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Variational approaches for bending and buckling of non-local stress-driven Timoshenko nano-beams for smart materials

R. Luciano[a], A. Caporale[b,*], H. Darban[a], C. Bartolomeo[a]

[a] Dipartimento di Ingegneria, Universitá degli Studi di Napoli Parthenope, Via Amm. F. Acton 38, Napoli - 80133, Italy
[b] Department of Civil and Mechanical Engineering, University of Cassino and Southern Lazio, Via G. Di Biasio 43, Cassino - 03043, Italy

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ABSTRACT

In this work, variational formulations are proposed for solving numerically the problem of bending and buckling of Timoshenko nano-beams. The present work belongs to research branch in which the non-local theory of elasticity has been used for analysis of beam-like elements in smart materials, micro-electro-mechanical (MEMS) or nano-electro-mechanical systems (NEMS). In fact, the local beam theory is not adequate to describe the behavior of beam-like elements of smart materials at the nano-scale, so that different non-local models have been proposed in last decades for nano-beams. The nano-beam model considered in this work is a convex combination (mixture) of local and non-local phases. In the non-local phase, the kinematic entities in a point of the nano-beam are expressed as integral convolutions between internal forces and an exponential kernel. The aim is to construct a functional whose stationary condition provides the solution of the problem. Two different functionals are defined: one for the pure non-local model, where the local fraction of the mixture is absent, and the other for the mixture with both local and non-local phases. The Euler equations of the two functionals are derived; then, attention focuses on the mixture model. The functional of the mixture depends on unknown Lagrange multipliers and the Euler equations of the functional provide not only the governing equations of the problem but also the relationships between these Lagrange multipliers and the other variables on which the functional depends. In fact, approximations of the variables of the functional can not be chosen arbitrarily in numerical analyzes but have to satisfy suitable conditions. The Euler equations involving the Lagrange multipliers are essential in the numerical analyzes and suggest the correct approximations that have to be adopted for Lagrange multipliers and the other unknown variables of the functional. The proposed method is verified by comparing numerical solutions with exact solutions in bending problem. Finally, the method is used to determine the buckling load of Timoshenko nano-beams with mixture of phases.

1. Introduction

Smart materials (such as transducers, sensor, actuators, etc.) are used in many practical applications. The requirement of smallness of devices made of smart materials has led to the production of micro-electro-mechanical systems (MEMS) made of components between 1 and 100 μm and nano-electro-mechanical systems (NEMS) with dimensions in nanoscale. Small size effects are expected to occur in small structures composing these systems and many authors have predicted these effects using non-local theory. Sedighi et al. [1] have employed Gurtin-Murdoch model and Eringen's elasticity in order to consider the surface energy and non-local effect, respectively, in the dynamic stability of double-sided NEMS. Ebrahimi and Dabbagh [2] have proposed a non-local strain gradient theory to capture size effects in wave propagation analysis of compositionally graded smart nano-plates. Mahinzare et al. [3] have modeled free vibration of a rotating circular nano-plate made of two directional functionally graded piezo materials (two directional FGPM) employing the modified couple stress theory. Rezazadeh et al. [4] have used a non-local EulerBernoulli beam model based on the theory of non-local elasticity to study the thermoelastic damping in a nano-beam considered as a beam-type of NEMS. Theory accounting for size effects and internal length of the material are widely used and applied to objects of different sizes [5].

Recently, Romano and Barretta [6] have introduced a stress-driven non-local integral model, where the bending field is the input variable so that the constitutive law is evaluated by

* Corresponding author.
E-mail addresses: raimondo.luciano@uniparthenope.it (R. Luciano), a.caporale@unicas.it (A. Caporale).

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convolution between bending field and an averaging kernel, solving a ill-posedness of non-local problems in nano-mechanics [7]. Barretta et al. [8] have improved the stress-driven model introducing a mixture of local and non-local phases for the bending problem of Timoshenko nano-beams. In this work, we present variational methods for the numerical solution of the stress-driven model with mixture of phases and apply the methods to the bending and buckling of Timoshenko nano-beams. The present work belongs to research branch in which the non-local theory of elasticity has been used for the analysis of beam-like elements in smart materials, MEMS or NEMS. The aim of this work is to construct and use simple functionals whose stationary condition provides the solutions for nano-beams in stress-driven approach. Our attention focuses on the functional for the mixture of local and non-local phases. The constitutive equations of the adopted stress-driven model are briefly described in Section 2. In the stress-driven model, bending curvature and shear deformation are expressed as integral convolutions between bending moment and shear force, respectively, and an averaging kernel. The convolutions involve a non-local behavior where kinematic entities at a point of the beam depend on internal forces in a neighborhood of that point. The integral formulation of the stress-driven mixture admits an equivalent differential formulation, which is described in Section 2 and on which the proposed variational methods are based. Sections 3 and 4 introduce two different functionals and derive their Euler equations. The functionals are constructed by adding the product between Lagrange multipliers and suitable expressions, representing the constitutive relations of the adopted stress-driven model, to the total potential energy (TPE). The first functional is presented in Section 3 for a stress-driven model without the local phase (no mixture): in this case, the constitutive laws of the stress-driven model are inserted directly in the strain energy of the TPE and only the constitutive boundary conditions of the stress-driven model have to be added in the functional through Lagrange multipliers. The second functional is defined in Section 4 for a stress-driven model with mixture of local and non-local phases: here, the constitutive laws of the stress-driven mixture model are added to the functional through Lagrange multipliers: this simple operation involves the addition of further unknown variables whose meaning has to be clearly determined through the derivation of the Euler Equations. In Section 5, numerical methods are used to find approximate solutions of the problem through the stationary conditions of the functional presented in previous Section 4. The functional depends on unknown variables approximated with linear combinations of functions belonging to a base. In the mixture model, the derivation of the Euler equations of the functional is necessary to choose correctly the functions of the linear combinations approximating the unknown variables, on which the functional depends on. In fact, the functions of the linear combinations for some unknown variables (e.g. the Lagrange multipliers) in the mixture model can not be chosen arbitrarily but they have to respect consistency conditions emerged from the Euler equations of the functional. The derivation of the Euler equations may be tedious, however the numerical method for imposing the stationary condition of the functional is easy to implement. The proposed method can be applied to cases where exact solutions are not yet available in literature or may be difficult to find. In Section 5, Timoshenko nano-beams based on stress-driven mixture and subject to transverse load and/or critical axial load are considered. For nano-beams in bending, the proposed numerical solutions are compared with exact solutions present in literature. Finally, the method is applied for evaluating the buckling load of Timoshenko nano-beams, for which exact solutions are not yet available in literature.

