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Necessary and Sufficient Conditions for State-Space Network Realization

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Realization

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ABSTRACT

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Realization

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This thesis presents the formulation and solution of a new problem in systems and control theory, called the *Network Realization Problem*. Its relationship to other problems, such as *State Realization* [1] and *Structural Identifiability* [2], is shown. The motivation for this work is the desire to completely quantify the conditions for transitioning between different mathematical representations of linear time-invariant systems. The solution to this problem is useful for theorists because it lays a foundation for quantifying the information cost of identifying a system's complete network structure from the transfer function.

ACKNOWLEDGMENTS

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Chapter 1

Introduction

In this work, we focus on linear time-invariant systems described by a state-space model in the following form:

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t),
\end{aligned}
\tag{1.1}
$$

where $x(t) \in \mathbb{R}^n$ is a vector of the states of the system, $u(t) \in \mathbb{R}^m$ is the vector of inputs, $y(t) \in \mathbb{R}^p$ is the vector of outputs, and $\dot{x}(t)$ indicates the time derivative of $x(t)$. We assume that (A, C) is observable and (A, B) is controllable, and therefore that the given state-space model is minimal [3]. A system's transfer function is derived from a state-space model by:

$$
G(s) = C(sI - A)^{-1}B + D.
\tag{1.2}
$$

In this expression s is the Laplace variable, and $G(s)$ is a $p \times m$ matrix of rational functions in s. We say the state-space model (A, B, C, D) is a realization of $G(s)$.

There is a one-to-many relationship between a transfer function and minimal state-space realizations because many state-space realizations can produce the same input-output behavior, which is captured by the transfer function. In particular, there are many other minimal state-space realizations besides that in Equation (1.1), characterized, for example, by $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$, that yield the same transfer function, i.e. such that $G(s) = C(sI - A)^{-1}B + D = \hat{C}(sI - \hat{A})^{-1}\hat{B} + \hat{D}$. These state-space realizations are related to system (1.1) by an invertible

state transformation, X, where

$$\hat{A} = XAX^{-1}, \ \hat{B} = XB, \ \hat{C} = CX^{-1}, \text{ and } \hat{D} = D, \tag{1.3}$$

and any state-space model, $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$, generated from some invertible transformation, X. This state-space system clearly shares the same transfer function $G(s)$ with the system (1.1),

$$
\begin{aligned}
G(s) &= \hat{C}(sI - \hat{A})^{-1}\hat{B} + \hat{D} = CX^{-1}(sXX^{-1} - XAX^{-1})^{-1}X^{-1}B + D \\
&= CX^{-1}X(sI - A)^{-1}XX^{-1}B + D = C(sI - A)^{-1}B + D.
\end{aligned} \tag{1.4}
$$

This thesis precisely characterizes the information about (A, B, C, D) that must be known, a priori, to recover it from $G(s)$, distinct from any of the other minimal realizations of $G(s)$.

Given rich input-output data, system identification algorithms obtain a transfer function. The problem of State Realization is usually then posed as: given a transfer function, $G(s)$, find any state-space realization of the system. Different state realization algorithms give different realizations.

Network structure is how the system is connected together internally, that is, how the states affect each other, affect the outputs, and are affected by the inputs. The network structure is revealed by, or encoded in, a set of state-space matrices, (A, B, C, D) that produced the input-output behavior.The network structure is not necessarily, and is very unlikely to be equal to an arbitrary minimal realization, $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$, that is obtained from $G(s)$ via a state realization algorithm as discussed above.

The State-Space Network Realization Problem is to identify the internal network structure of a system, (A, B, C, D), given its transfer function $G(s)$. More precisely, suppose a state-space system, (A, B, C, D), is fixed and its transfer function, $G(s)$, is given. From $G(s)$ you can obtain an arbitrary realization, $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ using any state realization algorithm, but from $G(s)$ alone one cannot tell whether $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ is the given system, (A, B, C, D), that generated $G(s)$. The question then becomes, what elements of (A, B, C, D), if known a priori,
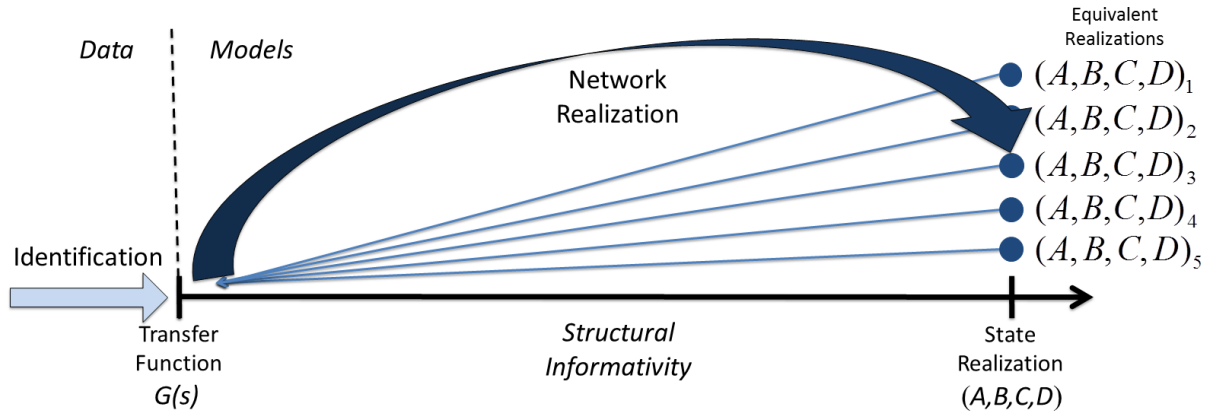
Figure 1.1: A graphical representation of the Network Realization Problem: System Identification produces a transfer function $G(s)$ from data. There are many state-space realizations of a given transfer function $G(s)$. The question is how much information about the true realization must be known a priori in order to recover the whole realization.

allow the whole system structure, (A, B, C, D), to be recovered from $G(s)$. A visualization of the problem can be seen in Figure 1.1. In this work we formulate the problem, first solve the problem for a certain important class of systems, and then expand these results to network realization of the general state-space model, solving the Network Realization Problem.

