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Nonlinear vibrations of beams and plates with fractional derivative elements subject to combined harmonic and random excitations

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ABSTRACT

This paper proposes an efficient approach for estimating reliably the second order statistics of the response of continua excited by combinations of harmonic and random loads. The problem is relevant in several engineering applications, where, for instance, the harmonic load is influenced by significant noise that cannot be neglected when computing the response statistics. The considered problems pertain to the vibration of beams and of plates endowed with fractional derivative elements. In both cases, it is shown that by representing the system response by the linear modes of vibration, systems of nonlinear fractional ordinary differential equations describing the time-dependent variation of the modes amplitudes are obtained. These equations are coupled and are treated by combining the harmonic balance and statistical linearization techniques, leading to the determination of the second-order statistics of the response. Relevant Monte Carlo data demonstrate the reliability of the proposed solution approach. The specific numerical examples considered pertain to simply supported beams, and plates with simply supported stress-free edges conditions.

1. Introduction

Although most random vibration analyses pertain to systems excited either by deterministic or by random loads, determining the response of a system to a combination of periodic (harmonic or not) and random forces is critical in several circumstances. A quite common situation occurs in case of periodic loadings influenced by noise. However, there are also other notable examples such as the helicopter rotor blade vibration or the vibration of turbine blades in a turbulent flow [1]. In all of these cases, the system response must be determined by including both excitations in describing the exciting load.

Numerical and approximate analytical techniques have been proposed in the literature for addressing the problem of determining the response statistics of nonlinear systems excited by combined periodic and random loadings. Numerical techniques, such as finite difference, finite elements, and path integration-based [2,3] are applicable to a broad class of nonlinear differential equations, and have been used for assessing the reliability of analytical approaches. For instance, Hawes and Langley [4] employed the global weighted residual method for the analysis of a Duffing oscillator. They confirmed by comparison with pertinent Monte Carlo data that it is reasonably reliable only in case of weakly nonlinear systems or in case of high damping.

In the context of analytical approaches, the common solution strategy involves combining techniques employed in the study of systems excited either by periodic or by random loads [5]. Both cases have been extensively investigated in the open literature, and a variety of related techniques are available. In case of deterministic excitations, one of the most popular techniques is the averaging method. The method was developed by Bogoliubov and Mitropolskii [6] and has found extensive applications also in random vibration applications, where it was extended by Stratonovich [7]. Typically, the method leads to the problem of determining the solution of a Fokker–Planck equation, which lends itself to exact analytical solutions [8–11] for a limited number of cases, but, in general, it can be solved by several approximate techniques [12–14]. In the case of random excitations, one of the most versatile techniques is the statistical linearization, see for instance Ref. [15]. The technique over a period of several decades has been utilized for solving a quite broad class of random vibration problems (see Ref. [16]), ranging from single-degree-of-freedom (SDOF) to multi-degree-of-freedom (MDOF) systems, and also to problems about the vibration of continua [17–21]. Other techniques widely discussed in the literature are moment closure [22–24], equivalent nonlinear equation [25–29], Wiener Path integral [30–33], perturbation technique [34].

The response to combined deterministic and random excitations was investigated by Nayfeh and Serhan [35], who used the method of multiple scales to determine the mean and mean-square response of a Duffing–Rayleigh oscillator. Huang et al. [12] employed stochastic

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averaging and path integration for estimating the response joint probability density of the amplitude and of the phase pertaining to a Duffing oscillator. The same system was investigated by Haiwu et al. [36] by combining harmonic balance and stochastic averaging. Cai and Lin [37] utilized a stochastic averaging method to obtain an appropriate Itô stochastic differential equation, which was approximated via a Markov vector, to derive an exact stationary probability density function. More recently, the problem of a Duffing oscillator under combined harmonic and random loads was studied by Zhu and Wu [1,38] with regards to the first passage problem. They first derived a set of Itô equations by stochastic averaging from the equation of motion. Then, they derived a backward Kolmogorov equation governing the reliability function, which was solved by finite-differences. A similar approach was also used by Chen et al. [39] for MDOF systems, and by Chen and Zhu [40] for a system with a fractional derivative element. The techniques of stochastic averaging and statistical linearization were combined by Anh and Hieu [41] and Anh et al. [5] for Duffing and Van der Pol oscillators, respectively, while harmonic balance and Gaussian closure were utilized by Zhu and Guo [42]. The response of nonlinear multi-degree-of-freedom systems subject to combined mono-frequency periodic and random excitations was presented by Spanos et al. [43]. Their approach combined statistical linearization and harmonic balance techniques.

This paper considers the problem of determining the response statistics of continua exposed to the combined action of harmonic and random loads. Specifically, the paper deals with the problem of moderately large vibration of beams and of plates endowed with fractional derivative elements. Approximate analytical techniques have been proposed in the open literature for estimating the response of these systems when excited either by deterministic or random loads (see, for instance, Refs. [44–52]). However, to the authors' knowledge, there are no solutions available to the problem involving their combination. For this purpose, herein the statistical linearization technique is employed in conjunction with the harmonic balance technique for estimating the system response statistics. The reliability of this approach is assessed vis-à-vis relevant Monte Carlo data obtained by Boundary Element Method (BEM) based techniques [53,54].

2. Moderately large beam displacement

2.1. Equation of motion

The moderately large vibration of a beam having length L is governed by the partial differential equation

$$\frac{EI\partial^4 v(x,t)}{\partial x^4}+\frac{\rho A\partial^2 v(x,t)}{\partial t^2}+c_0\partial_t^\alpha v(x,t)-N\frac{\partial^2 v(x,t)}{\partial x^2}=p(x)\left[f_d(t)+f_r(t)\right], \tag{1}$$

where

$$N=\frac{EA}{2L}\int_0^L\left(\frac{\partial v(x,t)}{\partial x}\right)^2 dx, \tag{2}$$

is the axial force; the symbols E, I, ρ and A on the left hand side denote elastic modulus, moment of inertia of the cross-section, mass density, and cross-sectional area, respectively; $p(x)$ is a deterministic function rendering the space-wise distribution of the load; $f_d(t)$ is the deterministic time-dependent part of the load; and $f_r(t)$ is the random part of the load with power spectral density function $S(\omega)$. The excitation of the system given in Eq. (1) involves the combination of a random load and of a deterministic load. Both quantities are supposed to have an identical space-wise distribution $p(x)$. Further, the deterministic load is considered to be harmonic with amplitude f_0 and frequency Ω_0, so that