2. Stress-driven model

We briefly describe the equations of the stress-driven model, for which we develop variational formulations in the next sections. The model is a convex combination (mixture) of local and non-local phases, with the non-local phases expressed as integral convolutions between internal forces and an exponential kernel depending on a scale parameter $L_c > 0$. Barretta et al. [9] have shown that the mixture integral relationship between bending curvature χ and bending moment M is equivalent to the second-order differential equation

$$\chi''(x) - \frac{1}{L_c^2}\chi(x) = \alpha\frac{M''(x)}{EI} - \frac{1}{L_c^2}\frac{M(x)}{EI}, \tag{1}$$

with the following constitutive boundary conditions (CBCs)

$$\begin{cases} \chi'(0) - \dfrac{1}{L_c}\chi(0) = \alpha\dfrac{M'(0)}{EI} - \dfrac{\alpha}{L_c}\dfrac{M(0)}{EI}, \\ \chi'(L) + \dfrac{1}{L_c}\chi(L) = \alpha\dfrac{M'(L)}{EI} + \dfrac{\alpha}{L_c}\dfrac{M(L)}{EI}, \end{cases} \tag{2}$$

where EI is the local bending stiffness, $x \in [0, L]$ in (1) and $L > 0$ is the length of the nano-beam. From integral equations, it appears that the parameter α governs the quantity of the phases in the mixture: $\alpha = 0$ provides a full non-local model without local phase, whereas $\alpha = 1$ provides the classical local model without non-local phase. In this work, the prime symbol $(')$ denotes derivatives with respect to the variable x.

Analogously, Barretta et al. [8] have shown that the mixture integral relationship between shear deformation γ and shear force Q is equivalent to the differential problem governed by the equation

$$\theta''(x) - \frac{\theta(x)}{L_c^2} = \alpha\frac{Q''(x)}{\kappa GA} - \frac{1}{L_c^2}\frac{Q(x)}{\kappa GA} + \phi''(x) - \frac{\phi(x)}{L_c^2}, \tag{3}$$

with the following CBCs

$$\begin{cases} \theta'(0) - \dfrac{\theta(0)}{L_c} = \alpha\dfrac{Q'(0)}{\kappa GA} - \dfrac{\alpha}{L_c}\dfrac{Q(0)}{\kappa GA} + \phi'(0) - \dfrac{\phi(0)}{L_c}, \\ \theta'(L) + \dfrac{\theta(L)}{L_c} = \alpha\dfrac{Q'(L)}{\kappa GA} + \dfrac{\alpha}{L_c}\dfrac{Q(L)}{\kappa GA} + \phi'(L) + \dfrac{\phi(L)}{L_c}, \end{cases} \tag{4}$$

where GA is the local shear stiffness and κ is the shear factor. The shear deformation is $\gamma(x) = \theta(x) - \phi(x)$, where $\theta(x) = v'(x)$ is the slope of the nano-beam deformed center-line and $\phi(x)$ is the rotation of the transverse section of the beam. The bending curvature is correlated to the rotation: $\chi(x) = \phi'(x)$.

3. Variational formulation of the non-local problem (no mixture)

In this section, we consider the local fractions of the mixtures equal to zero, by setting $\alpha = 0$. With this assumption, relation (1) provides the bending moment M as function of the curvature χ and its second derivative:

$$M(x) = EI\big(\chi(x) - L_c^2\chi''(x)\big) \tag{5}$$

and the relevant CBCs are

$$\chi'(0) - \frac{1}{L_c}\chi(0) = 0, \quad \chi'(L) + \frac{1}{L_c}\chi(L) = 0. \tag{6}$$

In a similar manner, relation (3) provides the shear force Q as function of the shear strain γ and its second derivative:

$$Q(x) = \kappa GA\big(\gamma(x) - L_c^2\gamma''(x)\big) \tag{7}$$

and the relevant CBCs are

$$\gamma'(0) - \frac{1}{L_c}\gamma(0) = 0, \quad \gamma'(L) + \frac{1}{L_c}\gamma(L) = 0. \tag{8}$$