1.1 Motivation

The main motivation for this work is that it is an open problem in systems theory. This is an important problem for system theorists because it will quantify the number of elements of the state-space matrices that must be known a priori to recover the complete network structure of a system from the transfer function. Also, this work can be connected to applications that have network structure. Specific examples of such applications include:

- Data communication networks: network administrators measure transmission rates at various nodes over time and want to understand the network structure relating them.

- Biochemical reaction networks: scientists measure concentrations of various chemical species over time and want to understand the network structure relating them.

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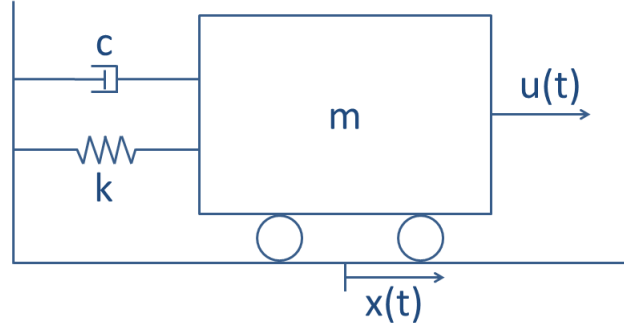
Figure 1.2: This a simple mass spring damper system where m is the mass, c is the dampening coefficient, k is the spring coefficient, x is relative position, and u is the input to the system.

- Financial networks: investors measure prices of various securities over time and want to understand the network structure relating them.

- Social networks: observers measure opinions of various agents on a particular topic over time and want to understand the network structure relating them.

Although these systems are probably best described by nonlinear dynamics, the theory developed in this thesis may be applicable to them near equilibria by virtue of Lyapunov's Indirect Method [3].

1.1.1 State-Space Realizations vs. Physical Systems

This work focuses on the relationship between the state-space representation of linear time-invariant systems and the transfer function. This is motivated by the idea that the state-space model has more physical meaning than the transfer function. However sometimes the physical interpretation does not specify the states.

Example 1. *Consider a simple mass spring damper system, shown in Figure 1.2. The dynamics are explained by Netwon's laws,*

$$m\ddot{x} + c\dot{x} + kx = u, \tag{1.5}$$

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where m is the mass of the block, c is the dampening coefficient, k is the spring coefficient, x is position, and u is the input. We can arbitrarily pick position and velocity as the states, which gives the system

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u
$$
$$
y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.
$$
(1.6)

While the choice of position and velocity is logical for this system, we could have just as easily picked position and position plus velocity as the states, giving the following realization

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ c/m - 1 - k/m & 1 - c/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u
$$
$$
y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.
$$
(1.7)

The construction of the mass spring damper system in Figure 1.2 prefers neither of these two realizations over the other.

While the construction of the system in Example 1 did not prefer one set of states, sometimes the physical system does give a specific set of states preference over other states. We now explore several examples where this occurs.

Example 2. *Consider a simple system with two water tanks in series, depicted in Figure 1.3. Consider the following state equations for this system,*

$$
\dot{x} = \begin{bmatrix} -k_1 & 0 \\ k_1 & -k_2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u,
$$
$$
y = \begin{bmatrix} 0 & k_2 \end{bmatrix} x,
$$
(1.8)

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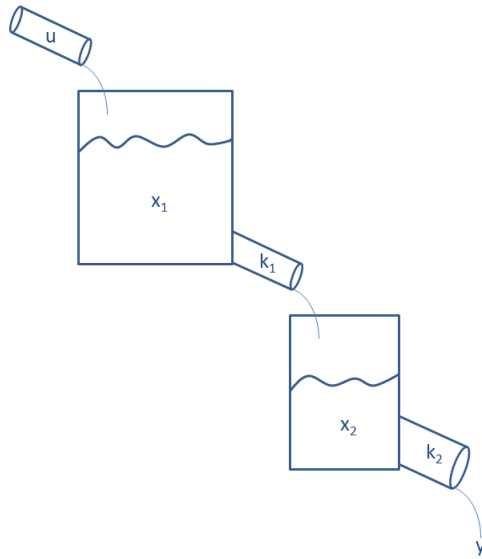
Figure 1.3: Two water tanks in series: This simple example illustrates how a system can have underlying structure by construction.

where the states are the quantity of water in each tank and the k_i's are the flow rates shown in Figure 1.3. These states are specified by the construction of the tanks that are in series and each piece of the system could be separated with each state treated as its own single-state subsystem. Then the realization in Equation 1.8 shows the interaction between these two subsystems. The states could be changed to $x_1 + x_2$ and $x_1 - x_2$ but these cannot be separated and the realization does not show how the two tanks interact.

The important idea to take from Example 2 is that the construction of the water tanks specifies a unique set of states unlike the mass spring damper system in Example 1 where neither of the systems in Equations 1.6 and 1.7 can be preferred over the other due to the physical structure. Since, in Example 2, the construction of the water tanks are separated and in series, choosing the quantities of water being stored in each tank shows the interaction of the two subsystems, that is the tanks, and other choices of states do not portray this structure. In Example 1, the physical construction of the system prefers no state-space realization over others. We now explore two compartment models of pharmacokinetics that, similar to Example 2, have structure that prefers a set of states over others.
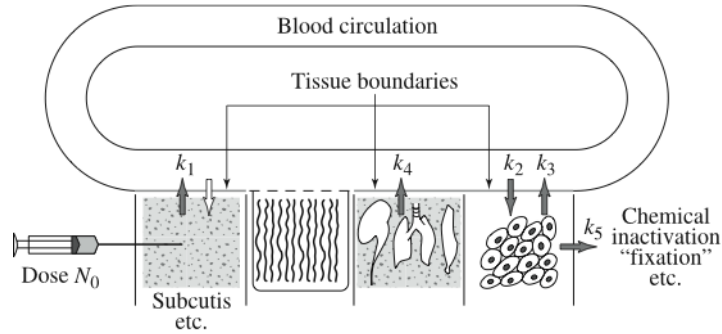
Figure 1.4: [3] This is an abstraction that explains how the body can be modeled as set of compartments separated by membranes and connected by blood flow based on ideas from [4].

