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Formulation of an efficient continuum mechanics-based model to study wave propagation in one-dimensional diatomic lattices

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ABSTRACT

In this paper, we formulate an efficient continuum mechanics-based model on the basis of a discrete lattice model. First, the dispersion relation of a lattice wave in a one-dimensional diatomic crystal lattice is derived. Then, the second- and fourth-order continuum models are obtained from the differential-difference equations of motion by using the Padé approximations. The results show that the proposed fourth-order continuum model can predict the dispersion behaviour of the one-dimensional diatomic crystal lattice very well in the first Brillouin zone. Furthermore, the applicability of the present model to the prediction of the dispersion behaviour of the one-dimensional diatomic lattice with internal resonator and inerter is examined. Finally, the vibration frequencies of finite diatomic lattices are calculated by both the discrete and the proposed continuum models.

1. Introduction

Wave propagation in one-dimensional periodic structures has a very long history, and it is of great importance in various fields of science and engineering [1]. The first attempt dates back from 1686 when Newton derived a formula for determining the sound velocity. Afterwards, a huge number of papers appeared dealing with this problem. The interested reader is referred to the reviews by Mead [2], Li and Wang [3], and Hussein et al. [4]. To study the wave propagation phenomena, one-dimensional monoatomic and diatomic lattice chains are widely used. The diatomic lattice chains are probably more important because they possess stop-band or band-gap frequency ranges. In addition, this chain is the simplest possible model to study the wave propagation through inhomogeneous media [5]. Therefore, the wave propagation in a one-dimensional diatomic crystal lattice has become a subject of extensive investigations [6–12]. Usually, wave propagation in lattice chains is investigated via discrete and continuum models. However, the following important question has been raised by some researchers.

- Is there a need to develop a continuum mechanics-based model to study a discrete problem for which the exact solution is already known?

This question can be answered as follows. It should be noted that although the simplicity of discrete problems allows us to obtain analytical solutions in many cases, however, their application to systems with a complex geometry and new artificial materials is rather limited [13]. Consequently, several researchers have attempted to develop continuum mechanics-based models to predict the static and dynamic behaviour of lattices. The continuum model has been largely developed for monoatomic lattice chains [14–18]. Unfortunately, few continualization schemes have been presented for diatomic lattice chains in the literature. One of the most and earliest quasi-continuum approximations was proposed by Askar [19]. The proposed model predicted both the acoustic and optical modes accurately, but only in a limited frequency range. In addition, Wattis [20] extended the quasi-continuum method to study the solitary waves in a diatomic lattice. Recently, some continuum mechanics-based models have been developed and applied to the study of the acoustic metamaterial [21].

In this paper, an effective continuum method is developed to analyse the wave propagation in one-dimensional diatomic lattices. To the best of our knowledge, there has been no attempt to tackle the problem described in this investigation. The paper is divided into three parts. In the first part, which consists of Sections 2 and 3, the classic and nonlocal continuum mechanics-based models are derived from the discrete model of the diatomic chain. Then, the dispersion curves obtained from the discrete and continuum models are compared with each other. Numerical results show that the proposed fourth-order continuum model is ac-

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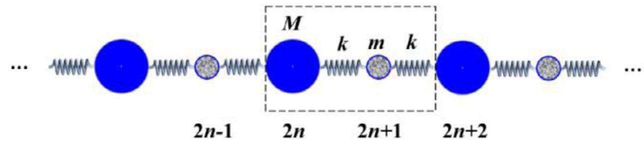

Fig. 1. A diatomic lattice chain.

curate and appropriate for the prediction of the dispersion behaviour of the diatomic crystal lattices in the first Brillouin zone. In the second part, i.e., Sections 4 and 5, the proposed continuum mechanics-based model will be used to predict the dispersion behaviour of various diatomic metamaterials. In Section 4, the dispersion curves obtained from the continuum mechanics-based model are compared with those obtained from the discrete model. In Section 5, the longitudinal wave propagation in a diatomic lattice with inerter in the local attachments will be investigated for the first time. In the last part, i.e., Section 6, the applicability of the proposed continuum models for vibration of finite diatomic lattices is examined. Finally, the summary and concluding remarks are presented in Section 7.

2. Discrete model

Let us consider a one-dimensional diatomic chain as shown in Fig. 1. Each unit cell contains a mass M connected by a massless linear spring k to a mass m. The differential-difference equations of motion for two adjacent odd and even atoms are [7]

$$M\frac{d^2u_{2n}}{dt^2}+k(2u_{2n}-v_{2n-1}-v_{2n+1})=0 \tag{1}$$

$$m\frac{d^2v_{2n+1}}{dt^2}+k(2v_{2n+1}-u_{2n}-u_{2n+2})=0 \tag{2}$$

where u_{2n} represents the displacement of the mass M at position $2n$ and v_{2n+1} represents the position of the mass m at position $2n+1$. For harmonic wave propagation, it is assumed that the displacements are expressed as

$$u_{2n}=U\exp([2n\kappa a-\omega t]i) \tag{3}$$

$$v_{2n+1}=V\exp([(2n+1)\kappa a-\omega t]i) \tag{4}$$

where a is the distance between the neighbouring atoms, κ denotes the wavenumber, ω is the angular frequency, and U and V are the complex wave amplitudes. By substituting Eqs. (3) and (4) into Eqs. (1) and (2), a set of homogeneous algebraic equations with respect to U and V is obtained

$$\begin{bmatrix} 2k-M\omega^2 & -2k\cos(\kappa a) \\ -2k\cos(\kappa a) & 2k-m\omega^2 \end{bmatrix}\begin{pmatrix} A \\ B \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5}$$