The objective is to construct a functional L_N whose stationary point is the solution satisfying the governing equations of the problem, namely the equilibrium equations and the constitutive relations (5) and (7), together with kinematic boundary conditions and CBCs (6) and (8). The total potential energy (TPE) for beams subject to distributed transverse load q and critical axial load P is

$$\Pi = U + W, \tag{9}$$

where the strain energy U is

$$U = \frac{1}{2}\int_0^L (M(x)\chi(x) + Q(x)\gamma(x))dx \tag{10}$$

and the potential energy W of the external loads is

$$W = -\int_0^L q(x)v(x)dx - \frac{P}{2}\int_0^L \left(v'(x)\right)^2 dx. \tag{11}$$

Replacing $M(x)$ and $Q(x)$ in (10) with the right-hand sides of (5) and (7), respectively, we obtain:

$$U = \frac{EI}{2}\int_0^L \left(\chi - L_c^2\chi''\right)\chi\, dx + \frac{\kappa GA}{2}\int_0^L \left(\gamma - L_c^2\gamma''\right)\gamma\, dx, \tag{12}$$

where the dependence on x of the variables χ and γ is omitted for brevity. Total potential energy Π with strain energy U given by (12) is a functional depending on $v(x)$ and $\phi(x)$, which are the unknowns of the problem. Denoting the first variation of Π as $\delta\Pi$, it can be shown that the condition $\delta\Pi = 0$ provides the equilibrium equations in terms of the kinematic entities $v(x)$ and $\phi(x)$, if $v(x)$ and $\phi(x)$ already satisfy CBCs as well as kinematic boundary conditions. Therefore, Π with substitutions (5) and (7) could be the searched functional. Imposing the satisfaction of the CBCs before solving the stationary problem $\delta\Pi = 0$ may be tedious. Therefore, we construct a simple functional L_N whose stationary point satisfies the governing equations of the problem and the CBCs. The functional L_N is obtained by adding suitable terms to the total potential energy Π:

$$L_N = \Pi + \sum_{i=1}^{4} \mu_i b_i, \tag{13}$$

where μ_i is the Lagrange multiplier associated with the constitutive boundary expression b_i and

$$b_1 = \chi'(0) - \frac{1}{L_c}\chi(0), \qquad b_2 = \chi'(L) + \frac{1}{L_c}\chi(L),$$
$$b_3 = \gamma'(0) - \frac{1}{L_c}\gamma(0), \qquad b_4 = \gamma'(L) + \frac{1}{L_c}\gamma(L).$$

Next, we write the stationary condition of the functional L_N with strain energy U given by (12). After several integrations by parts, the stationary condition becomes

$$\begin{aligned} 0 = \delta L_N = &-\int_0^L \left(M' + Q\right)\delta\phi\, dx - \int_0^L \left(Q' + q - Pv''\right)\delta v dx \\ &+ \left[M\delta\phi\right]_0^L + \left[\left(Q - Pv'\right)\delta v\right]_0^L + \sum_{i=1}^{4}\left(\delta\mu_i b_i + \mu_i \delta b_i\right) \\ &- \frac{EI}{2}L_c^2\left\{\left[\chi\delta\chi'\right]_0^L - \left[\chi'\delta\chi\right]_0^L\right\} \\ &- \frac{\kappa GA}{2}L_c^2\left\{\left[\gamma\delta\gamma'\right]_0^L - \left[\gamma'\delta\gamma\right]_0^L\right\}, \end{aligned} \tag{14}$$

where M and Q are given by the right-hand sides of (5) and (7), respectively. The Euler equations of the functional L_N are obtained by setting the coefficients of $\delta\phi$ and δv in the integrals and the coefficients of $\delta\mu_i$ to zero in (14) and by observing that Lagrange multipliers μ_i for $i = 1, \ldots, 4$ are arbitrary parameters so that the coefficients of $\delta\chi(\bar{x})$, $\delta\chi'(\bar{x})$, $\delta\gamma(\bar{x})$ and $\delta\gamma'(\bar{x})$ for $\bar{x} = 0, L$ also have to be equal to zero. Concluding, the Euler equations of L_N are:

$$\delta\phi: \quad M' + Q = 0, \quad \text{in } 0 < x < L, \tag{15}$$

$$\delta v: \quad Q' + q - Pv'' = 0, \quad \text{in } 0 < x < L, \tag{16}$$

$$\delta\mu_i: \quad b_i = 0 \quad \text{for } i = 1, \ldots, 4, \tag{17}$$

$$\delta\chi(0): \quad e\,\chi'(0) - \frac{\mu_1}{L_c} = 0, \tag{18}$$

$$\delta\chi'(0): \quad -e\,\chi(0) + \mu_1 = 0, \tag{19}$$

$$\delta\chi(L): \quad -e\,\chi'(L) + \frac{\mu_2}{L_c} = 0, \tag{20}$$

$$\delta\chi'(L): \quad e\,\chi(L) + \mu_2 = 0, \tag{21}$$

$$\delta\gamma(0): \quad g\,\gamma'(0) - \frac{\mu_3}{L_c} = 0, \tag{22}$$

$$\delta\gamma'(0): \quad -g\,\gamma(0) + \mu_3 = 0, \tag{23}$$

$$\delta\gamma(L): \quad -g\,\gamma'(L) + \frac{\mu_4}{L_c} = 0, \tag{24}$$

$$\delta\gamma'(L): \quad g\,\gamma(L) + \mu_4 = 0, \tag{25}$$

where $e = -\frac{EI}{2}L_c^2$ and $g = -\frac{\kappa GA}{2}L_c^2$. Relations (15) and (16) are the equilibrium equations of the Timoshenko nano-beam subject to distributed transverse load q and critical axial load P. In this work, Q is the shear force normal to the deformed beam axis. Equilibrium equations may be expressed in terms of a shear force V that is normal to the un-deformed beam axis and keeps this constant direction. The relation between Q and V is