Pharmacokinetics is the study of how a drug spreads through the body after it is administered. Compartment models are quite natural for the modeling of many different parts of the human body. The following example of drug administration in the human body is from [3] motivated by [4].

Example 3. [3] *The system that is represented here is the human body. Each part of the body that is important to drug flow through the body can be modeled as a compartment, depicted in Figure 1.4. Each is separated by a membrane and connected by blood circulation. The effect of a drug, e, can be modeled by a simple nonlinear relationship*

$$e = \frac{c_0}{c_0 + c} e_{max}, \tag{1.9}$$

where c is the concentration of the drug. The simplest dynamic model for drug administration is treating the body as one whole compartment

$$V \frac{dc}{dt} = -qc, \quad c \geq 0, \tag{1.10}$$
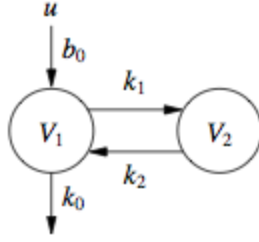
Figure 1.5: [3] A two compartment drug administration model: The first compartment is blood stream and the second the tissue. The accompanying state equations are in Equation 1.12.

where V is the volume and q the outflow rate. This can be generalized to two compartment model

$$
\begin{aligned}
V_1 \tfrac{dc}{dt} &= q(c_2 - c_1) - q_0 c_1 + u, \quad c_1 \geq 0, \\
V_2 \tfrac{dc}{dt} &= q(c_2 - c_1), \quad c_2 \geq 0, \\
y &= c_1.
\end{aligned}
\tag{1.11}
$$

Dividing by the volumes, the linear system becomes,

$$
\begin{aligned}
\tfrac{dx}{dt} &= \begin{bmatrix} -k_0 - k_1 & k_1 \\ k_2 & -k_2 \end{bmatrix} x + \begin{bmatrix} b_0 \\ 0 \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x,
\end{aligned}
\tag{1.12}
$$

where $k_0 = q_0/V_1$, $k_1 = q/V_1$, $k_2 = q/V_2$, $b_0 = c_0/V1$, and each state represents the concentration of the drug in each compartment, the first state being the blood stream where the measurement is taken and the second is the tissue. The structure of this system is shown in Figure 1.5.

Note for this example that how the system is connected together, or its network structure, has a specific meaning. The state-space matrices are linked to the physical system, encoding how the drug concentration in different compartments of the body affect each other (states), what compartments are directly measured (output), and what compartments are directly affected
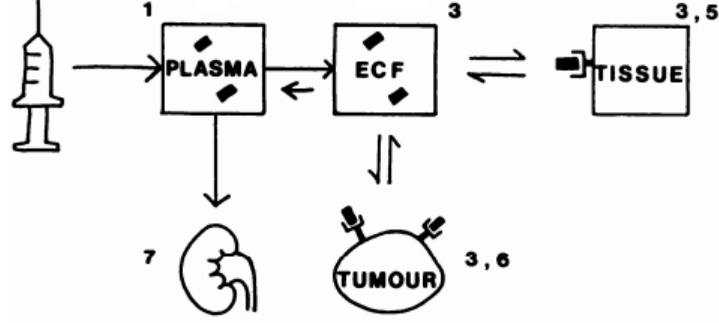
Figure 1.6: [5] This is a model of the disposition of antibodies (or another substance) that enter the blood plasma through injection and from there enter the extra cellular fluid (ECF) or urinary excretion. From the ECF, it can affect the normal tissue or tumor tissue. Normal arrows show linear interaction and half arrows show nonlinear interaction.

by a new dosage of drug (input). We now examine another similar but more interesting example.

Example 4. [6] *Here we use a compartment model to describe tumor targeting by antibodies, depicted in Figure 1.6. Antibodies enter the blood plasma through injection and from there enter the extra cellular fluid (ECF) or urinary excretion. From the ECF, the antibodies can affect the normal tissue or tumor tissue. We can simplify this to compartments and interactions between compartments as in Figure 1.7. Compartment 1 is the plasma, 3 is the ECF, 5 and 3,5 are the normal tissue, 6 and 3,6 are the tumor tissue, and 7 is the urinary excretion, which is the measured output. The dynamics are described as*

$$
\begin{aligned}
\dot{q}_1 &= k_4 q_3 - (k_3 + k_7)q_1 + u(t) \\
\dot{q}_3 &= k_3 q_1 - k_4 q_3 - k_5 q_3 (RV_3 - q_{3,5}) + k_6 q_{3,5} - k_5 q_3 \left(\frac{5V_{36}}{V_3}\right)(SV_{36} - q_{3,6}) + k_6 q_{3,6} \\
\dot{q}_{3,5} &= k_5 q_3 (RV_3 - q_{3,5}) - k_6 q_{3,5} \\
\dot{q}_{3,6} &= k_5 q_3 \left(\frac{5V_{36}}{V_3}\right)(SV_{36} - q_{3,6}) - k_6 q_{3,6} \\
\dot{q}_7 &= k_7 q_1 \\
y(t) &= q_7,
\end{aligned}
$$

$$(1.13)$$

where q_i is the quantity of the antibody in the ith compartment, the k_i's are the interactions between compartments shown in Figure 1.7, u is the antibody input into the plasma, R, S, V_3,
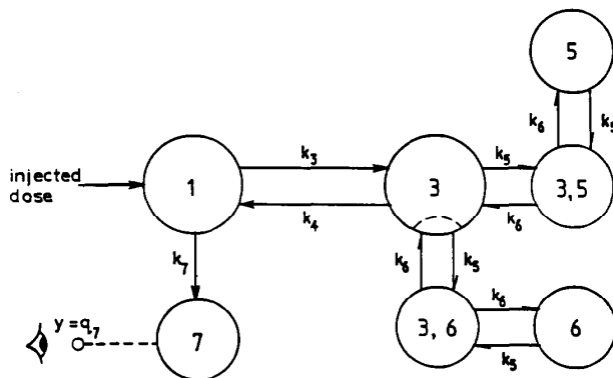
Figure 1.7: [6] A model describing tumor targeting by antibodies: The compartments are the same as Figure 1.6 and the input directly affects Compartment 1 and the output is measured directly from Compartment 7.

