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On nonlinear modeling of an acoustic metamaterial

A.V. Porubov[a,b,*], E.F. Grekova[a]

[a] Institute for Problems in Mechanical Engineering, Bolshoy 61, V.O., Saint-Petersburg, Russia

[b] Peter the Great St. Petersburg Polytechnic University (SPbPU), Polytechnicheskaya st., 29, Saint Petersburg, Russia

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ABSTRACT

Two kinds of nonlinearity are studied in the framework of a model of a discrete diatomic model of an acoustic metamaterial. It is shown that the continuum limit of a discrete model is similar to the equations obtained using a model of a reduced continuum with a microstructure. An asymptotic approach is developed to obtain a modulation nonlinear governing equation for dynamical processes in an acoustic metamaterial. New metamaterial features caused by nonlinearity are found.

1. Introduction

The interest to the study of the acoustic metamaterials has grown considerably in the recent years, see [1–6] and references therein. The linear acoustic metamaterials draw more attention [2,3,7,8], however, more and more works devoted to the nonlinear acoustic metamaterials appear in the last years [2,9–13]. An influence of nonlinearity on the frequency band gap, effective negative modulus and density allows to describe new features of metamaterials. The nonlinearities shift the double negative passband into the adjacent modulus single negative forbidden band and transform the metamaterial from an acoustic insulator into an acoustic conductor [12]. Novel nonlinear phenomena affecting the bandwidth are discovered in [9,10]. The amplitude dependent frequency band gaps are noted in [1,2,12].

Metamaterials are usually studied on the basis of discrete lattice models. Theoretical and experimental description of acoustic metamaterials on the basis of mas-in mass discrete system is shown in [2] where nonlinearity is introduced using a nonlinear rigidity of the internal spring in the model. The Helmholtz resonators and membranes periodically distributed along a pipe are considered in [12]. In [13] an acoustic metamaterial model contains a pure Duffing oscillator shell with the internal oscillator. Various metamaterial lattice models are considered in [9,10].

Continuum modeling have been developed in [3,7,8,14,15]. A three-dimensional continuum theory for fibrous mechanical metamaterials is proposed in [3], in which the fibers are assumed to be spatial Kirchhoff rods whose mechanical response is controlled by a deformation field and a rotation field. This leads naturally to a model based on the Cosserat elasticity. Rigidity constraints are introduced that effectively reduce the model to a variant of a second-gradient elasticity theory [3,4]. A metamaterial is modeled in [7] where a homogenization approach based on the asymptotic analysis establishes a connection between the different characteristics at micro- and macroscales like in composite materials with the inclusion embedded in the matrix. A metamaterial model studied in [8], consists of periodic cylindrical cavities carried out from an elastic matrix. In [14] it is shown that some classes of *reduced continua*, e.g. media, whose strain energy depends on the generalized co-ordinates, but does not depend on the gradient of part of them, are the acoustic metamaterials. Continuum modeling of media with a microstructure [14,15] gives rise to the linear model equations whose solutions allow frequency bandgaps (single negative acoustic metamaterials) and/or decreasing parts of the dispersion curves (double negative acoustic metamaterials). Continuum models are usually linear. At the same time nonlinear continuum generalization of the Cosserat model, see, e.g., [16,17], results in derivation of the governing nonlinear dispersion equations which model an influence of a microstructure and nonlinearity on the dispersion properties of a material. A similarity between the description of metamaterials and composites allows to employ the approaches used for nonlinear description of the composite materials, see, e.g., [19]

In this paper we study a nonlinear metamaterial model following both from a discrete consideration [9,10] and a continuum modeling of a microstructure [14,15]. Two kinds of nonlinearity are introduced in the model. An asymptotic transformation of the original equations results in a nonlinear governing modulation equation for a nonlinear acoustic metamaterial.

* Corresponding author at: Institute for Problems in Mechanical Engineering, Bolshoy 61, V.O., Saint-Petersburg, Russia.

E-mail address: alexey.porubov@gmail.com (A.V. Porubov).