$$Q = V\cos\theta + P\sin\theta \simeq V + P\theta = V + Pv'. \tag{26}$$

Eqs. (18) and (19) imply $\mu_1 = L_c\, e\,\chi'(0)$ and $\mu_1 = e\,\chi(0)$, respectively, and $L_c\,\chi'(0) = \chi(0)$, which is the first of the CBCs (6). Analogously, (20)–(25) imply the remaining CBCs introduced in this section. Therefore, Euler equations of the functional L_N are the governing equations of the non-local Timoshenko nano-beam subject to distributed transverse load and axial critical load in stress-driven mode characterized by $\alpha = 0$ (no mixture). In numerical methods, an approximate solution of this problem, which is governed by the above-mentioned equilibrium equations, the constitutive laws (5) and (7), and CBCs (for $\alpha = 0$), can be found by expressing the variables $\phi(x)$ and $v(x)$ as linear combination of given functions and then setting the partial derivatives of L_N, with respect to the unknown coefficients of the combination and the Lagrange multipliers μ_i, to zero. In the mixture model characterized by $0 < \alpha < 1$, it is no longer possible to express the bending moment as a function of only the curvature and its derivative through the constitutive law (5); the same holds for the shear force. In this case, a different formulation is required with the introduction of a new functional depending on $\phi(x)$, $v(x)$ and other new variables.

4. Variational formulation of the mixture problem

In previous Section 3, the strain energy U in (12) depends only on the kinematic entities ϕ and v, thanks to Eqs. (5) and (7), which appear like constitutive laws of the nano-beam. Assuming $0 < \alpha < 1$ in the mixture model defined in Section 2, we have to consider Eqs. (1) and (3) in place of (5) and (7). This task may be accomplished by introducing a new functional L_M where (1) and (3) are imposed through Lagrange multipliers:

$$L_M = \Pi + \int_0^L \left(\sum_{i=1}^{2}\lambda_i(x)\, c_i(x)\right)dx + \sum_{i=1}^{4}\mu_i b_i, \tag{27}$$

where $\lambda_i(x)$ is the Lagrange multiplier associated with the function $c_i(x)$ given by

$$c_1(x) = \chi''(x) - \frac{\chi(x)}{L_c^2} - \alpha\frac{M''(x)}{EI} + \frac{1}{L_c^2}\frac{M(x)}{EI},$$
$$c_2(x) = \gamma''(x) - \frac{\gamma(x)}{L_c^2} - \alpha\frac{Q''(x)}{\kappa GA} + \frac{1}{L_c^2}\frac{Q(x)}{\kappa GA},$$

and μ_i is the Lagrange multiplier associated with the boundary term b_i given by

$$b_1 = \chi'(0) - \frac{1}{L_c}\chi(0) - \alpha\frac{M'(0)}{EI} + \frac{\alpha}{L_c}\frac{M(0)}{EI},$$
$$b_2 = \chi'(L) + \frac{1}{L_c}\chi(L) - \alpha\frac{M'(L)}{EI} - \frac{\alpha}{L_c}\frac{M(L)}{EI},$$
$$b_3 = \gamma'(0) - \frac{1}{L_c}\gamma(0) - \alpha\frac{Q'(0)}{\kappa GA} + \frac{\alpha}{L_c}\frac{Q(0)}{\kappa GA},$$
$$b_4 = \gamma'(L) + \frac{1}{L_c}\gamma(L) - \alpha\frac{Q'(L)}{\kappa GA} - \frac{\alpha}{L_c}\frac{Q(L)}{\kappa GA}.$$

Here, M and Q in potential energy Π of (27) are two unknown functions. After several integrations by parts, the stationary condition of L_M becomes

$$\begin{aligned}0 = \delta L_M = \int_0^L \Bigg[&-\left(\frac{M'}{2} + \lambda_1''' - \frac{\lambda_1'}{L_c^2} + \frac{Q}{2} + \lambda_2'' - \frac{\lambda_2}{L_c^2}\right)\delta\phi \\ &- \left(\frac{Q'}{2} + \lambda_2''' - \frac{\lambda_2'}{L_c^2} + q - Pv''\right)\delta v + \left(\frac{\phi'}{2} - \alpha\frac{\lambda_1''}{EI} + \frac{1}{L_c^2}\frac{\lambda_1}{EI}\right)\delta M \\ &+ \left(\frac{v'-\phi}{2} - \alpha\frac{\lambda_2''}{\kappa GA} + \frac{1}{L_c^2}\frac{\lambda_2}{\kappa GA}\right)\delta Q + \sum_{i=1}^{2} c_i\delta\lambda_i\Bigg]dx \\ &+ \left[\left(\frac{M}{2} + \lambda_1'' - \frac{\lambda_1}{L_c^2}\right)\delta\phi\right]_0^L + \left[\left(\frac{Q}{2} + \lambda_2'' - \frac{\lambda_2}{L_c^2} - Pv'\right)\delta v\right]_0^L \\ &+ \left[\lambda_1\delta\phi'' - \lambda_1'\delta\phi'\right]_0^L - \frac{\alpha}{EI}[\lambda_1\delta M' - \lambda_1'\delta M]_0^L \\ &+ \left[\lambda_2\delta\gamma' - \lambda_2'\delta\gamma\right]_0^L - \frac{\alpha}{\kappa GA}[\lambda_2\delta Q' - \lambda_2'\delta Q]_0^L \\ &+ \sum_{i=1}^{4}(\delta\mu_i b_i + \mu_i\delta b_i). \end{aligned} \tag{28}$$