For a nontrivial solution of Eq. (5), the determinant of the matrix of coefficients must vanish. This condition gives the *dispersion relation*.

$$Mm\omega^4-2k\omega^2(M+m)+4k^2\sin^2(\kappa a)=0 \tag{6}$$

Solutions of the dispersion relation (6) are

$$\Omega_{1,2}^2=(1+\alpha)\pm\sqrt{(1+\alpha)^2-4\alpha\sin^2(\kappa a)} \tag{7}$$

where

$$\Omega_{1,2}^2=\frac{\omega_{1,2}^2}{\omega_0^2}\quad \omega_0^2=\frac{k}{m}\quad \alpha=\frac{m}{M} \tag{8}$$

where Ω_1 and Ω_2 are the frequencies of upper and lower branches respectively. The upper branch is known as the optical branch while the lower branch is the acoustic branch.

3. Continuum model

3.1. Second-order continualized model

To obtain an efficient continuum model from the differential-difference equations of motion, the discrete displacements must be replaced by their equivalent continuum counterparts. In Eq. (1), the discrete displacements are approximated as follows:

$$\begin{aligned} u_{2n}&=u(x_{2n})=u(x)\\ v_{2n+1}&=v(x_{2n}+a)=v(x)+\frac{a}{1!}\frac{\partial v}{\partial x}+\frac{a^2}{2!}\frac{\partial^2 v}{\partial x^2}+\frac{a^3}{3!}\frac{\partial^3 v}{\partial x^3}+\frac{a^4}{4!}\frac{\partial^4 v}{\partial x^4}+\cdots\\ v_{2n-1}&=v(x_{2n}-a)=v(x)-\frac{a}{1!}\frac{\partial v}{\partial x}+\frac{a^2}{2!}\frac{\partial^2 v}{\partial x^2}-\frac{a^3}{3!}\frac{\partial^3 v}{\partial x^3}+\frac{a^4}{4!}\frac{\partial^4 v}{\partial x^4}-\cdots \end{aligned} \tag{9}$$

Substituting Eq. (9) into Eq. (1), we have

$$M\frac{\partial^2 u}{\partial t^2}+2k(u-v)=ka^2\left(\frac{\partial^2 v}{\partial x^2}+\frac{a^2}{12}\frac{\partial^4 v}{\partial x^4}+\cdots\right) \tag{10}$$

In a similar manner, the continuum model of Eq. (2) is obtained as

$$m\frac{\partial^2 v}{\partial t^2}+2k(v-u)=ka^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{a^2}{12}\frac{\partial^4 u}{\partial x^4}+\cdots\right) \tag{11}$$

when Eqs. (10) and (11) are truncated at the second-order derivatives, the equations of motion associated with the second-order continuum model can be expressed as

$$M\frac{\partial^2 u}{\partial t^2}+2k(u-v)=ka^2\frac{\partial^2 v}{\partial x^2} \tag{12}$$

$$m\frac{\partial^2 v}{\partial t^2}+2k(v-u)=ka^2\frac{\partial^2 u}{\partial x^2} \tag{13}$$

These equations are similar to those presented by Askar [19]. For harmonic wave propagation, the solutions of Eqs. (12) and (13) can be written in complex forms as:

$$u(x,t)=\hat{U}\exp([\kappa x-\omega t]i) \tag{14}$$

$$v(x,t)=\hat{V}\exp([\kappa x-\omega t]i) \tag{15}$$

where $\hat{U}$ and $\hat{V}$ are complex amplitudes. Applying the same procedure as that used for the discrete model, we obtain the frequencies of upper and lower branches as

$$\Omega_{1,2}^2=(1+\alpha)\pm\sqrt{(1+\alpha)^2-\alpha(4\kappa^2a^2-\kappa^4a^4)} \tag{16}$$

The dispersion curves predicted by the second-order continuum model are shown in Fig. 2 and compared with the results of the discrete model. It is observed that the frequencies of both the acoustic and the optic modes are accurately predicted by the second-order continuum model only for $\kappa a<\pi/4$. It can be concluded that the second-order continuum model is applicable only for small wave numbers. This conclusion is consistent with previous observations.