$$f_d(t)=f_0\cos\left(\Omega_0 t\right); \tag{3}$$

and the random load is a stationary random process with autocorrelation function

$$\langle f_r(t-\tau_1)\, f_r(t-\tau_2)\rangle=\int_{-\infty}^{+\infty}S(\omega)\exp\left[i\omega\left(\tau_2-\tau_1\right)\right]d\omega, \tag{4}$$

in which i is the imaginary unit. Eq. (1) includes also a fractional derivative element with parameters α and c, that can be employed for incorporating other external actions, such as viscoelastic foundation, etc. [55,56]. This operator generalizes the classical operators of differentiation and of integration by one operator which inherits their properties [57]. In this regard, it is noted that the Fourier Transform $\mathcal{F}[\cdot]$ of the fractional derivative of a given function $w(t)$ will be expressed as

$$\mathcal{F}[{}_0\mathrm{D}_t^\alpha w(t)]=(i\omega)^\alpha\,\mathcal{F}[w(t)]. \tag{5}$$

In this equation, it is seen that for positive integer values of the fractional derivative order α, Eq. (5) renders the well-known relation between the Fourier Transform of a function and the one of its derivative.

2.2. Approximate response statistics

The proposed solution procedure relies on a response representation as the superposition of the linear modes of beam vibration as proposed by Spanos and Malara [53]. In this context, the vertical beam displacement is represented by the equation

$$v(x,t)=\sum_{m=1}^{\infty}w_m(t)\,\Phi_m(x), \tag{6}$$

in which $\Phi_m(x)$ and $w_m(t)$ are, respectively, the linear beam modes and the time-dependent amplitudes. Eq. (6) allows exploiting the orthogonality of these modes for deriving the equation governing the time variation of the amplitudes $w_m=w_m(t)$. Indeed, by projecting Eq. (1) on the space of the linear modes, by defining the quantities

$$K_{mn}=K_{nm}=\int_0^L\Phi'_m\Phi'_n\,dx, \tag{7}$$

$$R_{mn}=\int_0^L\Phi_m\Phi''_n\,dx, \tag{8}$$

and

$$P_m=\int_0^L p(x)\,\Phi_m(x)\,dx, \tag{9}$$

and by considering that,

$$\int_0^L\Phi_m\Phi_n\,dx=L\delta_{mn}, \tag{10}$$

where δ_{mn} is the Kronecker delta; the equation

$$\ddot{w}_m+\frac{c}{\rho A}\,{}_0\mathrm{D}_t^\alpha w_m+\omega_m^2 w_m-\frac{E}{2\rho L^2}\sum_n\sum_i\sum_j w_n w_i w_j K_{ij}R_{mn}=\frac{P_m}{\rho AL}\left[f_d(t)+f_r(t)\right], for\ \ m=1,2,\ldots \tag{11}$$

is derived. Eq. (11) is a set of nonlinear fractional ordinary differential equations, which can be solved approximately by combining the harmonic balance and the statistical linearization techniques. Note that the quantities ω_m are natural frequencies determined by the specified boundary conditions and are related to the beam modes by the equation

$$EI\Phi_m^{iv}=\rho A\omega_m^2\Phi_m. \tag{12}$$

The time-dependent amplitudes are represented as a combination of a mean amplitude $\overline{w}_m$ and a zero-mean random amplitude $\hat{w}_m$. That is,

$$w_m=\overline{w}_m+\hat{w}_m. \tag{13}$$

Substituting Eq. (13) into Eq. (11) and taking the ensemble average of the resulting equations, differential equations governing the variation of the deterministic amplitudes $\overline{w}_m$ are derived. That is,

$$\ddot{\overline{w}}_m + \frac{c}{\rho A}\ {}_0\mathrm{D}_t^{\alpha}\overline{w}_m + \omega_m^2\overline{w}_m - \frac{E}{2\rho L^2}\sum_n\sum_i\sum_j\left(\overline{w}_n\overline{w}_i\overline{w}_j + \overline{w}_n\sigma_{ij} + \overline{w}_i\sigma_{nj} + \overline{w}_j\sigma_{ni}\right)K_{ij}R_{mn} = \frac{P_m}{\rho AL}f_0\cos\left(\Omega_0 t\right), for\ m = 1,2,\dots \tag{14}$$

where

$$\sigma_{ij} = \mathrm{E}\left[\widehat{w}_i\widehat{w}_j\right], \tag{15}$$

with E[·] denoting the operator of mathematical expectation.

A harmonic balance solution of Eq. (14) is next sought by approximating the mean response components by the equation

$$\overline{w}_m = A_m\cos\left(\Omega_0 t\right) + B_m\sin\left(\Omega_0 t\right). \tag{16}$$

In this context, the fractional derivative involved in Eq. (14) can be calculated by the equation

$${}_0\mathrm{D}_t^{\alpha}\overline{w}_m \approx {}_{-\infty}\mathrm{D}_t^{\alpha}\overline{w}_m = A_m\Omega_0^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)\cos\left(\Omega_0 t\right) - A_m\Omega_0^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\sin\left(\Omega_0 t\right) + B_m\Omega_0^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)\sin\left(\Omega_0 t\right) + B_m\Omega_0^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\cos\left(\Omega_0 t\right), \tag{17}$$

where it is assumed that the analysis focuses only on the steady-state part of the response.