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2. Statement of the problem

Consider a discrete problem of a di-atomic chain studied previously in [9,10]. Let us modify this model be an inclusion of an additional nonlinearity caused by the non-Hookean interactions of the springs between masses m in the chain. The additional masses m_1 are attached by springs to each mass m in the chain, see [9,10]. Masses m_1 do not interact directly between themselves, thus corresponding on the continuum level to the distributed dynamic absorber, while interacting masses m, after continualization, will correspond to the bearing continuum [14]. The displacement of the mass, m, with the number n is denoted by x_n, while that of m_1 is denoted by y_n. Then the discrete equations of motion are

$$\ddot{x}_n = \beta_0(x_{n-1} - 2x_n + x_{n+1}) + \eta\beta_1(y_n - x_n) + \eta\beta_2(y_n - x_n)^2 +$$

$$\beta_3((x_{n+1} - x_n)^2 - (x_{n-1} - x_n)^2), \tag{1}$$

$$\ddot{y}_n = -\beta_1(y_n - x_n) - \beta_2(y_n - x_n)^2. \tag{2}$$

Here $\eta = m_1/m$, while the linear stiffness of the spring of the chain is $\beta_0 m$, the nonlinear stiffness is $\beta_3 m$. Corresponding linear and nonlinear stiffnesses of the attached spring are $\beta_1 m_1$, $\beta_2 m_1$ respectively. We exclude dissipative parts in the model of [9,10] for simplicity.

It is known that discrete nonlinear models are difficult for an analysis, and we proceed with a continuum limit of Eqs. (1) and (2). Following the standard procedure we introduce the continuum functions $u(x, t)$, $v(x, t)$ for description of the displacements of the masses m, m_1 with the number n while the continuum displacements of the neighboring masses are sought using the Taylor series around u. Retaining only the first nonzero term in the expansion we obtain

$$u_{tt} = \beta_0 h^2 u_{xx} + \eta\beta_1(v - u) + \eta\beta_2(v - u)^2 + 2\beta_3\ h u_x u_{xx}, \tag{3}$$

$$v_{tt} = -\beta_1(v - u) - \beta_2(v - u)^2, \tag{4}$$

where h is a distance between the masses m in the chain. The linearized version of Eqs. (3), (4) is similar to that of the equations for a reduced Cosserat model for metamaterials obtained in [14,15].

3. Derivation of governing nonlinear modulation equation

We consider a weakly nonlinear case. For this purpose a small parameter ε is introduced, and the solution is sought in the form

$$u = \varepsilon u_0 + \varepsilon^2 u_1 + \varepsilon^3 u_2 + \ldots,$$

$$v = \varepsilon v_0 + \varepsilon^2 v_1 + \varepsilon^3 v_2 + \ldots.$$

Moreover we introduce fast and slow variables, so as $u_i = u_i(x, t, T, X, \tau)$, $v_i = v_i(x, t, T, X, \tau)$, $T = \varepsilon t$, $X = \varepsilon x$, $\tau = \varepsilon^2 t$. Then we obtain coupled linear equations from Eqs. (3) and (4) at order ε,

$$u_{0,tt} = \beta_0 h^2 u_{0,xx} + \eta\beta_1(v_0 - u_0), \tag{5}$$

$$v_{0,tt} = -\beta_1(v_0 - u_0). \tag{6}$$

The solution to Eqs. (5) and (6) is sught in the form,

$$u_0 = A(X, T, \tau)\exp(\imath(px - \omega t)) + (^*),$$

$$v_0 = B(X, T, \tau)\exp(\imath(px - \omega t)) + (^*), \tag{7}$$

where $(^*)$ is a complex conjugate. Substitution of Eq. (7) into Eqs. (5) and (6) allows us to resolve A from Eq. (6),

$$A = \frac{\beta_1 - \omega^2)}{\beta_1}\ B. \tag{8}$$

Eq. (7) with the use of Eq. (8) gives rise to the dispersion relation,

$$\omega^2 = \frac{\beta_1(1+\eta) + \beta_0 p^2\ h^2}{2} \pm$$

$$\frac{1}{2}\sqrt{(\beta_1(1+\eta) + \beta_0 p^2 h^2)^2 - 4\beta_0\beta_1 p^2 h^2}. \tag{9}$$