3.2. Fourth-order continualized model

We propose another continualization scheme for the wave propagation in a one-dimensional diatomic crystal lattice. Truncating the infinite series in Eqs. (10) and (11) at the fourth-order

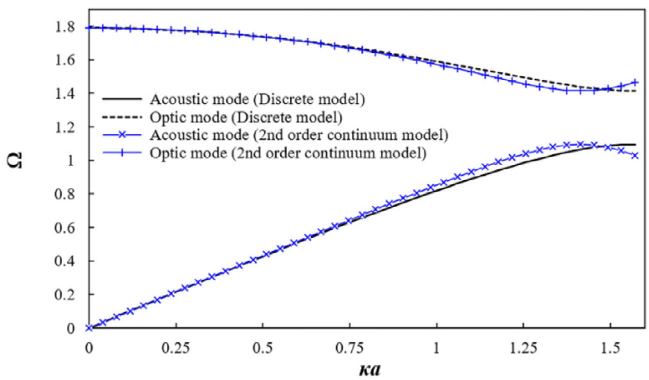

Fig. 2. Comparison between the second-order continuum mechanics-based model and the discrete model for α =0.6.

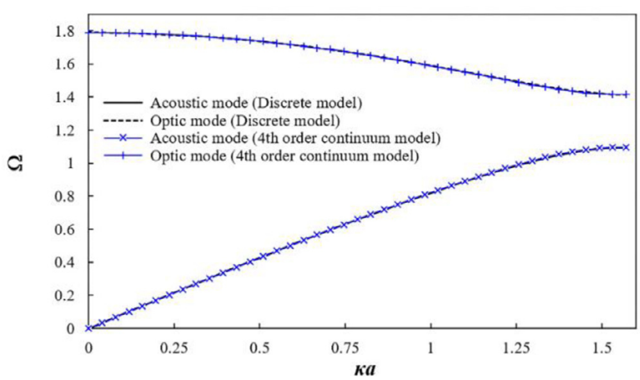

Fig. 3. Comparison between the fourth-order continuum mechanics-based model and the discrete model for α =0.6.

derivatives, the equations of motion associated with the fourth-order continuum model are obtained:

$$M\frac{\partial^2 u}{\partial t^2} + 2k(u - v) = ka^2 L(v) \tag{17}$$

$$m\frac{\partial^2 v}{\partial t^2} + 2k(v - u) = ka^2 L(u) \tag{18}$$

where L is the differential operator that can be efficiently approximated via the Padé approximation [15] as

$$L = \frac{\partial^2}{\partial x^2} + \frac{a^2}{12}\frac{\partial^4}{\partial x^4} \approx \frac{\frac{\partial^2}{\partial x^2}}{1 - \frac{a^2}{12}\frac{\partial^2}{\partial x^2}} \tag{19}$$

Substituting Eq. (19) into Eqs. (17) and (18) and multiplying the resulting equation by $[1 - (a^2/12)(\partial^2/\partial x^2)]$, Eqs. (17) and (18) can then be efficiently approximated by the following linear differential equations

$$\left(1 - \frac{a^2}{12}\frac{\partial^2}{\partial x^2}\right)\left\{M\frac{\partial^2 u}{\partial t^2} + 2k(u - v)\right\} = ka^2\frac{\partial^2 v}{\partial x^2} \tag{20}$$

$$\left(1 - \frac{a^2}{12}\frac{\partial^2}{\partial x^2}\right)\left\{m\frac{\partial^2 v}{\partial t^2} + 2k(v - u)\right\} = ka^2\frac{\partial^2 u}{\partial x^2} \tag{21}$$

Using the non-dimensional parameters defined in Eq. (8) and substituting Eqs. (14) and (15) into Eqs. (20) and (21), the frequencies of upper and lower branches are obtained as

$$\Omega_{1,2}^2 = (1+\alpha) \pm \sqrt{(1+\alpha)^2 - \alpha\frac{12\kappa^2 a^2(48 - 8\kappa^2 a^2)}{(12+\kappa^2 a^2)^2}} \tag{22}$$

Fig. 3 displays the dispersion curves obtained from the fourth-order continuum model. The figure demonstrates a perfect agreement between the results of the fourth-order continuum model and the discrete lattice model. As a result, we can conclude that the proposed continuum method is accurate and reliable for predicting the wave propagation in the diatomic lattice chains in the first Brillouin zone. Although the results are expected, the fourth-order continuum model has not been developed so far.

4. Application of continuum mechanics-based model to diatomic metamaterials

Acoustic metamaterials are artificial materials designed to achieve novel physical properties not commonly observed in nature. This new class of materials has attracted significant attention in the research community because they have remarkable number of potential applications in vibration mitigation and isolation, impact absorption and wave guides [22,23]. In this section, the applicability of the present continuum mechanics-based model to diatomic metamaterials is investigated. At the first step, the dispersion relation for a diatomic lattice with internal resonator is obtained by using the discrete lattice model. In this connection, we consider a one-dimensional infinite diatomic chain of metamaterial as shown in Fig. 4a. Each unit contains two macro-materials with mass M_1 and mass m_1 which are connected by a massless spring k. In addition, a micro-material with mass M_2 is connected by a massless spring k_1 to mass M_1. Similarly, mass m_2 is connected by a massless spring k_2 to mass m_1. The differential-difference equations of motion for a unit cell are