and $V_{3,6}$ are constants, and y is the output directly measured from the urinary excretion. This model directly shows how the different parts of the body affect each other. While it does have nonlinear parts, it can be linearized around a equilibrium.

The states of this system have an exact physical meaning. They are the quantity of antibody in each separate component of the body. Each quantity q_i at a given time could, in theory, be measured directly, if the right sensors were in place.

Examples 2-4 have physical structures that prefer some realizations over others. This motivates the idea that for some systems the physical structure can prefer one realization over others. However even in these examples there exist some ambiguity as to which exact realization should be used.

Permutation

First, consider the simplest transformation on Example 2, a permutation of states, which gives

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} -k_2 & k_1 \\ 0 & -k_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\
y &= \begin{bmatrix} k_2 & 0 \end{bmatrix} x.
\end{aligned}
\tag{1.14}
$$

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This system is clearly equivalent to the system in Equation 1.8 and can be separated as previously mentioned, still encoding the network structure of the system. So the physical system cannot distinguish between this realization and the realization in Equation 1.8.

Change of Units

Another ambiguity arises when considering the units of the states. Consider a transformation of a system that changes the units that the states are measured in but keeps the input-output behavior the same. This is encoded by a diagonal transformation of the system where $X \neq I$ (See Equation 1.3). Consider that in Example 2 the states are originally measured in cubic meters and we change them to cubic feet. This is encoded by the transformation

$$
X = \begin{bmatrix} 35.3 & 0 \\ 0 & 35.5 \end{bmatrix}, \tag{1.15}
$$

giving the system

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} -k_1 & 0 \\ k_1 & -k_2 \end{bmatrix} x + \begin{bmatrix} 35.5 \\ 0 \end{bmatrix} u, \\
y &= \begin{bmatrix} 0 & \frac{k_2}{35.5} \end{bmatrix} x.
\end{aligned} \tag{1.16}
$$

This system is equivalent to the system in Equation 1.8 and still portrays the physical structure of the system. So the physical system cannot distinguish between this realization and the realization in Equation 1.8.

Relevance to this Work

It is beyond the scope of this work to tie the physical structure of a system to a specific state-space realization. However, as is shown in Examples 2-4, the structure of a system can reduce the set of acceptable realizations. Nevertheless since this is mainly a theoretical piece of work, as mentioned before, we will assume that there is an underlying system (A, B, C, D) that created the input-output data and we want to know what information is necessary and

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sufficient to to recover this system, (A, B, C, D). In so doing, we will solve the Network Realization Problem.

1.2 Notation

The following is some notation we will use throughout this work:

- I_n is the $n \times n$ identity matrix.

- For a matrix A, A^T is its transpose and A^{-1} is its inverse.

- Given a linear transformation A, the four fundamental subspaces are the range or column space $R(A) = \{ x \mid Ax = b \}$, the nullspace $N(A) = \{ x \mid Ax = 0 \}$, the row space $R(A^T) = \{ b \mid A^T b = x \}$, and the left nullspace $N(A^T) = \{ b \mid A^T b = 0 \}$ of the operator A.

- $vec(\cdot)$ stacks the columns of the argument into a vector.

- $A \otimes B$ is the Kronecker product of A and B.

- $A \oplus B$ is the Kronecker sum of A and B, defined by $A \otimes I + I \otimes B$.

- $A(1:k,:)$ indicates that A is truncated to only including the first k rows and all the columns of A. Similarly, the first k columns of A are denoted by $A(:,1:k)$.

- $A^\dagger$ is the Moore-Penrose psuedoinverse of A.

Chapter 2

Background

2.1 Background

Model classes are parameterized sets of models that attempt to constrain manifest variables in a manner consistent with our observations or beliefs about how a system behaves. Learning problems choose a particular element of this set, indexed by a unique vector of parameters, that best explains our observational data.

In this work, $G(s)$ represents our observational data, and the minimal state-space realizations of $G(s)$ is the model class of interest. We assume no noise, that is to say, we assume that the network structure, (A, B, C, D), is the system that generated $G(s)$ is, in fact, one of its minimal realizations. Our problem is to characterize the conditions when we can recover (A, B, C, D) from $G(s)$.

Solving this problem begins with the characterization of the appropriate model class. In this work, we begin with a state-space model as in Equation (1.1), so we start by parameterizing all systems of this form. We then define subsets of this model class characterized by fixing certain "known" elements of (A, B, C, D).