Next order equations at ε^2 are

$$u_{1,tt} = \beta_0 h^2 u_{1,xx} + \eta\beta_1(v_1 - u_1) -$$

$$2u_{0,tT} + 2\beta_0 h^2 u_{0,xX} + \eta\beta_2(v_0 - u_0)^2 + 2\beta_3\ h u_{0,x} u_{0,xx}, \tag{10}$$

$$v_{1,tt} = -\beta_1(v_1 - u_1) - 2v_{0,tT} - \beta_2(v_0 - u_0)^2. \tag{11}$$

The solution to Eqs. (10) and (11) is sought in the form,

$$u_1 = A_1(X, T, \tau)\exp(\imath(px - \omega t)) + Q_1(X, T, \tau)\exp(2\imath(px - \omega t)) +,$$

$$(^*) + F_1(X, T, \tau), \tag{12}$$

$$v_1 = B_1(X, T, \tau)\exp(\imath(px - \omega t)) + Q_2(X, T, \tau)\exp(2\imath(px - \omega t)) +$$

$$(^*) + F_2(X, T, \tau). \tag{13}$$

Substituting Eqs. (12) and (13) into Eqs. (10) and (11) and equating to zero the terms at corresponding exponents, we obtain the coupled equations for finding A_1, B_1.F_1, F_2, Q_1, Q_2. We obtain using Eq. (8)

$$F_1 = F_2 + \frac{\beta_2\omega^4}{\beta_1^3} BB^*, \tag{14}$$

$$Q_1 = \frac{\beta_1 - 4\omega^2}{\beta_1} Q_2 + \frac{\beta_2\omega^4}{\beta_1^3} B^2. \tag{15}$$

$$Q_2 = \frac{\left(\eta\beta_2\omega^4 - 2\imath\beta_3\ h p^3(\omega^2 - \beta_1)^2\right)B^2}{\beta_1(16\omega^4 - 4[\beta_1(1+\eta) + 4\beta_0 h^2 p^2]\omega^2 + 4\beta_0\beta_1 h^2 p^2)}. \tag{16}$$

However, equating terms at $\exp(\imath(px - \omega t))$ to zero in Eqs. (10) and (11) does not result in the solutions for A_1, B_1 due to Eq. (9). Indeed, it follows from Eq. (11) that the terms at $\exp(\imath(px - \omega t))$ being equating to zero, give rise to the equation

$$(\beta_1 - \omega^2)B_1 - \beta_1 A_1 - 2\imath\omega B_T = 0, \tag{17}$$

whose solution is

$$A_1 = \frac{\beta_1 - \omega^2}{\beta_1} B_1 - \frac{2\imath\omega}{\beta_1} B_T. \tag{18}$$

Substitution of Eq. (18) into equating to zero $\exp(\imath(px - \omega t))$ part of Eq. (10),

$$(\beta_0 h^2 p^2 + \eta\beta_1 - \omega^2)A_1 - \beta_1\eta B_1 = 2\imath(\omega A_T + \beta_0 h^2 p A_X), \tag{19}$$

results in the equation for B. The part proportional to B_1 disappears because of Eq. (9) and we obtain from Eq. (19),

$$\omega(\beta_1(1+\eta) - 2\omega^2 + \beta_0 h^2 p^2)B_T + (\beta_1 - \omega^2)\beta_0 h^2 p B_X = 0. \tag{20}$$

Then $B = B(\theta, \tau)$ where $\theta = X - WT$,

$$W = \frac{(\beta_1 - \omega^2)\beta_0 h^2 p}{\omega(\beta_1(1+\eta) - 2\omega^2 + \beta_0 h^2 p^2)}, \tag{21}$$

and the solution at this order does not contain secular or growing terms.

To complete the solution and define B, we consider at order ε^3 only the problem of suppression the secular terms at $\exp(\imath(px-\omega t))$. Assume that

$$u_2 = A_2(X,T,\tau)\exp(\imath(px-\omega t)) + (^*) + \ldots,$$

$$v_2 = B_2(X,T,\tau)\exp(\imath(px-\omega t)) + (^*) + \ldots$$

Then the same as before algebraic manipulations with the terms at $\exp(\imath(px-\omega t))$ result in the equation for B

$$\imath B_\tau + (\alpha_1 + \imath\alpha_2)B^2B^* + \gamma B_{\theta\theta} = 0, \tag{22}$$

where

$$\alpha_1 = \left(\beta_1^4\omega[\beta_1(\eta-2) + 3\omega^2 - \beta_2 h^2p^2\right)^{-1}$$

$$\left(\omega^2[\beta_1(1+\eta) + 4(\beta_0h^2p^2\beta_1 - \omega^2)] - \beta_0h^2p^2\beta_1\right)^{-1}$$

$$\left(2\beta_1^2\beta_3^2\ \ h^2p^6(\beta_1 - 4\omega^2)(\beta_1-\omega^2) - 2\beta_2^2\omega^6(2\beta_1\eta + \omega^2 - \beta_0h^2p^2)\right.$$