$$M_1\frac{d^2 u_{2n}^{(1)}}{dt^2} + k\left(2u_{2n}^{(1)} - v_{2n-1}^{(1)} - v_{2n+1}^{(1)}\right) + k_1\left(u_{2n}^{(1)} - u_{2n}^{(2)}\right) = 0 \tag{23}$$

$$M_2\frac{d^2 u_{2n}^{(2)}}{dt^2} + k_1\left(u_{2n}^{(2)} - u_{2n}^{(1)}\right) = 0 \tag{24}$$

$$m_1\frac{d^2 v_{2n+1}^{(1)}}{dt^2} + k\left(2v_{2n+1}^{(1)} - u_{2n}^{(1)} - u_{2n+2}^{(1)}\right) + k_2\left(v_{2n+1}^{(1)} - v_{2n+1}^{(2)}\right) = 0 \tag{25}$$

$$m_2\frac{d^2 v_{2n+1}^{(2)}}{dt^2} + k_2\left(v_{2n+1}^{(2)} - v_{2n+1}^{(1)}\right) = 0 \tag{26}$$

The solutions of the lattice wave are of the form

$$u_{2n}^{(1)} = U^{(1)}\exp\left([2n\kappa a - \omega t]i\right) \tag{27}$$

$$u_{2n}^{(2)} = U^{(2)}\exp\left([2n\kappa a - \omega t]i\right) \tag{28}$$

$$v_{2n+1}^{(1)} = V^{(1)}\exp\left([(2n+1)\kappa a - \omega t]i\right) \tag{29}$$

$$v_{2n+1}^{(2)} = V^{(2)}\exp\left([(2n+1)\kappa a - \omega t]i\right) \tag{30}$$

where $U^{(1)}$, $U^{(2)}$, $V^{(1)}$ and $V^{(2)}$ are the complex wave amplitudes. By substituting Eqs. (27)–(30) into Eqs. (23)–(26), a set of homogeneous algebraic equations with respect to $U^{(1)}$, $U^{(2)}$, $V^{(1)}$ and $V^{(2)}$ is obtained

$$\begin{bmatrix} 2k + k_1 - M_1\omega^2 & -k_1 & -2k\cos(\kappa a) & 0 \\ -k_1 & k_1 - M_2\omega^2 & 0 & 0 \\ -2k\cos(\kappa a) & 0 & 2k + k_2 - m_1\omega^2 & -k_2 \\ 0 & 0 & -k_2 & k_2 - m_2\omega^2 \end{bmatrix} \times \begin{pmatrix} U^{(1)} \\ U^{(2)} \\ V^{(1)} \\ V^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{31}$$

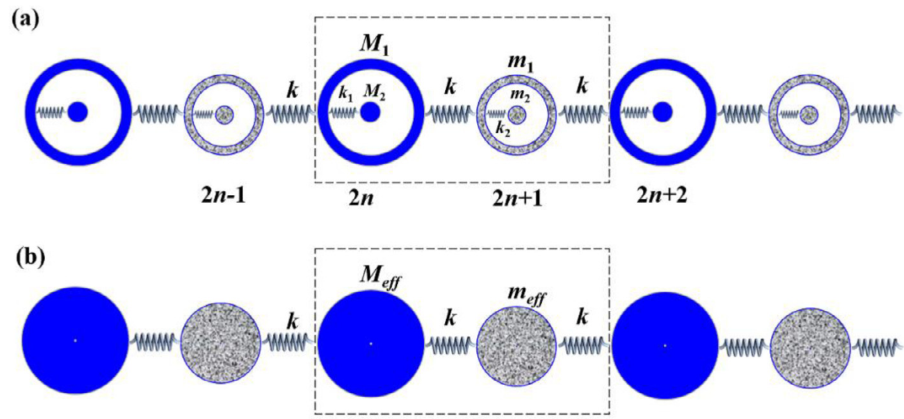

Fig. 4. Diatomic metamaterial chain; (a) mass-in-mass system (b) effective mass-spring system.

For a nontrivial solution of Eq. (31), the determinant of this set of equations must be zero, i.e.

$$\begin{vmatrix} 2k+k_1-M_1\omega^2 & -k_1 & -2k\cos(\kappa a) & 0 \\ -k_1 & k_1-M_2\omega^2 & 0 & 0 \\ -2k\cos(\kappa a) & 0 & 2k+k_2-m_1\omega^2 & -k_2 \\ 0 & 0 & -k_2 & k_2-m_2\omega^2 \end{vmatrix} = 0 \tag{32}$$

Eq. (32) is the dispersion relation of the lattice wave in the diatomic metamaterial. Let $\alpha_0 = m_1/M_1$, $\alpha_1 = M_2/M_1$, $\alpha_2 = m_2/m_1$, $\beta_1 = k_1/k$, $\beta_2 = k_2/k$, $\omega_0^2 = k/M_1$ and $\Omega^2 = \omega^2/\omega_0^2$, Eq. (32) can be rewritten as

$$\begin{vmatrix} 2+\beta_1-\Omega^2 & -\beta_1 & -2\cos(\kappa a) & 0 \\ -\beta_1 & \beta_1-\alpha_1\Omega^2 & 0 & 0 \\ -2\cos(\kappa a) & 0 & 2+\beta_2-\alpha_0\Omega^2 & -\beta_2 \\ 0 & 0 & -\beta_2 & \beta_2-\alpha_0\alpha_2\Omega^2 \end{vmatrix} = 0 \tag{33}$$

By solving Eq. (33), the wave dispersion diagrams of diatomic metamaterials can be determined with given wavenumbers.