Next, we show that the Euler equations of L_M are the governing equations of the problem based on the mixture defined in Section 2 for $0 \le \alpha \le 1$. Before proceeding with the derivation of the Euler equations, we note that these equations also have another purpose. In next section, the functional L_M will be used to find approximate solutions to some non-local problems. To this end, the functions of the linear combinations approximating the unknown variables $\lambda_1(x)$ and $\lambda_2(x)$ in the functional L_M defined in (27) can not be chosen arbitrarily, but they have to satisfy certain conditions, which depend on the physical meaning of $\lambda_1(x)$ and $\lambda_2(x)$. The Euler equations of L_M in (27) also provide the relationships between the unknown functions $M(x)$, $Q(x)$, $\lambda_1(x)$ and $\lambda_2(x)$, on which L_M depends, and the physical meaning of $\lambda_1(x)$ and $\lambda_2(x)$. The Euler equations of L_M are

$$\delta\phi: \quad \frac{M'}{2} + \lambda_1''' - \frac{\lambda_1'}{L_c^2} + \frac{Q}{2} + \lambda_2'' - \frac{\lambda_2}{L_c^2} = 0, \quad \text{in } 0 < x < L, \tag{29}$$

$$\delta v: \quad \frac{Q'}{2} + \lambda_2''' - \frac{\lambda_2'}{L_c^2} + q - Pv'' = 0, \quad \text{in } 0 < x < L, \tag{30}$$

$$\delta M: \quad \frac{\phi'}{2} - \frac{1}{EI}\left(\alpha\lambda_1'' - \frac{\lambda_1}{L_c^2}\right) = 0, \quad \text{in } 0 < x < L, \tag{31}$$

$$\delta Q: \quad \frac{v'-\phi}{2} - \frac{1}{\kappa GA}\left(\alpha\lambda_2'' - \frac{\lambda_2}{L_c^2}\right) = 0, \quad \text{in } 0 < x < L, \tag{32}$$

$$\delta\lambda_1: \quad c_1 = 0, \quad \text{in } 0 < x < L, \tag{33}$$

$$\delta\lambda_2: \quad c_2 = 0, \quad \text{in } 0 < x < L, \tag{34}$$

in the interior points and

$$\delta\mu_i: \quad b_i = 0 \quad \text{for } i = 1, \dots, 4, \tag{35}$$

$$\delta\chi(0): \quad -\frac{\mu_1}{L_c} + \lambda_1'(0) = 0, \tag{36}$$

$$\delta\chi(L): \quad \frac{\mu_2}{L_c} - \lambda_1'(L) = 0, \tag{37}$$

$$\delta\chi'(0): \quad \mu_1 - \lambda_1(0) = 0, \tag{38}$$

$$\delta\chi'(L): \quad \mu_2 + \lambda_1(L) = 0, \tag{39}$$

$$\delta M(0): \quad \frac{\alpha}{EI}\left(\frac{\mu_1}{L_c} - \lambda_1'(0)\right) = 0, \tag{40}$$

$$\delta M(L): \quad \frac{\alpha}{EI}\left(-\frac{\mu_2}{L_c} + \lambda_1'(L)\right) = 0, \tag{41}$$

$$\delta M'(0): \quad \frac{\alpha}{EI}(-\mu_1 + \lambda_1(0)) = 0, \tag{42}$$

$$\delta M'(L): \quad \frac{\alpha}{EI}(-\mu_2 - \lambda_1(L)) = 0, \tag{43}$$

$$\delta\gamma(0): \quad -\frac{\mu_3}{L_c} + \lambda_2'(0) = 0, \tag{44}$$

$$\delta\gamma(L): \quad \frac{\mu_4}{L_c} - \lambda_2'(L) = 0, \tag{45}$$

$$\delta\gamma'(0): \quad \mu_3 - \lambda_2(0) = 0, \tag{46}$$

$$\delta\gamma'(L): \quad \mu_4 + \lambda_2(L) = 0, \tag{47}$$

$$\delta Q(0): \quad \frac{\alpha}{\kappa GA}\left(\frac{\mu_3}{L_c} - \lambda_2'(0)\right) = 0, \tag{48}$$

$$\delta Q(L): \quad \frac{\alpha}{\kappa GA}\left(-\frac{\mu_4}{L_c} + \lambda_2'(L)\right) = 0, \tag{49}$$

$$\delta Q'(0): \quad \frac{\alpha}{\kappa GA}(-\mu_3 + \lambda_2(0)) = 0, \tag{50}$$

$$\delta Q'(L): \quad \frac{\alpha}{\kappa GA}(-\mu_4 - \lambda_2(L)) = 0, \tag{51}$$

at the boundary points. Next, we show that Euler equations of the functional L_M are the governing equations of the non-local Timoshenko nano-beam subject to distributed transverse load and axial critical load in stress-driven mode characterized by $0 \le \alpha \le 1$ (mixture of local and non-local phases). Expressing χ as function of λ_1 through (31), it results

$$\chi'' - \frac{1}{L_c^2}\chi = \frac{2\alpha}{EI}\left(\lambda_1'' - \frac{\lambda_1}{L_c^2}\right)'' - \frac{2}{L_c^2 EI}\left(\lambda_1'' - \frac{\lambda_1}{L_c^2}\right). \tag{52}$$