By substituting Eqs. (16) and (17) into Eq. (14), and neglecting the contribution due to high order harmonics it is seen that the amplitudes A_m and B_m are the solutions of the following set of nonlinear algebraic equations:

$$\begin{cases} -\Omega_0^2A_m + \frac{c}{\rho A}A_m\Omega_0^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right) + \frac{c}{\rho A}B_m\Omega_0^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right) + \omega_m^2A_m + \\ -\frac{E}{2\rho L^2}\sum_n\sum_i\sum_j\left(\frac{3}{4}A_nA_iA_j + \frac{1}{4}A_nB_iB_j + \frac{1}{4}B_nB_iA_j + \frac{1}{4}B_nA_iB_j + \right. \\ \left. + A_n\sigma_{ij} + A_i\sigma_{nj} + A_j\sigma_{ni}\right)K_{ij}R_{mn} = \frac{P_m}{\rho AL}f_0, \qquad for\ m = 1,2,\dots \\ -\Omega_0^2B_m - \frac{c}{\rho A}A_m\Omega_0^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right) + \frac{c}{\rho A}B_m\Omega_0^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right) + \omega_m^2B_m + \\ -\frac{E}{2\rho L^2}\sum_n\sum_i\sum_j\left(\frac{1}{4}A_nB_iA_j + \frac{1}{4}A_nA_iB_j + \frac{1}{4}B_nA_iA_j + \frac{3}{4}B_nB_iB_j + \right. \\ \left. + B_n\sigma_{ij} + B_i\sigma_{nj} + B_j\sigma_{ni}\right)K_{ij}R_{mn} = 0, \qquad for\ m = 1,2,\dots \end{cases} \tag{18}$$

Clearly the system of Eq. (18) involves coupling of the deterministic amplitudes A_m and B_m, and of the statistics of the random components of the response. For the random components, a statistical linearization approach is utilized. Specifically, Eq. (14) is subtracted to Eq. (11) for deriving the equation governing the time-variation of $\widehat{w}_m$. That is,

$$\ddot{\widehat{w}}_m + \frac{c}{\rho A}\ {}_0\mathrm{D}_t^{\alpha}\widehat{w}_m + \omega_m^2\widehat{w}_m - \frac{E}{2\rho L^2}\sum_n\sum_i\sum_j\left(\overline{w}_n\overline{w}_i\widehat{w}_j + \overline{w}_n\widehat{w}_i\overline{w}_j + \overline{w}_n\widehat{w}_i\widehat{w}_j + \widehat{w}_n\overline{w}_i\overline{w}_j + \widehat{w}_n\overline{w}_i\widehat{w}_j + \widehat{w}_n\widehat{w}_i\overline{w}_j + \widehat{w}_n\widehat{w}_i\widehat{w}_j - \overline{w}_n\sigma_{ij} - \overline{w}_i\sigma_{nj} - \overline{w}_j\sigma_{ni}\right) \times K_{ij}R_{mn} = \frac{P_m}{\rho AL}f_r, for\ m = 1,2,\dots \tag{19}$$

Next, Eq. (19) is replaced by a set of surrogate linear differential equations used for estimating approximately the response statistics. The set of linear equations is

$$\ddot{\widehat{w}}_m + \frac{c}{\rho A}\ {}_0\mathrm{D}_t^{\alpha}\widehat{w}_m + \omega_{eq,m}^2\widehat{w}_m = \frac{P_m}{\rho AL}f_r(t), for\ m = 1,2,\dots \tag{20}$$

whose stiffness values $\omega_{eq,\ m}$ are equivalent stiffness parameters selected by minimizing the mean square error between the original equation and the linear one. Specifically, the error between Eqs. (20) and (19) is,

$$\varepsilon_m = \omega_{eq,m}^2\widehat{w}_m - \omega_m^2\widehat{w}_m + \frac{E}{2\rho L^2}\sum_n\sum_i\sum_j\left(\overline{w}_n\overline{w}_i\widehat{w}_j + \overline{w}_n\widehat{w}_i\overline{w}_j + \overline{w}_n\widehat{w}_i\widehat{w}_j + \widehat{w}_n\overline{w}_i\overline{w}_j + \widehat{w}_n\overline{w}_i\widehat{w}_j + \widehat{w}_n\widehat{w}_i\overline{w}_j + \widehat{w}_n\widehat{w}_i\widehat{w}_j - \overline{w}_n\sigma_{ij} - \overline{w}_i\sigma_{nj} - \overline{w}_j\sigma_{ni}\right) \times K_{ij}R_{mn}, for\ m = 1,2,\dots \tag{21}$$

and the equivalent stiffness values are calculated by the equations [16],

$$\frac{\partial}{\partial\omega_{eq,m}^2}\mathrm{E}\left[\varepsilon_m^2\right] = 0, for\ m = 1,2,\dots \tag{22}$$

That is,

$$\omega_{eq,m}^2\mathrm{E}\left[\widehat{w}_m^2\right] = \omega_m^2\mathrm{E}\left[\widehat{w}_m^2\right] - \frac{E}{2\rho L^2}\sum_n\sum_i\sum_j\left(\overline{w}_n\overline{w}_i\mathrm{E}\left[\widehat{w}_j\widehat{w}_m\right] + \overline{w}_n\overline{w}_j\mathrm{E}\left[\widehat{w}_i\widehat{w}_m\right] + \overline{w}_i\overline{w}_j\mathrm{E}\left[\widehat{w}_n\widehat{w}_m\right] + \mathrm{E}\left[\widehat{w}_n\widehat{w}_i\widehat{w}_j\widehat{w}_m\right]\right) \times K_{ij}R_{mn}, for\ m = 1,2,\dots \tag{23}$$

The expected values in Eq. (23) are determined by utilizing the input–output relationships for the linear system (20). Specifically, denoting the impulse response function and the transfer function associated with the system (20) by the symbols $h_m(t)$ and $H_m(\omega)$, it is found that the output of Eq. (20) is given by the equation

$$\widehat{w}_m = \frac{P_m}{\rho AL}\int_{-\infty}^{+\infty}h_m(\tau)f_r(t-\tau)d\tau, \tag{24}$$

the frequency response function is

$$H_m(\omega) = \frac{1}{-\omega^2 + \beta(i\omega)^{\alpha} + \omega_{eq,m}^2}, \tag{25}$$

and $h_m(t)$ and $H_m(\omega)$ constitute a Fourier transform pair. That is,

$$h_m(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}H_m(\omega)e^{i\omega t}d\omega,\ \text{and}\ H_m(\omega) = \int_{-\infty}^{+\infty}h_m(t)e^{-i\omega t}dt. \tag{26}$$