Next, consider an effective diatomic lattice system in which the mass-in-mass system is represented by a simple diatomic mass-spring system (Fig. 4b) with stiffness k and two effective masses M_{eff} and m_{eff}. The two effective masses in the equivalent mass-spring model are frequency dependent and are given by [21]

$$M_{eff} = M_1 + \frac{k_1 M_2}{k_1 - M_2\omega^2} \quad m_{eff} = m_1 + \frac{k_2 m_2}{k_2 - m_2\omega^2} \tag{34}$$

Substituting Eq. (34) into Eqs. (20) and (21), the continuum equations of motion for the diatomic metamaterials are obtained as

$$\left(M_1 + \frac{k_1 M_2}{k_1 - M_2\omega^2}\right)\frac{\partial^2 u}{\partial t^2} + 2k(u-v) - ka^2\frac{\partial^2 v}{\partial x^2} - \frac{a^2}{12}\left\{\left(M_1 + \frac{k_1 M_2}{k_1 - M_2\omega^2}\right)\frac{\partial^4 u}{\partial t^2 \partial x^2} + 2k\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 v}{\partial x^2}\right)\right\} = 0 \tag{35}$$

$$\left(m_1 + \frac{k_2 m_2}{k_2 - m_2\omega^2}\right)\frac{\partial^2 v}{\partial t^2} + 2k(v-u) - ka^2\frac{\partial^2 u}{\partial x^2} - \frac{a^2}{12}\left\{\left(m_1 + \frac{k_2 m_2}{k_2 - m_2\omega^2}\right)\frac{\partial^4 v}{\partial t^2 \partial x^2} + 2k\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial x^2}\right)\right\} = 0 \tag{36}$$

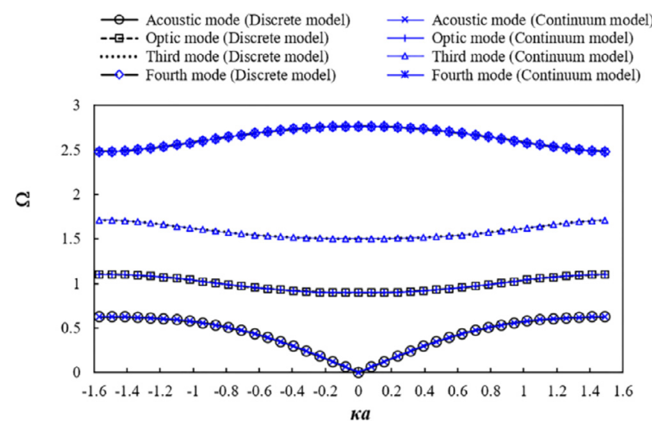

Fig. 5. Dispersion curves of the diatomic metamaterials obtained from the discrete and the continuum mechanics-based models for $\alpha_0 = 0.5$, $\alpha_1 = 1.3$, $\alpha_2 = 0.8$ and $\beta_1 = \beta_2 = 0.75$.

Substituting from Eqs. (14) and (15) into Eqs. (35) and (36) leads to the dispersion equation

$$\begin{vmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{vmatrix} = 0 \tag{37}$$

where

$$\begin{aligned} S_{11} &= 2(\beta_1 - \alpha_1\Omega^2) - (\beta_1 - \alpha_1\Omega^2 + \alpha_1\beta_1)\Omega^2 \\ S_{12} &= -\left(\frac{24 - 10\kappa^2 a^2}{12 + \kappa^2 a^2}\right)(\beta_1 - \alpha_1\Omega^2) \\ S_{21} &= -\left(\frac{24 - 10\kappa^2 a^2}{12 + \kappa^2 a^2}\right)(\beta_2 - \alpha_0\alpha_2\Omega^2) \\ S_{22} &= 2(\beta_2 - \alpha_0\alpha_2\Omega^2) - (\alpha_0\beta_2 - \alpha_0^2\alpha_2\Omega^2 + \alpha_0\alpha_2\beta_2)\Omega^2 \end{aligned} \tag{38}$$

For each value of κa, Eq. (37) has four distinct real roots implying four natural frequencies. A comparison between the dispersion curves of the diatomic metamaterials obtained from the discrete and the continuum mechanics-based models is plotted in Fig. 5. For numerical calculations in this figure, we adopted $\alpha_0 = 0.5$, $\alpha_1 = 1.3$, $\alpha_2 = 0.8$ and $\beta_1 = \beta_2 = 0.75$. It is found that the dispersion curves corresponding to the continuum mechanics-based model is fitting well with that predicted by the discrete lattice model in the first Brillouin zone. As can be seen from Fig. 5, there are three band-gaps for the diatomic metamaterial lattices.