Euler Eq. (33) and relation (52) imply

$$\lambda_1'' - \frac{\lambda_1}{L_c^2} = \frac{M}{2}. \tag{53}$$

Similarly, the Lagrange multiplier λ_2 satisfies the relation

$$\lambda_2'' - \frac{\lambda_2}{L_c^2} = \frac{Q}{2}. \tag{54}$$

From (53) and (54), Euler Eqs. (29) and (30) represent the equilibrium equations of the problem, namely

$$M' + Q = 0, \quad Q' + q - Pv'' = 0, \tag{55}$$

and the boundary terms with variations $\delta\phi$ and δv in (28) become

$$\left[\left(\frac{M}{2} + \lambda_1'' - \frac{\lambda_1}{L_c^2}\right)\delta\phi\right]_0^L = [M\delta\phi]_0^L$$
$$\left[\left(\frac{Q}{2} + \lambda_2'' - \frac{\lambda_2}{L_c^2} - Pv'\right)\delta v\right]_0^L = \left[\left(Q - Pv'\right)\delta v\right]_0^L \tag{56}$$

and vanish: e.g. either the transverse displacement v or the shear force $V \simeq Q - Pv' = 0$ can be assigned at each boundary point (the only external transverse load different from zero is the distributed load q).

5. Numerical solutions through the stationary condition of the functional L_M

In this section, approximate solutions of the Timoshenko nano-beams subject to distributed transverse load and/or critical axial load are determined by imposing the stationary condition of the functional L_M defined in Section 4. This functional depends on unknown functions $\phi(x)$, $v(x)$, $M(x)$, $Q(x)$, $\lambda_1(x)$, $\lambda_2(x)$ and unknown constants μ_i for $i = 1, \ldots, 4$. In the proposed numerical method, we express each of the unknown functions as a finite linear combination of given basis functions, paying attention to the fact that some unknown functions are not independent, as shown by relationships (53) and (54). Approximations adopted for rotation ϕ, displacement v, bending moment M, shear force Q and Lagrange multipliers λ_1 and λ_2 are

$$\phi_{n_1}(x) = \psi_0^{(1)}(x) + \sum_{i=1}^{n_1} a_i\, \psi_i^{(1)}(x), \tag{57}$$

$$v_{n_2}(x) = \psi_0^{(2)}(x) + \sum_{i=1}^{n_2} a_{n_1+i}\, \psi_i^{(2)}(x), \tag{58}$$

$$M_{n_3}(x) = \sum_{i=1}^{n_3} a_{\bar{n}_2+i}\, \psi_i^{(3)}(x), \tag{59}$$

$$Q_{n_4}(x) = \sum_{i=1}^{n_4} a_{\bar{n}_3+i}\, \psi_i^{(4)}(x), \tag{60}$$

$$\lambda_{1,n_5}(x) = \sum_{i=1}^{n_5} a_{\bar{n}_4+i}\, \psi_i^{(5)}(x), \tag{61}$$

$$\lambda_{2,n_6}(x) = \sum_{i=1}^{n_6} a_{\bar{n}_5+i}\, \psi_i^{(6)}(x), \tag{62}$$

respectively, where $\psi_i^{(j)}(x)$ for $j = 1, \ldots, 6$ are the basis functions, $\bar{n}_j = \sum_{k=1}^{j} n_k$ and a_i for $i = 1, \ldots, \bar{n}_6$ are unknown real constants. Functions $\psi_i^{(j)}(x)$ for $i = 1, \ldots, n_j$ and $j = 1, 2$ satisfy the homogeneous form of the kinematic boundary conditions, e.g. if $v(0) = 0$ then $\psi_i^{(2)}(0) = 0$ for $i = 1, \ldots, n_2$. The choice of the approximations in (57)–(62) has to take into account the relationships established by Eqs. (33), (34), (53) and (54). This task is accomplished in next subsections after the choice of the basis functions $\psi_i^{(j)}(x)$ for $j = 1, 2$.

The stationary condition of L_M is

$$\delta L_M = \sum_{i=1}^{\bar{n}_6+4} \frac{\partial L_M}{\partial a_i}\delta a_i = 0, \tag{63}$$

where $a_{\bar{n}_6+i} = \mu_i$ for $i = 1, \ldots, 4$. Relation (63) involves the following linear system

$$\frac{\partial L_M}{\partial a_i} = 0 \quad \text{for} \quad i = 1, \ldots, \bar{n}_6 + 4, \tag{64}$$

in the unknowns a_i. Next, numerical solutions obtained by solving the system (64) are compared with exact solutions [8] for Timoshenko nano-beams subject to a distributed transverse load. Finally, the proposed numerical method is used to determine the buckling load of Timoshenko nano-beams based on stress-driven mixture, for which solutions are not yet available in literature.