After obvious algebraic manipulations, using linear input–output relationships, and the property of Gaussian random processes [51]

$$\begin{aligned}&\mathrm{E}\left[f_r\left(t-\tau_1\right)f_r\left(t-\tau_2\right)f_r\left(t-\tau_3\right)f_r\left(t-\tau_4\right)\right]\\ &= \mathrm{E}\left[f_r\left(t-\tau_1\right)f_r\left(t-\tau_2\right)\right]\mathrm{E}[f_r\left(t-\tau_3\right)f_r\left(t-\tau_4\right)]\\ &+ \mathrm{E}\left[f_r\left(t-\tau_1\right)f_r\left(t-\tau_3\right)\right]\mathrm{E}[f_r\left(t-\tau_2\right)f_r\left(t-\tau_4\right)]\\ &+ \mathrm{E}\left[f_r\left(t-\tau_1\right)f_r\left(t-\tau_4\right)\right]\mathrm{E}[f_r\left(t-\tau_2\right)f_r\left(t-\tau_3\right)],\end{aligned} \tag{27}$$

the expected values are determined by the equations

$$\mathrm{E}\left[\widehat{w}_m\widehat{w}_n\right] = \sigma_{mn} = \frac{P_mP_n}{(\rho AL)^2}\int_{-\infty}^{+\infty}H_m(\omega)S(\omega)H_n(-\omega)d\omega, \tag{28}$$

and

$$\mathrm{E}\left[\widehat{w}_m\widehat{w}_n\widehat{w}_j\widehat{w}_i\right] = \frac{P_mP_nP_jP_i}{(\rho AL)^4}\left(S_{mn}S_{ij} + S_{mi}S_{nj} + S_{mj}S_{ni}\right), \tag{29}$$

where

$$S_{mn} = \int_{-\infty}^{+\infty}H_m(\omega)S(\omega)H_n(-\omega)d\omega. \tag{30}$$

Thus, the equations used for determining the equivalent stiffness parameters are

$$\omega_{eq_m}^2 = \omega_m^2 - \frac{E}{2\rho^3A^2L^4}\frac{1}{P_mS_{mm}}\sum_n\sum_i\sum_j\left[\overline{w}_n\overline{w}_iP_jS_{jm} + \overline{w}_n\overline{w}_jP_iS_{im} + \overline{w}_i\overline{w}_jP_nS_{nm} + P_nP_iP_j\left(S_{mn}S_{ij} + S_{mi}S_{nj} + S_{mj}S_{ni}\right)\right]K_{ij}R_{mn},\ \text{for}\ m = 1,2,\dots \tag{31}$$

It is worth mentioning that Eq. (31) provides time-dependent values of the equivalent stiffness parameters because the mean response components $\overline{w}_m$ are time-dependent. In this context, a further approximation is introduced for producing an approximate set of constant equivalent parameters. Specifically, by averaging Eq. (31) over the period T (= $2\pi/\Omega_0$) of the harmonic load f_d, and considering that

$$\frac{1}{T}\int_0^T\overline{w}_m\overline{w}_ndt = \frac{1}{2}\left(A_mA_n + B_mB_n\right), \tag{32}$$

the system of Eq. (31) reduces to,

$$\overline{\omega}_{eq,m}^2 = \omega_m^2 - \frac{E}{2\rho^3 A^2 L^4}\frac{1}{P_m S_{mm}}\sum_n\sum_i\sum_j\left[\frac{1}{2}\left(A_nA_i + B_nB_i\right)P_jS_{jm} + \frac{1}{2}\left(A_nA_j + B_nB_j\right)P_iS_{im} + \frac{1}{2}\left(A_iA_j + B_iB_j\right)P_nS_{nm} + P_nP_iP_j\left(S_{mn}S_{ij} + S_{mi}S_{nj} + S_{mj}S_{ni}\right)\right]K_{ij}R_{mn}, for\ m = 1,2,\ldots \tag{33}$$

The numerical determination of the equivalent stiffness values is pursued by an iterative scheme. Indeed, Eqs. (33) and (18) involve simultaneously the harmonic and the random parts of the response. Clearly, no explicit solution of these equations can, in general, be derived. Thus, iteration can be initialized by setting $\omega_{eq,m} = \omega_m$. At each iteration, the algebraic system (18) is solved, then, new values of the equivalent stiffness parameters are calculated by Eq. (33) and are utilized as input values for the next iteration. The procedure is repeated until no significant improvement occurs within two consecutive iterations.

The second-order statistical moments of the beam displacement are calculated by relying on the deterministic response component and on the linearized system. Specifically, by relying on the representation of Eq. (6), taking the mean square value and then averaging over one period, the response variance is calculated as

$$\sigma^2(x) = \sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}\Phi_m(x)\Phi_n(x)\left\{\frac{1}{2}\left(A_mA_n + B_mB_n\right) + \frac{P_mP_n}{(\rho AL)^2}S_{mn}\right\}. \tag{34}$$

Eq. (34) is a generalization of the formula derived in Spanos and Malara [53]. The critical element introduced in this problem is the harmonic part of the response that is added to the term computed by the linearized system. Further it is important to note that the harmonic balance/statistical linearization solution schemes are coupled and cannot be pursued separately.

3. Large plate displacement

3.1. Equation of motion

Next a plate vibration problem is considered. The solution procedure and its implementation are similar to the one described in the preceding section. Nevertheless, appropriate changes must be made due to the different elements involved in the equation of motion. Indeed, in this context the partial differential equation [54]

$$\rho h\frac{\partial^2 u}{\partial t^2} + c_0\partial_t^\alpha u + D\nabla^4 u - h\left(\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2\phi}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - 2\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial^2 u}{\partial x\partial y}\right) = p(x,y)\left[f_d(t) + f_r(t)\right], \tag{35}$$

governs the vibration of a rectangular plate having sides a and b, thickness h and flexural stiffness D. The operator $\nabla^4 = (\partial^4/\partial x^4 + \partial^4/\partial y^4 + 2\partial^4/\partial x^2\partial y^2)$ is the biharmonic operator, and $\phi = \phi\ (x, y, t)$ is the Airy stress function related to the plate displacement by the equation

$$\nabla^4\phi = E\left[\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2 - \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}\right]. \tag{36}$$

Again, in this problem the load is assumed to be of a separable format with deterministic space-wise distribution $p(x, y)$, and harmonic $f_d(t)$ plus random $f_r(t)$ time-dependent part.

3.2. Approximate response statistics

The vertical displacement $u = u(x, y, t)$ and the stress function are expanded as

$$u = \sum_{m,n} w_{mn}(t)U_{mn}(x,y), \tag{37}$$

and

$$\phi = \frac{P_x y^2}{2bh} + \frac{P_y x^2}{2ah} + \sum_{m,n} w_{mn}^{(2)}(t)\varphi_{mn}(x,y), \tag{38}$$

where $U_{mn}\ (x, y)$ and φ_{mn} are the modes associated with specific boundary conditions; and P_x and P_y are the total tension loads applied on the sides $x = (0, a)$ and $y = (0, b)$ of the plate; and for compactness of notation, $\sum_{m,n} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}$ denotes a double summation.