Definition 1. [2] *A general* parameterization *of the system defined by Equation* (1.1) *is a continuously differentiable function* $P(\alpha) : \Omega \subset \mathbb{R}^q \to \mathbb{R}^N$, *where q is the number of unknown parameters,* $N = n(n + m + p) + mp$, *and Ω is a set of admissible values of the parameters.*

A general parameterization allows one to choose q parameters depicted by α, and it will return $P(\alpha) = \beta \in R(P) \subseteq \mathbb{R}^N$ where $N = n(n + m + p) + mp$. Each element of $\mathbb{R}^N$

specifies the entries of (A, B, C, D) matrices giving a particular realization of the form (1.1). That is to say, we interpret each element of a parameterization's range as the vectorization, $\left[vec(A(\alpha))^T \quad vec(B(\alpha))^T \quad vec(C(\alpha))^T \quad vec(D(\alpha))^T \right]^T \in \mathbb{R}^N$, of the associated system matrices $(A(\alpha), B(\alpha), C(\alpha), D(\alpha))$. So an element in the range of the parameterization, $\beta(\alpha) = P(\alpha) \in R(P)$ for some $\alpha \in \Omega$, gives

$$
\begin{aligned}
A(\alpha) &= \begin{bmatrix} \beta_1(\alpha) & \beta_{n+1}(\alpha) & \cdots & \\ \beta_2(\alpha) & \ddots & & \vdots \\ \vdots & & \cdots & \beta_{n^2}(\alpha) \end{bmatrix}, \\
B(\alpha) &= \begin{bmatrix} \beta_{n^2+1}(\alpha) & \cdots & & \vdots \\ \vdots & & \cdots & \beta_{n^2+nm}(\alpha) \end{bmatrix}, \\
C(\alpha) &= \begin{bmatrix} \beta_{n^2+nm+1}(\alpha) & \cdots & & \vdots \\ \vdots & & \cdots & \beta_{N-mp}(\alpha) \end{bmatrix}, \text{ and} \\
D(\alpha) &= \begin{bmatrix} \beta_{N-mp+1}(\alpha) & \cdots & & \vdots \\ \vdots & & \cdots & \beta_N(\alpha) \end{bmatrix},
\end{aligned} \tag{2.1}
$$

where these are the state-space matrices from Equation (1.1). Most parameterizations do not span the set of all realizations because by restricting some parameters it makes some realizations unobtainable. This is illustrated in Figure 2.1.

Fixing a particular parameterization function specifies a particular model class of state-space models (the range of this function). When this parameterization function is injective, each element of the range (i.e. each model in the model class) is indexed by a unique parameter vector in the function's domain, $\alpha \in \Omega$. Globally identifiability is a stricter condition:

Definition 2. *Let* $P(\alpha) : \Omega \subset \mathbb{R}^q \rightarrow \mathbb{R}^N$ *be a parameterization of system matrices* (A, B, C, D). *The parameterization* $P(\alpha)$ *is* globally identifiable from the transfer function $G(s)$ *if for all* $\alpha_1, \alpha_2 \in \Omega$ *if* $G(s) = C(\alpha_1)(sI - A(\alpha_1))^{-1}B(\alpha_1) + D(\alpha_1) = C(\alpha_2)(sI -$
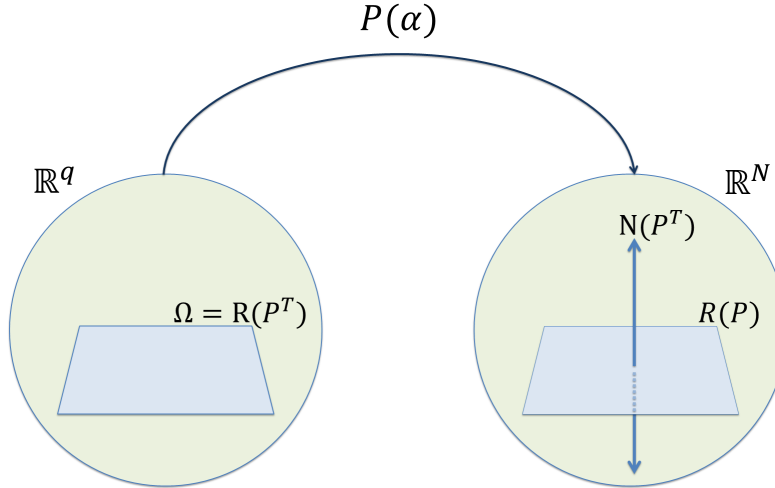
Figure 2.1: The parameterization $P(\alpha)$ is a mapping from $\mathbb{R}^q$ to $\mathbb{R}^N$. The arrow in the co-domain depicts that the range of the operator does not span the whole space $\mathbb{R}^N$, that is, there is a nontrivial left nullspace. This means there are some realizations that $P(\alpha)$ does not map to.

$A(\alpha_2))^{-1}B(\alpha_2) + D(\alpha_2)$ *then* $\alpha_1 = \alpha_2$. *That is to say, the concatenation map of the transfer function mapping and* $P(\alpha)$ *is one-to-one.*

Global identifiability says that any element in the range of such a parameterization, and therefore in the admissible model class, that corresponds to a realization of the transfer function $G(s)$ must be the only element in the model class with that same transfer function.

2.2 Related Work

The System Identification problem takes input-output data and, if it is rich enough, produces a transfer function. This problem is thoroughly reviewed by Lennart and Ljung in [7]. The problem of State Realization, that is, given a transfer function find a state-space model consistent with that input-output behavior, has been studied extensively [8–13]. In [8], Gilbert gives an algorithm for finding a minimal realization based on partial fraction expansion and under the assumption of distinct poles in the transfer function. Kalman proposes an alternative algorithm in [9], that reduces a realization iteratively until it is minimal. He also presents the observable and controllable canonical forms in this paper. In [10, 11], Ho and Kalman present

an alternative algorithm similar to [9] in that it iterates but it uses the Markov parameters instead of the transfer function. A thorough overview of the Minimal State Realization Problem is provided in [1].

The work done by Ho and Kalman in [10, 11] led to another problem called Subspace Identification. This problem takes input-output data and produces the Hankel operator, which is a matrix of the Markov parameters, and factors it into the observability and controllability grammians, which are then used to obtain a minimal state-space realization. Subspace Identification can be thought of as an alternative to the combination of the System Identification and the State Realization Problems because it goes from input-output data to a state-space model, but it differs in that it avoids the frequency domain. Similar to State Realization, Subspace Identification finds any state-space model consistent with the input-output behavior of the Hankel operator, that is to say, different subspace identification algorithms produce different state-space realizations and the likelihood that the realization produced by the algorithm is (A, B, C, D), or the network structure, sought by the Network Realization Problem is very low. Subspace identification gives no insight into finding the state transformation from an arbitrary realization to the network structure.