These results again show that the proposed continuum mechanics-based model is a promising tool to explore the wave propagation in one-dimensional diatomic lattices. In addition, the

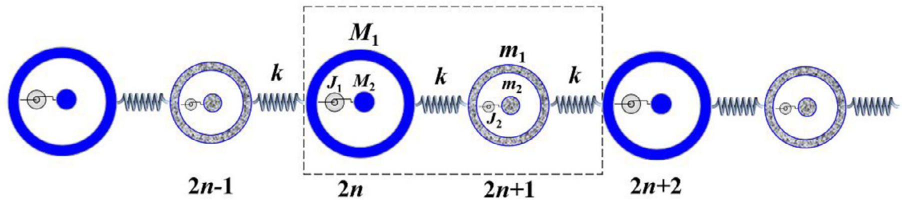

Fig. 6. Discrete representation of an infinite diatomic inertant acoustic metamaterial.

band-gaps can be predicted by the proposed continuum model accurately.

5. Application of continuum mechanics-based model to diatomic inertant acoustic metamaterials

A new modified acoustic metamaterial system with inerter was introduced by Kulkarni and Manimala [24]. An inerter is a mass-amplifying mechanical device that provides a force response proportional to the relative acceleration of its two ends [25]. As was stated above, the dispersion behaviour of the diatomic inertant acoustic metamaterials has not been studied so far. Therefore, in this section, the wave propagation in the diatomic acoustic metamaterials with inerters is investigated by using the proposed continuum mechanics-based model. A one-dimensional discrete element lattice representation for a diatomic inertant acoustic metamaterial is shown in Fig. 6.

In comparison to the mass-in-mass system, the internal springs k_1 and k_2 are respectively replaced by two inerters of inertance J_1 and J_2. In this case, the two effective masses in the equivalent mass-spring model are given by [24]

$$M_{eff} = M_1 + \frac{J_1 M_2}{J_1 + M_2} \quad m_{eff} = m_1 + \frac{J_2 m_2}{J_2 + m_2} \tag{39}$$

Using Eqs. (14), (15), (20), (21) (39), the dispersion relation is obtained as

$$\Omega^4\left(1 + \frac{\gamma_1\alpha_1}{\gamma_1 + \alpha_1}\right)\left(\alpha_0 + \frac{\gamma_2\alpha_0\alpha_2}{\gamma_2 + \alpha_0\alpha_2}\right) - 2\Omega^2\left(1 + \alpha_0 + \frac{\gamma_1\alpha_1}{\gamma_1 + \alpha_1} + \frac{\gamma_2\alpha_0\alpha_2}{\gamma_2 + \alpha_0\alpha_2}\right) + \left(4 - \left(\frac{24 - 10\kappa^2 a^2}{12 + \kappa^2 a^2}\right)^2\right) = 0 \tag{40}$$

where

$$\alpha_0 = \frac{m_1}{M_1} \quad \alpha_1 = \frac{M_2}{M_1} \quad \alpha_2 = \frac{m_2}{m_1} \quad \gamma_1 = \frac{J_1}{M_1}$$
$$\gamma_2 = \frac{J_2}{M_1} \quad \omega_0^2 = \frac{k}{M_1} \quad \Omega^2 = \frac{\omega^2}{\omega_0^2} \tag{41}$$

If the coefficients $\gamma_1 = 0$ and $\gamma_2 = 0$, Eq. (38) is reduced to the dispersion relation of diatomic crystal lattices. Fig. 7 shows the effect of the inertance ratio (γ_1 or γ_2) upon the dispersion curves while Fig. 8 demonstrates the effect of the mass ratio (α_1 and α_2) upon the dispersion curves. It is found that the lower and the upper branches of the dispersion curves shift towards low frequency with an increase of the both the inertance and the mass ratios.

6. Vibration of finite diatomic lattice chains

In contrast to the study of the infinite one-dimensional lattices, the vibration of the finite lattices by using a continuum nonlocal model has been barely investigated. Challamel and co-workers [26,27] investigated the vibration of finite monatomic lattices with direct and indirect neighbouring interactions by using the discrete

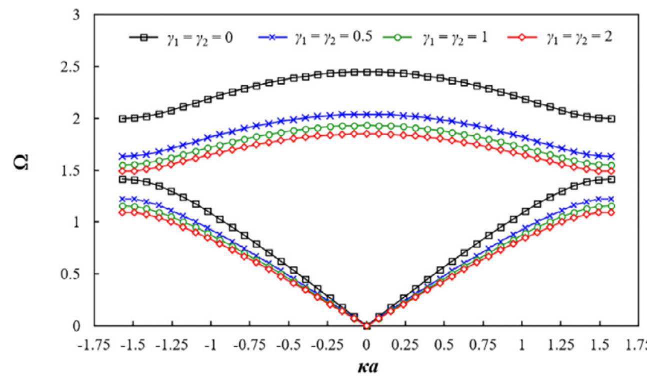

Fig. 7. The effect of the inertance ratio on the dispersion curves for $\alpha_0 = 0.5$ and $\alpha_1 = \alpha_2 = 1$.