The modes are orthogonal to each other, and the amplitudes $w_{mn}^{(2)}$ are functions of the amplitudes w_{mn}. Therefore, by projecting Eq. (35) on the space of modes and exploiting the orthogonality conditions ensuring that

$$\iint_A U_{mn}U_{kl}\,dA = \iint_A \varphi_{mn}\varphi_{kl}\,dA = \frac{4}{ab}\delta_{mk}\delta_{nl}, \tag{39}$$

the following set of nonlinear fractional ordinary differential equations is derived (see Ref. [54]),

$$\ddot{w}_{MN} + \frac{c}{\rho h}{}_0\mathrm{D}_t^\alpha w_{MN} + \omega_{MN}^2 w_{MN} - \frac{4}{ab\rho h}\left(\frac{P_x}{b}\sum_{m,n} w_{mn}R_{xx}(M,N,m,n) + \frac{P_y}{a}\sum_{m,n} w_{mn}R_{yy}(M,N,m,n)\right) - \frac{4}{ab}\frac{E}{\rho}\sum_{m,n}\sum_{k,l}\sum_{p,q} w_{mn}w_{kl}w_{pq} \times I(M,N,m,n,k,l,p,q) = \frac{4}{ab\rho h}P_{MN}\left[f_d(t) + f_r(t)\right], for\ M,N, = 1,2,\ldots \tag{40}$$

where

$$R_{xx}(M,N,m,n) = \iint_A \frac{\partial^2 U_{mn}}{\partial x^2}U_{MN}\,dA, \tag{41}$$

$$R_{yy}(M,N,m,n) = \iint_A \frac{\partial^2 U_{mn}}{\partial y^2}U_{MN}\,dA, \tag{42}$$

$$P_{MN} = \iint_A p(x,y)U_{MN}\,dA, \tag{43}$$

and

$$I(M,N,m,n,k,l,p,q) = \sum_{i,j}\iint_A \frac{\partial^2 U_{mn}}{\partial x^2}\frac{\partial^2\varphi_{ij}}{\partial y^2}U_{MN} + \frac{\partial^2 U_{mn}}{\partial y^2}\frac{\partial^2\varphi_{ij}}{\partial x^2}U_{MN} - 2\frac{\partial^2 U_{mn}}{\partial x\partial y}\frac{\partial\varphi_{ij}}{\partial x\partial y}U_{MN}\,dA \times \frac{\iint_A \frac{\partial^2 U_{kl}\partial^2 U_{pq}}{\partial x\partial y\partial x\partial y}\varphi_{ij} - \frac{\partial^2 U_{kl}\partial^2 U_{pq}}{\partial x^2\partial y^2}\varphi_{ij}\,dA}{\iint_A \frac{\partial^4\varphi_{ij}}{\partial x^2\partial y^2}\varphi_{ij} + \frac{\partial^4\varphi_{ij}}{\partial x^2\partial y^2}\varphi_{ij} + 2\frac{\partial^2\varphi_{ij}}{\partial x\partial y}\varphi_{ij}\,dA}. \tag{44}$$

The frequencies ω_{MN} are natural frequencies associated with the plate boundary conditions, which are determined by considering the linear plate vibration problem

$$D\nabla^4 U_{MN} = \rho h\omega_{MN}^2 U_{MN}. \tag{45}$$

The decomposition of the amplitudes w_{MN} in a deterministic component $\overline{w}_{MN}$ and in a zero-mean random component $\hat{w}_{MN}$ is pursued in this problem, as well. Then, the proposed approach is implemented as described in the beam vibration problem and leads to similar equations for the amplitudes of the deterministic part of the response, and for the determination of the equivalent stiffness parameters. Specifically, in this case the amplitudes A_{MN} and B_{MN} defining the response component $\overline{w}_{MN}$ by the equation

$$\overline{w}_{MN} = A_{MN}\cos\left(\Omega_0 t\right) + B_{MN}\sin\left(\Omega_0 t\right), \tag{46}$$

are solutions of the system of nonlinear algebraic equations

$$\begin{cases} -\Omega_0^2 A_{MN} + \frac{c}{\rho h} A_{MN} \Omega_0^{\alpha} \cos\left(\frac{\alpha\pi}{2}\right) + \frac{c}{\rho h} B_{MN} \Omega_0^{\alpha} \sin\left(\frac{\alpha\pi}{2}\right) + \omega_{MN}^2 A_{MN} + \\ -\frac{4}{ab\rho h}\left(\frac{P_x}{b}\sum_{m,n} A_{mn} R_{xx}(M,N,m,n) + \frac{P_y}{a}\sum_{m,n} A_{mn} R_{yy}(M,N,m,n)\right) + \\ -\frac{4}{ab}\frac{E}{\rho}\sum_{m,n}\sum_{k,l}\sum_{p,q} I(M,N,m,n,k,l,p,q)\{A_{mn}\sigma_{kl,pq} + A_{kl}\sigma_{mn,pq} + A_{pq}\sigma_{mn,kl} + \\ \quad +\frac{3}{4}A_{mn}A_{kl}A_{pq} + \frac{1}{4}A_{mn}B_{kl}B_{pq} + \frac{1}{4}B_{mn}B_{kl}A_{pq} + \frac{1}{4}B_{mn}A_{kl}B_{pq}\} = \\ \quad = \frac{4}{ab\rho h}P_{MN} f_0; \quad for\ M,N = 1,2,\dots \\ -\Omega_0^2 B_{MN} - \frac{c}{\rho h} A_{MN} \Omega_0^{\alpha} \sin\left(\frac{\alpha\pi}{2}\right) + \frac{c}{\rho h} B_{MN} \Omega_0^{\alpha} \cos\left(\frac{\alpha\pi}{2}\right) + \omega_{MN}^2 B_{MN} + \\ -\frac{4}{ab\rho h}\left(\frac{P_x}{b}\sum_{m,n} B_{mn} R_{xx}(M,N,m,n) + \frac{P_y}{a}\sum_{m,n} B_{mn} R_{yy}(M,N,m,n)\right) + \\ -\frac{4}{ab}\frac{E}{\rho}\sum_{m,n}\sum_{k,l}\sum_{p,q} I(M,N,m,n,k,l,p,q)\{B_{mn}\sigma_{kl,pq} + B_{kl}\sigma_{mn,pq} + B_{pq}\sigma_{mn,kl} + \\ \quad +\frac{1}{4}A_{mn}B_{kl}A_{pq} + \frac{1}{4}A_{mn}A_{kl}B_{pq} + \frac{1}{4}B_{mn}A_{kl}A_{pq} + \frac{3}{4}B_{mn}B_{kl}B_{pq}\} = 0; \\ \quad for\ M,N, = 1,2,\dots \end{cases} \tag{47}$$