$$\left.[12\omega^4 - \beta_1\omega^2(3+4\eta) + 3\beta_0h^2p^2(\beta_1 - 4\omega_2)]\right),$$

$$\alpha_2 = \left(\beta_1^4\omega[\beta_1(\eta-2) + 3\omega^2 - \beta_2h^2p^2\right)^{-1}$$

$$\left(\omega^2[\beta_1(1+\eta) + 4(\beta_0h^2p^2\beta_1 - \omega^2)] - \beta_0h^2p^2\beta_1\right)^{-1}$$

$$\left(\beta_1\beta_2\beta_3\ hp^3\omega^4(\beta_1-\omega^2)[20\omega^2 - 7\beta_1^2\eta - 8\beta_1\omega +\right.$$

$$\left.4\beta_0h^2p^2(2\beta_1 - 5\omega^2)]\right),$$

$$\gamma = \frac{1}{\omega^3(\beta_1(\eta-2) + 3\omega^2 - \beta_0h^2p^2)(\beta_1(1+\eta) - 2\omega^2 + \beta_0h^2p^2)^2}$$

$$\left(\beta_0h^2p^4[\beta_0^2\ \ h^4[\beta_1^2 + \beta_1\omega^2 - 2\omega^4] - \omega^2(\beta_1(1+\eta) - 2\omega^2]^2(\beta_1-\omega^2) +\right.$$

$$\left.\beta_0h^2p^2(\omega^2-\beta_1)[\beta_1^2(\eta-1) - 3\beta_1(1+\eta)\omega^2 + 6\omega^4]\right).$$

At $\alpha_2 = 0$ Eq. (22) is the well-known Nonlinear Schrōdinger equation.

4. Discussion

Analysis of dispersion relation (9) reveals two dispersion curves, one starting from zero at $p=0$ (acoustic branch) and another one at $\omega = \sqrt{\beta_1(1+\eta)}$ (optic branch). At $p\to\infty$ the acoustic branch tends to $\omega \to \sqrt{\beta_1\eta}$, hence a bandgap between the curves always exists. This feature is typical for a linear acoustic metamaterial.

Nonlinearity brings new features in the solution but makes analytical study more complicated. Only particular solutions can be obtained. The solution to Eq. (22) is sought in the form

$$B = P(\theta,\tau)\exp(\imath\varphi(\theta,\tau)), \tag{23}$$

where P and φ are real functions. Substituting Eq. (23) into Eq. (22) and separating real and imaginary parts we obtain

$$\gamma P_{\theta\theta} - P\varphi_\tau + \alpha_1P^3 - \gamma P\varphi_\theta^2 = 0, \tag{24}$$

$$P_\tau + \alpha_2P^3 + 2\gamma P_\theta\varphi_\theta + \gamma P\varphi_{\theta\theta} = 0. \tag{25}$$

For the nonlinear Schrōdinger equation, $\alpha_2 = 0$, or only one of nonlinearities is taken into account in the model, either β_2 or β_3 is zero. In this case bright or dark solitary wave solution exists depending on the sign of the product $\alpha_1\gamma$,

$$P = a\ \mathrm{sech}\left(\sqrt{-\frac{2\alpha_1}{\gamma}}\ a(\theta - 2\gamma q\tau)\right),\ \ \varphi = q(\theta - 2\gamma q\tau),$$

$$P = a\ \tanh\left(\sqrt{-\frac{2\alpha_1}{\gamma}}\ a(\theta - 2\gamma q\tau)\right),\ \ \varphi = q(\theta - 2\gamma q\tau).$$

The solution describes a dependence between the amplitude and the wave number, also the shape of modulation of the harmonic wave profile is defined by the sign of the equation coefficients. This is not covered within the linearized consideration. These solutions do not exist in the case of nonzero α_2 corresponding to a degenerate case of the Ginzburg- Landau equation (GLE) see [18] and references therein. Nonzero α_2 appears due to a mutual influence of nonlinearities of the chain, $\beta_3 \neq 0$, and of the internal oscillator, $\beta_2 \neq 0$. The solutions similar to the known localized solutions to the GLE require complex coefficient at the dispersion term and the presence of the linear term proportional to B in Eq. (22). This may be achieved by further modification of the original model by including the dissipative factors in it.

The approach developed for obtaining governing nonlinear equations in the paper may be used for a modeling of more complicated metamaterials like those based on a pantographic model [20,21].

Declaration of Competing Interest

There is no any conflict of interest.

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