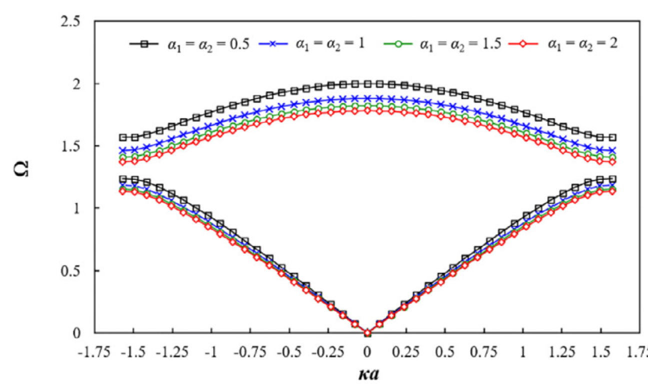

Fig. 8. The effect of the mass ratio on the dispersion curves for $\alpha_0 = 0.6$ and $\gamma_1 = \gamma_2 = 0.75$.

and nonlocal continuum models. Since the vibration of the finite diatomic lattices by using a continuum model has not been presented so far, in this section, the applicability of the present continuum mechanics-based model to this problem is examined. First, the vibration frequencies of a finite diatomic lattice are calculated by using the discrete lattice model. Let us consider a fixed-fixed diatomic lattice consisting of N unit cells (Fig. 9). The equations of motion for the discrete system are given as follows

$$m\frac{d^2 v_1}{dt^2} + k(2v_1 - u_2) = 0 \tag{42}$$

$$M\frac{d^2 u_{2j-2}}{dt^2} + k\left(2u_{2j-2} - v_{2j-3} - v_{2j-1}\right) = 0 \quad \text{for } j = 2, 3, \ldots, N \tag{43}$$

$$m\frac{d^2 v_{2j-1}}{dt^2} + k\left(2v_{2j-1} - u_{2j-2} - u_{2j}\right) = 0 \quad \text{for } j = 2, 3, \ldots, N \tag{44}$$

$$M\frac{d^2 u_{2N}}{dt^2} + k(2u_{2N} - v_{2N-1}) = 0 \tag{45}$$

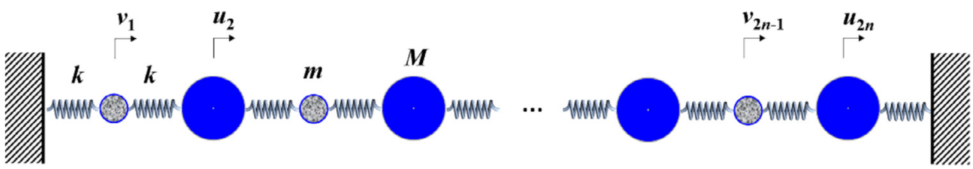

Fig. 9. Finite diatomic lattice chain with fixed ends.

Normal-mode solutions of Eqs. (42)–(45) have the following form:

$$u_{2n} = U_{2n}\cos(\omega t - \delta)$$
$$v_{2n-1} = V_{2n-1}\cos(\omega t - \delta) \text{ for } n = 1, 2, 3, \ldots, N \tag{46}$$

where δ is an arbitrary phase constant. Substituting Eq. (46) into Eqs. (42)–(45) and using Eq. (8), leads to

$$(2 - \Omega^2)V_1 - U_2 = 0 \tag{47}$$

$$\left(2 - \frac{\Omega^2}{\alpha}\right)U_{2j-2} - (V_{2j-3} + V_{2j-1}) = 0 \tag{48}$$

$$(2 - \Omega^2)V_{2j-1} - (U_{2j-2} + U_{2j}) = 0 \tag{49}$$

$$\left(2 - \frac{\Omega^2}{\alpha}\right)U_{2N} - V_{2N-1} = 0 \tag{50}$$

Eqs. (48) and (49) can be rewritten as

$$\begin{pmatrix} V_{2j-1} \\ U_{2j} \end{pmatrix} = [\mathbf{A}]\begin{pmatrix} V_{2j-3} \\ U_{2j-2} \end{pmatrix} \tag{51}$$

where

$$[\mathbf{A}] = \begin{pmatrix} -1 & \Gamma_1 \\ -\Gamma_2 & \Gamma_1\Gamma_2 - 1 \end{pmatrix}, \ \Gamma_1 = \left(2 - \frac{\Omega^2}{\alpha}\right), \ \Gamma_2 = \left(2 - \Omega^2\right) \tag{52}$$

On iterating Eq. (51), we have

$$\begin{pmatrix} V_{2j-1} \\ U_{2j} \end{pmatrix} = [\mathbf{A}]^{j-1}\begin{pmatrix} V_1 \\ U_2 \end{pmatrix} \tag{53}$$