Further, the stiffness parameters associated with the equivalent linear system

$$\ddot{\hat{w}}_{MN} + \frac{c}{\rho h}{}_0\mathrm{D}_t^{\alpha}\hat{w}_{MN} + \omega_{eq,MN}^2 w_{MN} = \frac{4}{\rho hab}P_{MN} f_r(t), M, N = 1,2,\dots \tag{48}$$

are calculated by the equation

$$\begin{aligned} \omega_{eq,MN}^2 &= \omega_{MN}^2 - \frac{4}{ab\rho h}\frac{1}{P_{MN}S_{MN,MN}}\sum_{m,n} P_{mn}S_{MN,mn}\left[\frac{P_x}{b}R_{xx}(M,N,m,n)\right. \\ &\left. + \frac{P_y}{a}R_{yy}(M,N,m,n)\right] - \left(\frac{4}{ab}\right)^3 \frac{E}{\rho^3 h^2}\frac{1}{P_{MN}S_{MN,MN}}\sum_{m,n}\sum_{k,l}\sum_{p,q} P_{mn}P_{kl}P_{pq} \\ &\times \left(S_{MN,mn}S_{kl,pq} + S_{MN,kl}S_{mn,pq} + S_{MN,pq}S_{mn,kl}\right) I(M,N,m,n,k,l,p,q) \\ &- \frac{4}{ab}\frac{E}{\rho}\frac{1}{P_{MN}S_{MN,MN}}\sum_{m,n}\sum_{k,l}\sum_{p,q} I(M,N,m,n,k,l,p,q) \\ &\times \left[\overline{w}_{mn}\overline{w}_{kl}P_{pq}S_{MN,pq} + \overline{w}_{mn}\overline{w}_{pq}P_{kl}S_{MN,kl} + \overline{w}_{kl}\overline{w}_{pq}P_{mn}S_{MN,mn}\right], for\ M,N = 1,2,\dots \end{aligned} \tag{49}$$

and by averaging over one period T for deriving a set of time-invariant equivalent stiffness parameters using the equation

$$\begin{aligned} \overline{\omega}_{eq,MN}^2 &= \omega_{MN}^2 - \frac{4}{ab\rho h}\frac{1}{P_{MN}S_{MN,MN}}\sum_{m,n} P_{mn}S_{MN,mn}\left[\frac{P_x}{b}R_{xx}(M,N,m,n)\right. \\ &\left. + \frac{P_y}{a}R_{yy}(M,N,m,n)\right] - \left(\frac{4}{ab}\right)^3 \frac{E}{\rho^3 h^2}\frac{1}{P_{MN}S_{MN,MN}}\sum_{m,n}\sum_{k,l}\sum_{p,q} P_{mn}P_{kl}P_{pq} \\ &\times \left(S_{MN,mn}S_{kl,pq} + S_{MN,kl}S_{mn,pq} + S_{MN,pq}S_{mn,kl}\right) I(M,N,m,n,k,l,p,q) \\ &- \frac{4}{ab}\frac{E}{\rho}\frac{1}{P_{MN}S_{MN,MN}}\sum_{m,n}\sum_{k,l}\sum_{p,q} I(M,N,m,n,k,l,p,q)\frac{1}{2}\left[\left(A_{mn}A_{kl}\right.\right. \\ &\left. + B_{mn}B_{pq}\right)P_{pq}S_{MN,pq} + \left(A_{mn}A_{pq} + B_{mn}B_{pq}\right)P_{kl}S_{MN,kl} \\ &\left. + \left(A_{kl}A_{pq} + B_{kl}B_{pq}\right)P_{mn}S_{MN,mn}\right], for\ M,N = 1,2,\dots \end{aligned} \tag{50}$$

where

$$S_{MN,mn} = \int_{-\infty}^{+\infty} H_{MN}(-\omega)\, S(\omega)\, H_{mn}(\omega)\, d\omega, \tag{51}$$

and

$$H_{MN}(\omega) = \frac{1}{-\omega^2 + \frac{c}{\rho h}(i\omega)^{\alpha} + \omega_{eq,MN}^2}. \tag{52}$$

By this approach, the second-order statistics of the response are calculated using Eq. (37). That is,

$$\sigma^2(x,y) = \sum_{m,n}\sum_{k,l} U_{mn}U_{kl}\left\{\frac{1}{2}\left(A_{mn}A_{kl} + B_{mn}B_{kl}\right) + \left(\frac{4}{ab\rho h}\right)^2 P_{mn}P_{kl}U_{mn}U_{kl}S_{mn,kl}\right\}, \tag{53}$$

where, again in this case, it is recognized that Eq. (53) generalizes the results of Malara and Spanos [54] by considering the additional contribution to the second-order response statistics associated with $\overline{w}_{MN}$.

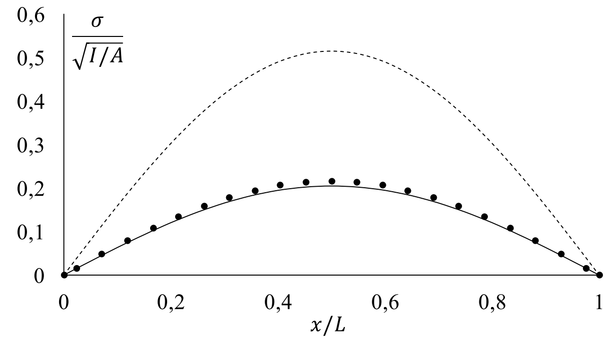

Fig. 1. Normalized standard deviation of the beam displacement. Continuous line: approximate analytical solution; dotted line: linear solution; circles: Monte Carlo data.