In [14], Kung proposed the first modification of [10, 11], using the singular value decomposition (SVD) of the estimated Hankel operator. In [15], Juang presents the eigen-system realization approach, which uses a truncation of the SVD of the Hankel operator to obtain a state-space model, and verifies it using data from the Galileo spacecraft. In [16–18], a realization algorithm is presented that matches measurements of impulse response parameters and the auto-covariance of the output to their corresponding models to provide a parameterization of all minimal stable realizations. At the end of [16] they explicitly state that a "method to choose a specific realization to satisfy other modeling considerations has not been determined." The work here addresses this problem. In [19], Van Der Veen uses both known and new subspace identification algorithms that employ SVD to discriminate

between desired and noisy signals. A good overview of Subspace Identification is provided in [20].

There is some confusion, or perhaps lack of rigor at times, surrounding Subspace Identification. In [21], when they formulate the problem they say, given input-output data, find the order n and the state-space matrices in Equation (1.1). This is not what Subspace Identification does; it finds a set of state-space matrices $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$, but not the exact ones that produced the data (A, B, C, D). The two are related by a state transformation but are different. In order to recover the exact matrices (A, B, C, D) that produced the input-output data some a priori knowledge of the matrices is required, which Subspace Identification does not assume.

The Network Realization Problem, the State Realization Problem, and Subspace Identification are all related to the problem of Structural Identifiability, which is the ability to identify the internal network structure of a system given that you know some underlying structure of the system. This allows the problem to be posed more generally but does not enable necessary and sufficient conditions for global identifiability. Bellman and Astrom introduce and formulate this problem in [22]. They also present the necessary and sufficient conditions for several classes of systems; including single–input single–output single–state, diagonal dynamics matrix, observable canonical form, and full–state measurement $(C = I)$. Rothenberg provides conditions for local identifiability for a general stochastic model in [23] which depends on the non-singularity of the information matrix. In [2], Glover and Willems provide necessary and sufficient conditions for local identifiability. They also provide a sufficient condition for global identifiability but fail to provide necessary conditions. In [24], Milanese provides necessary and sufficient conditions for observability of an unidentifiable parameterization. A great discussion on the different definitions of parameter and structural identifiability is provided in [25]. The Structural Identifiability problem can be motivated by several physiology applications in [26].

The theory for structural identifiability was originally developed in the 1970's. However, the expansion of research in the field of Systems Biology ([27]) over the last twenty five years has begun to highlight the importance of the identifiability results as shown in [6, 28–32]. These papers try to recover the network structure of systems in the applications of kinetics of substances like drugs and tracers, biochemical reaction networks, pancreatic cancer, and metabolic networks.

Another piece of related work is the dynamical structure function (DSF), introduced in [33]. This is a dynamic multi-scale representation of systems, that scales with the number of measured states. If one state is measured, the DSF is the transfer function. If all the states are measured, it provides a representation that is equivalent to the state-space matrices. Since structural information is encoded in the representation, some a priori knowledge of the structure is required for reconstruction similar to the state-space matrices. The necessary and sufficient conditions for reconstruction of the DSF from the transfer function are given in [34, 35]. These results differ from this work because they provide conditions for recovering the DSF from the transfer function while this work provides conditions for recovering the state-space matrices from the transfer function.

2.3   Thesis Contribution

In this work, we provide necessary and sufficient conditions for global identifiability from the transfer function for a specific type of parameterization, which solves the Network Realization Problem. Because we provide necessary and sufficient conditions for a certain type of parameterization, this problem can be thought of as a sub-problem of the Structural Identifiability Problem.

Chapter 3

Problem Formulation

For the Network Realization problem we focus on a specific type of parameterization that enables us to fix the known elements of the network structure, represented by (A, B, C, D). In order to formulate the problem we present essential definitions that refine Definition 1.

Definition 3. *The identity parameterization $I(\theta) : \mathbb{R}^N \to \mathbb{R}^N$, where $N = n(n+m+p)+mp$, maps θ to itself where θ is a vectorization of the system matrices, $\theta = \begin{bmatrix} vec(A)^T & vec(B)^T & vec(C)^T & vec(D)^T \end{bmatrix}^T$, so that the parameters in θ are exactly the elements of the system matrices, stacked appropriately.*

Definition 4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a function from a set $\mathcal{X}$ to a set $\mathcal{Y}$. If $\mathcal{F}$ is a subset of $\mathcal{X}$, then the restriction of f to $\mathcal{F}$ is the function*

$$f|_{\mathcal{F}} : \mathcal{F} \to \mathcal{Y}, \tag{3.1}$$

meaning $f|_{\mathcal{F}}(x) = f(x) \; \forall x \in \mathcal{F} \subset \mathcal{X}$.

Definition 5. *Consider $f : \mathcal{X} \to \mathcal{Y}$ for some linear vector spaces $\mathcal{X}$ and $\mathcal{Y}$, and let $\mathcal{B} = \{b_1, b_2, ..., b_r, ...\}$ be a basis of $\mathcal{X}$ with indicator function $\Theta_{\mathcal{R}}$. Let $\mathcal{R} = \mathrm{span}\{b_1, b_2, ..., b_r\} \subset \mathcal{X}$ and $\mathcal{F} = \mathcal{X}/\mathcal{R}$. Then, for some $\theta_{\mathcal{R}}^* \in \mathcal{R} \subset \mathcal{X}$, the restriction of f given by*

$$f|_{\{\theta_{\mathcal{R}}^* + \mathcal{F}\}} = \{\theta_{\mathcal{R}}^* + \mathcal{F}\} \to \mathcal{Y} \tag{3.2}$$

is called affine, *and we write* $f|_{\theta_{\mathcal{R}}^*}(\theta_{\mathcal{F}})$, *where* $\theta_{\mathcal{F}} \in \mathcal{F}$, *to mean* $f|_{\{\theta_{\mathcal{R}}^* + \mathcal{F}\}}$. *Moreover, the basis elements* $\{b_1, b_2, ..., b_r\}$ *are called* restricted, *while the rest of the basis elements are called* free.