If λ_+ and λ_- are two distinct eigenvalues of the matrix $[\mathbf{A}]$, then we have [28]

$$[\mathbf{A}]^{j-1} = \frac{(\lambda_+\lambda_-^{j-1} - \lambda_-\lambda_+^{j-1})}{\lambda_+ - \lambda_-}[\mathbf{I}] + \frac{(\lambda_+^{j-1} - \lambda_-^{j-1})}{\lambda_+ - \lambda_-}[\mathbf{A}] \tag{54}$$

The eigenvalues of the matrix $[\mathbf{A}]$ are obtained from

$$\begin{vmatrix} -1 - \lambda & \Gamma_1 \\ -\Gamma_2 & \Gamma_1\Gamma_2 - 1 - \lambda \end{vmatrix} = 0 \tag{55}$$

Thus, it is concluded that

$$\lambda_+\lambda_- = 1 \tag{56}$$

$$\lambda_+ + \lambda_- = \Gamma_1\Gamma_2 - 2 \tag{57}$$

Using Eq. (53), we obtain

$$\begin{pmatrix} V_{2N-1} \\ U_{2N} \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}\begin{pmatrix} V_1 \\ U_2 \end{pmatrix} \tag{58}$$

where g_{11}, g_{12}, g_{21} and g_{22} are calculated from Eq. (54). Using Eqs. (47), (50), (56)–(58), we obtain

$$\frac{\lambda_+^{2N+1} - 1}{\lambda_+ - 1} = 0 \tag{59}$$

Table 1
Comparison of non-dimensional frequencies for $\alpha = 0.5$ and $q = N$.

N	Discrete model		Continuum model	
	Ω_1	Ω_2	Ω_1	Ω_2
10	0.9945	1.4181	0.9968	1.4165
20	0.9985	1.4152	0.9997	1.4144
30	0.9993	1.4147	1.0000	1.4142
40	0.9996	1.4145	1.0000	1.4142
50	0.9998	1.4144	1.0000	1.4142

Eq. (59) has 2 N roots;

$$\lambda_{+1} = \exp\left(i\frac{2\pi q}{2N+1}\right)$$
$$\lambda_{+2} = \exp\left(-i\frac{2\pi q}{2N+1}\right) \text{ for } q = 1, 2, 3, \ldots, N \tag{60}$$

Using Eqs. (52), (57) and (60), the characteristic equation is obtained as

$$\Omega^4 - 2\Omega^2(1+\alpha) + 4\alpha sin^2\left(\frac{\pi q}{2N+1}\right) = 0 \tag{61}$$

Solutions of Eq. (61) are

$$(\Omega_{1,2}^2)_q = (1+\alpha) \pm \sqrt{(1+\alpha)^2 - 4\alpha\sin^2\left(\frac{\pi q}{2N+1}\right)} \tag{62}$$

In the following, the vibration frequencies are evaluated by using the fourth-order continuum mechanics-based model. For free vibration, the solutions of Eqs. (12) and (13) can be written as

$$u(x,t) = \sum_{q=1}^{N} U_q \sin\left(\frac{q\pi x}{L^*}\right)\cos\left(\omega_q t - \delta_q\right) \tag{63}$$

$$v(x,t) = \sum_{q=1}^{N} V_q \sin\left(\frac{q\pi x}{L^*}\right)\cos\left(\omega_q t - \delta_q\right) \tag{64}$$

where U_q, V_q, ω_q and δ_q are the amplitudes, the frequencies and the phase constant of the qth mode. In addition, $L^* = (N + +1)a$ is the total length of the chain. Using the non-dimensional parameters defined in Eq. (8) and substitution of the qth solutions into Eqs. (20) and (21), the non-dimensional frequencies of qth mode are obtained as

$$(\Omega_{1,2}^2)_q = (1+\alpha) \pm \sqrt{(1+\alpha)^2 - \alpha\frac{576(2N+1)^2q^2\pi^2 - 96q^4\pi^4}{\left(12(2N+1)^2 + q^2\pi^2\right)^2}} \tag{65}$$

Table 1 compares the frequencies of the diatomic lattice chain obtained from the discrete and the continuum mechanics-based models. For numerical calculations in this table, we adopted $\alpha = 0.5$ and $q = N$. Excellent agreement between the reslts is observed. Therefore, it can be concluded that the proposed continuum mechanics-based model is also a promising tool to explore the vibration of finite diatomic lattice chains.

7. Conclusions

In this paper we have proposed a fourth-order continuum model for the study of wave propagation in the diatomic lattice chains. This is a modification of the continuum model developed by Askar [19]. Here, we showed that the second-order continuum model cannot accurately predict the dispersion behaviour of the diatomic lattice chain. In addition, the results indicate that the fourth-order continuum model can predict physically realistic dispersion curves in the first Brillouin zone. In addition, the proposed continuum mechanics-based model was used to predict the wave propagation in the classical and the inertant acoustic metamaterials. Furthermore, we have shown that the proposed fourth-order continuum model is also applicable for predicting the vibration frequencies of the finite diatomic lattices. In the future, this investigation can be extended to two- and three-dimensional diatomic crystal lattices.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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