5.1. Accuracy of the numerical solutions in bending test

Here, we consider Timoshenko cantilever nano-beams with mixture of phases subject to a constant distributed transverse load q. Adopting the following base of functions

$$\psi_i(x) = x^{i-1} \quad \text{for} \quad i \geq 1, \tag{65}$$

the approximations of ϕ and v satisfying the kinematic boundary conditions ($v(0) = \phi(0) = 0$) are

$$\phi_{n_1}(x) = \sum_{i=1}^{n_1} a_i \psi_{i+1}(x), \tag{66}$$

$$v_{n_2}(x) = \sum_{i=1}^{n_2} a_{n_1+i} \psi_{i+1}(x). \tag{67}$$

Since $\phi = v'$ in absence of shear deformations (Euler-Bernoulli beam), we assume $n_2 = n_1 + 1$. Approximations adopted for M and Q have to satisfy relationships (33) and (34), respectively, and are given by (59) and (60) with $n_3 = n_1$, $n_4 = n_2$ and $\psi_i^{(j)}(x) = \psi_i(x)$ for $i = 1, \ldots, n_j$ and $j = 3, 4$. The approximations of the Lagrange multipliers $\lambda_1(x)$ and $\lambda_2(x)$ have to satisfy relationships (53) and (54), respectively. This means that basis functions used for approximating $\lambda_1(x)$ and $\lambda_2(x)$ are the same used for $M(x)$ and $Q(x)$, respectively. Therefore, the approximations of the Lagrange multipliers $\lambda_1(x)$ and $\lambda_2(x)$ are given by (61) and (62), respectively, with $n_5 = n_3$, $n_6 = n_4$ and $\psi_i^{(j)}(x) = \psi_i(x)$ for $i = 1, \ldots, n_j$ and $j = 5, 6$. The numerical solution is obtained from (27) setting $P = 0$ (buckling is not considered in this example).

The solution of the Timoshenko nano-beam subject to a constant distributed transverse load q depends on the following four dimensionless parameters: $p_1 = L_c/L$, $p_2 = \kappa GAL^2/(EI)$, $p_3 = qL/(\kappa GA)$ and α. Next, we assume $p_2 = p_3 = 1$, $\alpha = 0.6$ and compare the numerical solutions with the exact solutions [8] for different values of p_1. In Fig. 1 , the dimensionless transverse displacement v_{n_2}/L is plotted against x/L for $n_2 = n = 3, 4, 5, 6$ and $p_1 = 0.1$. The figure shows that the convergence of the numerical solution improves as n increases, being n the number of basis functions used to approximate the displacement. Same objects are plotted in Fig. 2 but $p_1 = 0.4$. Comparing Figs. 1 and 2, one notes that convergence is better for larger scale parameter $p_1 = L_c/L$. In fact, the blue dashed curve ($n = 5$) practically coincides with the exact solution (black continuous curve) for $p_1 = 0.4$ (Fig. 2) and is quite far from the exact solution for $p_1 = 0.1$ (Fig. 1). A slow convergence is observed for very small values of $p_1 = L_c/L$, but this circumstance is not significant as the solution for L_c/L tending to zero is the local solution. The figures also bring out an important aspect of the stress-driven model: the dimensionless displacements decrease with increasing the scale parameter L_c/L, in accordance with experimental evidences. Omitting the case of very small values of

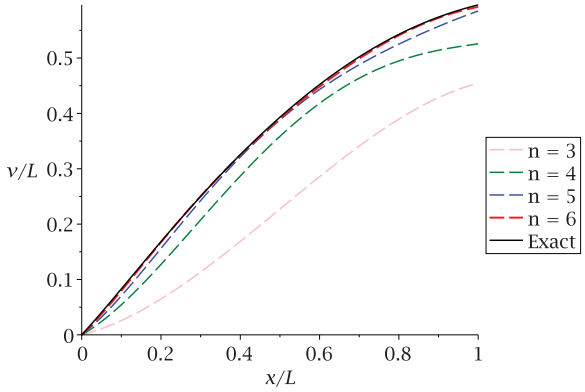

Fig. 1. Comparison between dimensionless approximate and exact displacements for $\alpha = 0.6$ and $p_1 = 0.1$.

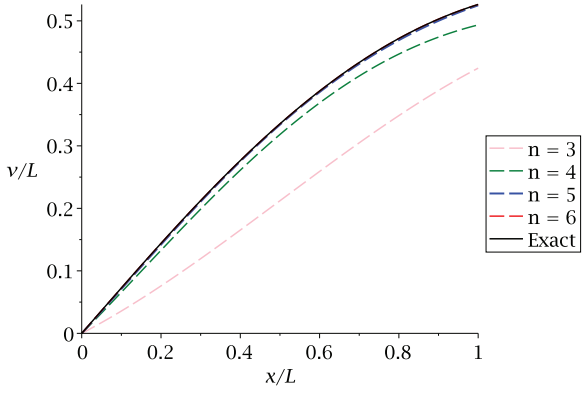

Fig. 2. Comparison between dimensionless approximate and exact displacements for $\alpha = 0.6$ and $p_1 = 0.4$.