$\Theta_{\mathcal{R}}$ is a binary vector that indicates which coordinates of $\mathcal{X}$ are restricted. $\theta_{\mathcal{R}}^*$ is a vector in $\mathcal{X}$ with specific values for the restricted coordinates and zeros in the positions of the free coordinates. In this way, $\theta_{\mathcal{R}}^*$ encodes the information about a fixed realization, (A, B, C, D), that is known a priori. Likewise, $\theta_{\mathcal{F}}$ is a vector in $\mathcal{X}$ with specific values for the free coordinates and zeros in the positions of the restricted coordinates. Sometimes we abuse notation by eliminating the zeros in the positions of the restricted coordinates of $\theta_{\mathcal{F}}$, and view it as an element of $\mathcal{F}$, not as an element of $\mathcal{F}$ embedded in $\mathcal{X}$. The following example illustrates the restriction of the identity parameterization.

Example 5. *Consider the system characterized by the state equations*

$$
\begin{aligned}
\dot{x} &= ax + bu \\
y &= cx.
\end{aligned}
\tag{3.3}
$$

The identity parameterization takes a vector in $\mathbb{R}^3$ and associates a particular system with it, so $\theta = [-1\ 1\ 2]^T$ becomes

$$
\begin{aligned}
\dot{x} &= -x + u \\
y &= 2x.
\end{aligned}
\tag{3.4}
$$

The restriction of the identity given by $I|_{[0\ 0\ 0]^T}(\theta_{\mathcal{F}})$, with $\Theta_{\mathcal{R}} = [1\ 0\ 0]^T$, generates the class of systems parameterized by $\theta_{\mathcal{F}} = [0\ b\ c]^T$ (or, equivalently, $\theta_{\mathcal{F}} = [b\ c]^T$) as

$$
\begin{aligned}
\dot{x} &= bu \\
y &= cx.
\end{aligned}
\tag{3.5}
$$

While the restriction of the identity given by $I|_{[-3\ 0\ 5]^T}(\theta_{\mathcal{F}})$, with $\Theta_{\mathcal{R}} = [1\ 0\ 1]^T$, generates the class of systems parameterized by $\theta_{\mathcal{F}} = [0\ b\ 0]^T$ (or, equivalently, $\theta_{\mathcal{F}} = b$) as

$$
\begin{aligned}
\dot{x} &= -3x + bu \\
y &= 5x.
\end{aligned}
\tag{3.6}
$$

Definition 6. *Consider a system (A, B, C, D) given by Equation (1.1), and let $\theta^* = \left[vec(A)^T \quad vec(B)^T \quad vec(C)^T \quad vec(D)^T\right]^T$. An affine restriction of the identity parametrization, $I|_{\theta_{\mathcal{R}}^*}$, with arguments characterized by $\mathcal{F}$, is* consistent *with θ^* (or (A, B, C, D)) if there exists some $\theta_{\mathcal{F}} \in \mathcal{F}$ such that $\theta^* = \theta_{\mathcal{R}}^* + \theta_{\mathcal{F}}$. In other words, $I|_{\theta_{\mathcal{R}}^*}$ is consistent with θ^* if $\theta^* \in R(I|_{\theta_{\mathcal{R}}^*})$, the range of the parameterization $I|_{\theta_{\mathcal{R}}^*}$.*

Thus, a restriction of the identity is *consistent* with a particular system if the restricted parameters and the associated values in the system's state representation are equal.

Lemma 1. *If $I|_{\theta_{\mathcal{R}}^*}$ is consistent with two unequal realizations with the same transfer function $G(s)$ then $I|_{\theta_{\mathcal{R}}^*}$ is not globally identifiable from $G(s)$.*

Proof. Assume $I|_{\theta_{\mathcal{R}}^*}$ is consistent with θ_1 and θ_2, $\theta_1 \neq \theta_2$, with the same transfer function $G(s)$. This means there exist $\theta_{\mathcal{F}_1}, \theta_{\mathcal{F}_2} \in \mathcal{F}$, $\theta_{\mathcal{F}_1} \neq \theta_{\mathcal{F}_2}$, where $\theta_1 = I|_{\theta_{\mathcal{R}}^*}(\theta_{\mathcal{F}_1})$ and $\theta_2 = I|_{\theta_{\mathcal{R}}^*}(\theta_{\mathcal{F}_2})$, such that $G(s) = C(\theta_{\mathcal{F}_1})(sI - A(\theta_{\mathcal{F}_1}))^{-1}B(\theta_{\mathcal{F}_1}) = C(\theta_{\mathcal{F}_2})(sI - A(\theta_{\mathcal{F}_2}))^{-1}B(\theta_{\mathcal{F}_2})$. But since $\theta_1 \neq \theta_2$, then by definition $I|_{\theta_{\mathcal{R}}^*}$ is not globally identifiable from $G(s)$. $\qquad\square$

3.1   Problem Statement

Loosely speaking, our objective is to characterize the conditions under which one can recover (A, B, C, D) from knowledge of its transfer function. This goal is made precise in the following problem formulation.

State-Space Network Realization Problem. *Consider a system (A, B, C, D) as in Equation (1.1), with (A, B) controllable and (A, C) observable, and suppose*

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$G(s) = C(sI - A)^{-1}B + D$ is given. Find an affine restriction of the identity parameterization, $I|_{\theta_{\mathcal{R}}^*}$, consistent with (A, B, C, D), that is globally identifiable from $G(s)$.

Note that the identity parameterization is not globally identifiable from any $G(s)$ by Lemma 1, since every $G(s)$ has multiple realizations and $I(\theta)$ is consistent with all of them. Finding a consistent affine restriction that is globally identifiable means identifying $\theta_{\mathcal{R}}^*$, or elements of (A, B, C, D), that, if known, enable one to recover the rest of (A, B, C, D) from $G(s)$.