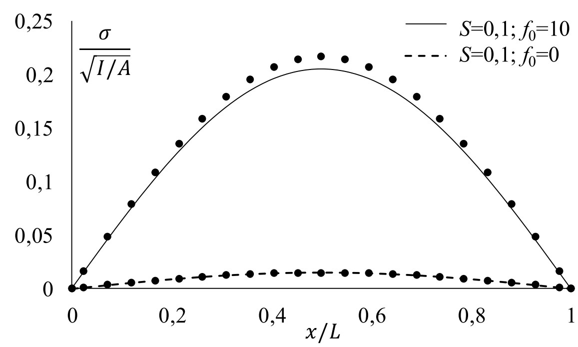

Fig. 2. Normalized standard deviation of the beam displacement under combined random and harmonic load (continuous line) and under random load only (dotted line). The circles denote results from relevant Monte Carlo data.

Table 1
Beam system parameters.

E	$2.1 \cdot 10^{11}$ N/m^2
ρ	2355 kg/m^3
c	10 N/(m/s)$^{\alpha}$
L	20 m
A	0.03 m^2
I	$2.25 \cdot 10^{-4}$ m^4

4. Numerical results

Two numerical examples are discussed next. The first example concerns vibrations of a simply supported beam; the second one concerns vibration of a simply supported plate. In both cases, the results of the numerical calculations are validated by comparison versus relevant Monte Carlo data. These data are obtained by generating spectrum compatible excitations and, then, integrating the equations of motion by the BEM based approach discussed in Refs. [53,54].

The beam vibration problem considered in this numerical example concerns a simply supported structure with the material and geometrical properties shown in Table 1. The modes of vibration of a simply supported beam are given by the equation

$$\Phi_m = \sqrt{2}\sin\left(\frac{\pi m x}{L}\right) \text{ for } m = 1,2,\dots \tag{54}$$

while the natural frequencies ω_m are calculated as

$$\omega_m^2 = \frac{EI}{\rho A}\left(\frac{m\pi}{L}\right)^4. \tag{55}$$

The random part of the excitation is compatible with a white noise power spectral density function having spectral level $S(\omega) = S_0 = 0,1$ (N/m)^{2}s. The harmonic part of the excitation has amplitude $f_0 = 10$ N/m, while the frequency Ω_0 is proportional to the first natural frequency of the system ω_1.

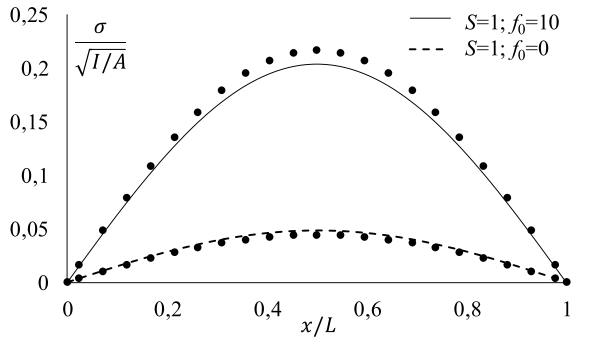

Fig. 3. Normalized standard deviation of the beam displacement under combined random and harmonic load (continuous line) and under random load only (dotted line). The circles denote results from relevant Monte Carlo data.

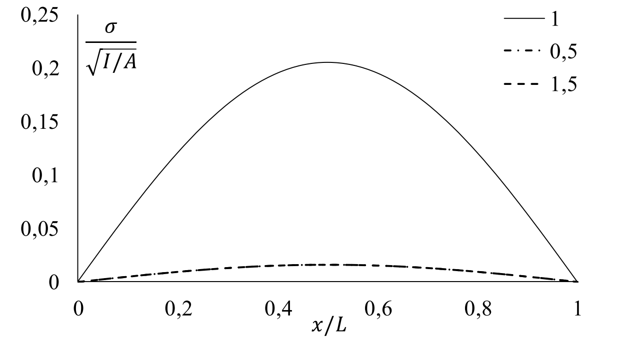

Fig. 5. Normalized standard deviation of the beam displacement for different ratios $\Omega_0/\omega_1, \omega_1$ being the first natural frequency of the system.

A first test of the proposed approach is captured in Fig. 1. The figure shows the approximate response standard deviation (continuous line) vis-à-vis the Monte Carlo data (circles) and the response calculated by a linear solution obtained by neglecting the nonlinear contributions (dotted line). In this numerical example the fractional derivative element has order $\alpha = 1$, and the frequency of the harmonic component of the excitation is $\Omega_0 = \omega_1$. It is seen that the method is able to capture the second-order statistics of the response quite reliably.

The influence of the harmonic component is captured in Fig. 2. The figure shows the results computed by calculating the response to either combined loads or to only random loads. Obviously, the only random case reduces to the problem investigated by Spanos and Malara [53]. The figure highlights the fact that the harmonic component of the load induces significant amplifications of the beam response that are well predicted by the approximate method. Fig. 3 shows a similar comparison, but with a spectral level $S_0 = 1$ (N/m)^{2}s. The spectral level is one order of magnitude higher than in the previous example, but the agreement between the numerical results and the approximate ones is again manifested.

The influence of the fractional derivative element is captured in Fig. 4. The figure shows the results associated with various fractional derivative orders. In this context, it is seen that there are limited discrepancies between the data and the approximate solution. Therefore, it is concluded that the approach can be applied irrespective of the fractional derivative order of the system.

A final numerical example is shown in Fig. 5. The figure highlights the relevance of the frequency of the harmonic excitation. It compares results associated with three different values of the frequency Ω_0: $\Omega_0 = \omega_1$; $\Omega_0 = 0.5\omega_1$; and $\Omega_0 = 1.5\omega_1$. The result emphasizes the fact that significant amplifications of the response are associated with excitations close to the natural frequencies of the system.