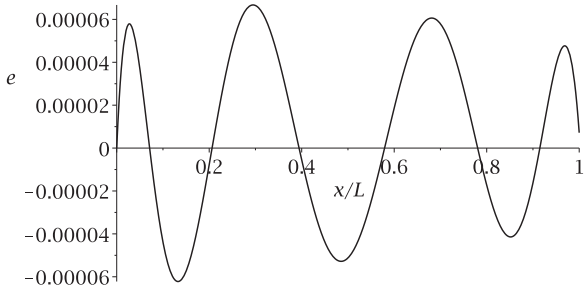

Fig. 3. Difference between dimensionless exact displacement and the approximation v_9/L.

L_c/L, good approximations are already achieved with a small number n of the basis functions. Moreover, the numerical solution approximates the exact solution to any degree of accuracy, as shown in Fig. 3, where the error e defined as the difference between dimensionless exact and approximate displacements,

$$e(x/L) = \frac{v_{exact}(x) - v_{n_2}(x)}{L}, \tag{68}$$

is plotted against x/L for $n_2 = 9$, $\alpha = 0.6$ and $p_1 = 0.1$.

5.2. Buckling load of Timoshenko nano-beams with mixture of phases

Next, we consider simply supported nano-beams. Remembering that Eqs. (33), (34), (53) and (54) represent constraints for the approximations of the variables ϕ, v, M, Q, λ_1 and λ_2, we proceed as done in previous Section 5.1. Firstly, we define the approximations of the kinematic variables ϕ and v. Then, we determine the approximations of the remaining variables, so that constraints imposed by the above-mentioned equations can not be violated. The basis functions $\psi_i^{(2)}$ satisfying the kinematic boundary conditions $(v(0) = v(L) = 0)$ of the simply supported nano-beams are

$$\psi_0^{(2)}(x) = 0, \quad \psi_i^{(2)}(x) = x^i(L - x) \quad \text{for} \quad i = 1, \ldots, n_2. \tag{69}$$

Therefore, consistent approximations (not violating constraints of the problem) for ϕ, v, M, Q, λ_1 and λ_2 are given by (57)–(62), respectively, with $n_2 = n_3 = n_5 = n_1 - 1$, $n_4 = n_6 = n_1$ and $\psi_i^{(j)} = \psi_i(x)$ for $i = 1, \ldots, n_j$ and $j = 1, 3, 4, 5, 6$, where $\psi_i(x)$ is given by (65). We determine the buckling load of Timoshenko nano-beams in stress-driven mixture. The distributed transverse load is absent in this example; therefore, the numerical solution is obtained from (27) setting $q = 0$. Let A_b denote the matrix of the coefficients of the system (64). The only unknown in matrix A_b is the axial force P. The buckling load is denoted by P_{cr} and is the smallest root of the equation $\det(A_b) = 0$ in the unknown P. We express the results in dimensionless form: the solution depends on the four dimensionless parameters $p_1 = L_c/L$, $p_2 = \kappa GAL^2/(EI)$, $p_3 = P/(\kappa GA)$ and α. We assume $\alpha = 0.5$ and $p_2 = 1$. Table 1 reports the dimensionless buckling load $p_{3,cr} = P_{cr}/(\kappa GA)$ for different values of $p_1 = L_c/L$. The table also reports the value of $p_{3,cr}$ for the local beam (LB), i.e in local elasticity; $p_{3,cr}$ for the local beam is

$$p_{3,cr,loc} = \frac{\pi^2 EI}{L^2 \kappa GA} \frac{1}{1 + \frac{\pi^2 EI}{L^2 \kappa GA}} = \frac{1}{1 + \frac{p_2}{\pi^2}}. \tag{70}$$

Table 1
Dimensionless buckling load $p_{3,cr}$ for different values of $p_1 = L_c/L$.

	LB	$p_1 = 0.1$	$p_1 = 0.2$	$p_1 = 0.3$	$p_1 = 0.4$
$p_{3,cr}$	0.908	1.0185	1.176	1.308	1.408

Table 1 shows that the dimensionless buckling load increases with increasing the scale parameter $p_1 = L_c/L$. This trend also appears in other load cases of the stress-driven model and is in accordance with experimental evidences.

6. Conclusions

This work proposes functionals whose stationary condition is the solution of the bending and buckling problems of Timoshenko nano-beams that can be considered as beam-like element in smart materials. The scale effects in nano-beams are considered through a stress-driven model, which overcomes a ill-posedness emerged in non-local treatments of nano-mechanics. Two different functionals are defined: one for the pure non-local model and the other for the mixture with both local and non-local phases. Attention focuses on the functional for the mixture model. This functional also depends on unknown Lagrange multipliers, whose physical meaning and relationships with the other unknown variables of the functional have to be found. In fact, the numerical solution is sought by imposing the stationary condition of the functional depending on unknown variables approximated with finite linear combinations of basis functions. Approximations of the variables of the functional can not be chosen arbitrarily for the mixture model but have to satisfy suitable conditions. The Euler equations of the functional provide the governing equations of the problem and the necessary information to choose correctly the approximations of the unknown variables. The proposed method is verified by comparing numerical solutions with exact solutions in bending problem. Omitting the unimportant case of very small values of the scale parameter L_c/L, good approximations are already achieved with a small number n of the basis functions. Finally, the numerical method is used to determine the buckling load of Timoshenko nano-beams with mixture of phases, for which solutions are not yet available in literature.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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