation including a fractional Laplacian operator. Such a problem is relevant in a number of practical situations, as in the propagation of waves in random or disordered media. Specifically, the paper has considered the nonlinear fractional heat equation. To the authors' knowledge, no analytical solution is available for this class of problems. Therefore, an approximate approach based on the concept of statistical linearization has been developed. The critical feature of the studied problem is the treatment of the fractional Laplacian operator. Indeed, it is shown that a series expansion representation of the response can be sought by utilizing the linear modes associated with the classical heat equation. In this manner, by projecting the original equation on the mode space, a system of nonlinear ordinary differential equations describing the time variation of the mode amplitudes is obtained. The influence of the fractional Laplacian is seen in the inclusion of off-diagonal elements associated with the linear term in the resulting differential equation, which are null in the case of the classical Laplacian operator. Next, the equation governing the mode amplitudes has been linearized in a mean square sense. That is, a surrogate (equivalent) linear system for the nonlinear one, has been identified by minimizing the mean square error between the two systems. Then, the equivalent linear system has been utilized for estimating the nonlinear system response statistics.

Numerical results have been used to assess the reliability of the method. Specifically, the statistical linearization outputs have been juxtaposed with relevant Monte Carlo data. It has been shown that the approach provides a good estimate of the response statistics even for strongly nonlinear term. Further, note that the numerical studies have not revealed a significant deteriorations of the reliability of the estimates when varying the fractional derivative order.

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