Similar conclusions can be drawn for the plate vibration problem. In this case, the system parameters are shown in Table 2. Further, shown in Fig. 6 are results for the case of simply supported stress free edges. In this context, the modes of vibration are given by the equation

$$U_{mn} = \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right), \text{ for } M, N = 1, 2, \ldots \quad (56)$$

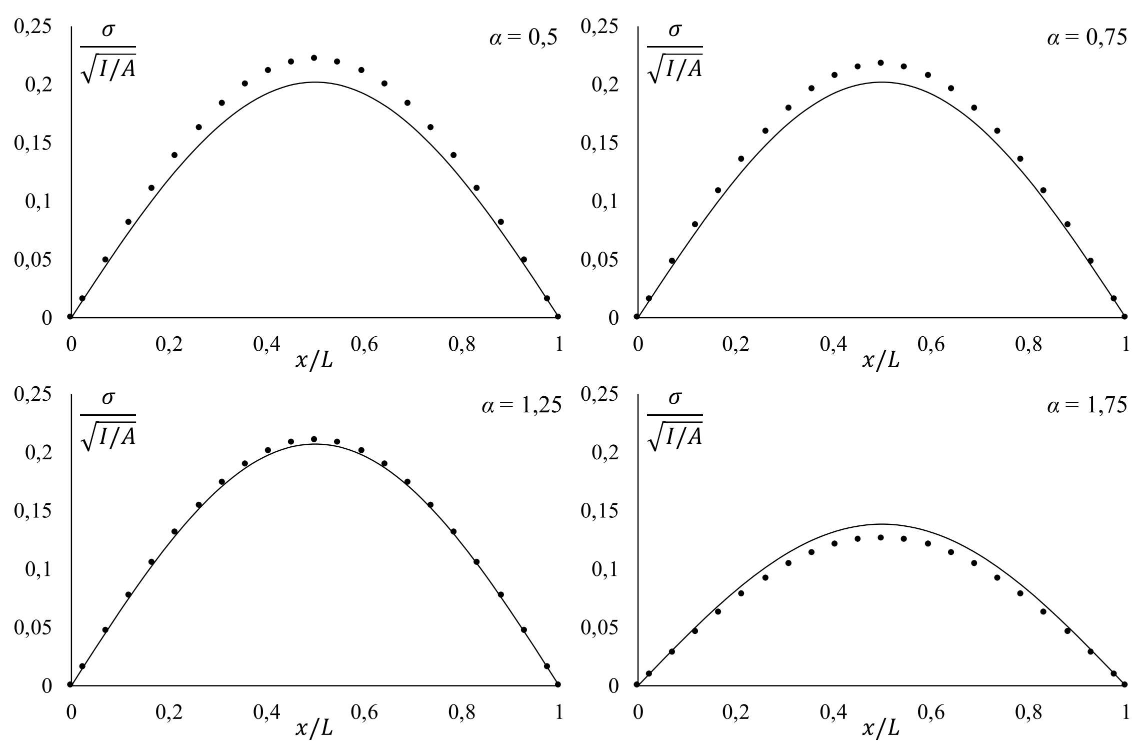

Fig. 4. Comparison between approximate (continuous lines) and numerical (circles) standard deviations of the beam response for various fractional derivative orders α.

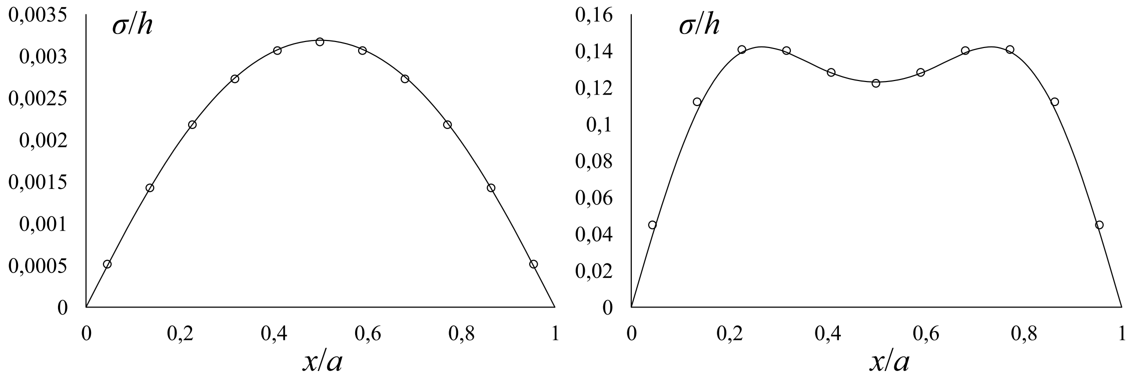

Fig. 6. Vertical displacement of the plate centre. Left panel: plate excited by a random excitation; right panel: plate excited by a combination of harmonic and random excitation.

Table 2
Plate system parameters.

$a = b$	10 m
h	0.1 m
E	2.1×10^{11} Pa
ρ	2355 kg/m^3
D	1.92×10^7 Pa m^3
c	5×10^5 N/(m/s)$^\gamma$

and the natural frequencies are calculated as

$$\omega_{mn}^2 = \frac{D}{\rho h}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]. \tag{57}$$

The amplitude of the harmonic load is $f_0 = 50 \times 10^4$ N/m^2 and its frequency is $\Omega_0 = \omega_{11}$. The random load is compatible with a white noise spectrum having spectral level $S_0 = 5000$ (N/m^2)^{2}s. Note that Fig. 6 compares the results obtained by considering only random load (left panel) and combined load (right panel). It demonstrates that even in this case the method can capture quite well the response statistics. Moreover, it emphasizes the fact that introducing a harmonic component in the system excitation may lead to significant changes of the system response. Indeed, not only the magnitude of the maximum response changes, but also its distribution over the plate domain.

5. Concluding remarks

The vibrations of continua excited by combinations of harmonic and random loads have been considered. Specifically, a vibrating beam and a vibrating plate excited by uniformly distributed loads have been examined. Both systems are described by nonlinear partial differential equations and are endowed with fractional derivative elements. In this regard an iterative approximate approach for estimating their response statistics has been developed. The approach is based on a combination of the harmonic balance and of the statistical linearization concepts. Specifically, the response has been expanded via orthogonal linear modes with time-dependent amplitudes. Then, by exploiting the mode orthogonality property, a system of nonlinear ordinary differential equations governing the time-variation of the amplitudes has been derived. Next, these amplitudes have been represented as a combination of harmonic and random components. The harmonic part of the response has been treated by relying on the harmonic balance technique, while the random part of the response has been treated by utilizing the statistical linearization technique. This last step allows determining linear systems equivalent to the original ones in a mean square error minimization sense that are used for estimating the second-order response statistics in conjunction with the harmonic part of the response.

Numerical examples involving the vibration of a simply supported beam and of a plate with simply supported stress-free edges have confirmed the reliability of the proposed approach by juxtaposition with pertinent Monte Carlo studies.

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