3.2 Parameterization Example

This example illustrates global identifiability and the Network Realization Problem.

Example 6. *Consider the transfer function*

$$G(s) = \frac{1}{s - 3} \tag{3.7}$$

with the model

$$\begin{aligned} \dot{x} &= ax + bu \\ y &= cx, \end{aligned} \tag{3.8}$$

which has a transfer function

$$G(s) = \frac{bc}{s - a}. \tag{3.9}$$

There are different pieces of knowledge that could be known a priori. For instance, consider that $a = 3$ was known. This would yield the parameterization, $I|_{[3\ 0\ 0]^T}(\theta_{\mathcal{F}})$, with indicator function, $\Theta_{\mathcal{R}} = [1\ 0\ 0]^T$, which generates the class of systems parameterized by $\theta = [3\ b\ c]^T$ (or, equivalently, $\theta_{\mathcal{F}} = [b\ c]^T$) as

$$\begin{aligned} \dot{x} &= 3x + bu \\ y &= cx. \end{aligned} \tag{3.10}$$

This parameterization has an infinite number of pairs of (b, c) that give the same $G(s)$, illustrated in Figure 3.1 a). So $I|_{[3\ 0\ 0]^T}(\theta_{\mathcal{F}})$ is not globally identifiable from $G(s)$.

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$$G(s) = \frac{1}{s-3}$$

Network Realization

$$\dot{x} = ax + bu$$
$$y = cx$$
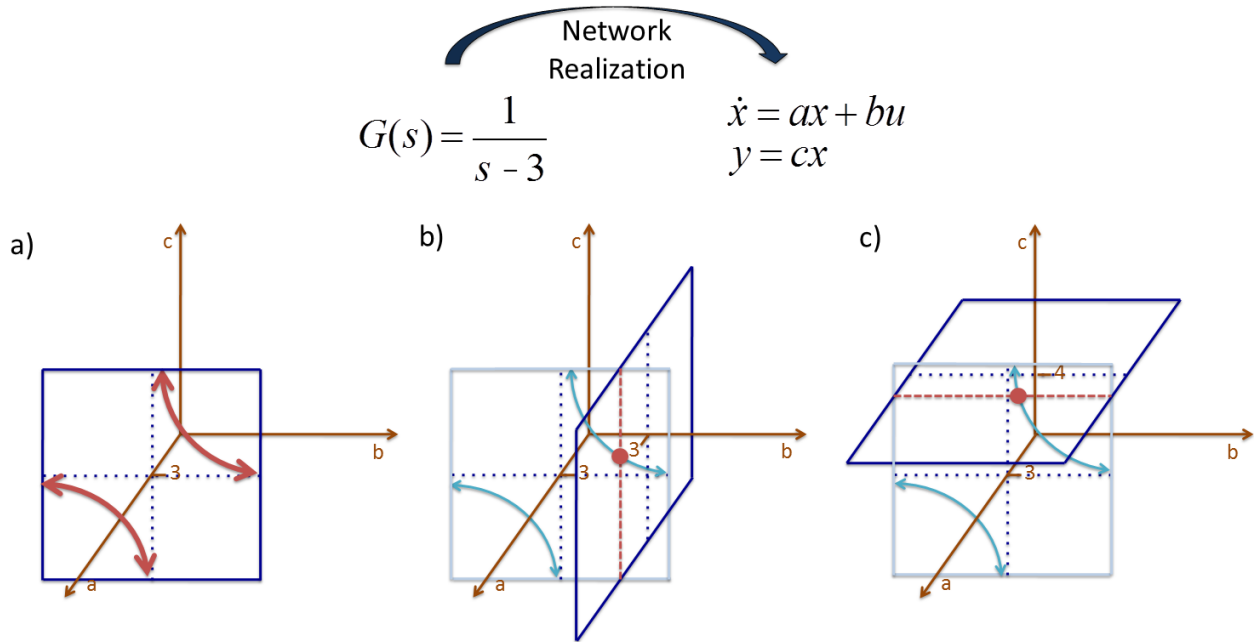
Figure 3.1: A simple parameterization illustrating the Network Realization Problem: In the state-space model parameter space, there is a set of equivalent systems that all generate the same input-output behavior, $G(s)$. Fixing one parameter in the model gives a restriction of the identity parameterization. Notice the parameterization in a), created by fixing $a = 3$, gives many realizations that yield $G(s)$ (red curves). However, the parameterizations in b) and c), created by fixing $b = 3$ and $c = 4$, respectively, each give only one realization that yields $G(s)$ (red dots).

Alternatively, assume $b = 3$ is known, yielding the parameterization $I|_{[0\ 3\ 0]^T}(\theta_{\mathcal{F}})$, with $\Theta_{\mathcal{R}} = [0\ 1\ 0]^T$, which generates the class of systems parameterized by $\theta = [a\ 3\ c]^T$ (or, equivalently, $\theta_{\mathcal{F}} = [a\ c]^T$) as

$$\begin{aligned} \dot{x} &= ax + 3u \\ y &= cx. \end{aligned} \tag{3.11}$$

The same amount of prior information was known but there is only one point in the parameter space that gives $G(s)$, illustrated in Figure 3.1 b). So $I|_{[0\ 3\ 0]^T}(\theta_{\mathcal{F}})$ is globally identifiable from $G(s)$.

Similarly, assume $c = 4$ was known, yielding the parameterization $I|_{[0\ 0\ 4]^T}(\theta_{\mathcal{F}})$, with $\Theta_{\mathcal{R}} = [0\ 0\ 1]^T$, which generates the class of systems parameterized by $\theta = [a\ b\ 4]^T$ (or,

equivalently, $\theta_{\mathcal{F}} = [a \; b]^T$) as

$$\dot{x} = ax + bu$$
$$y = 4x.$$

(3.12)

Again, there is only one point in the parameter space that gives $G(s)$, illustrated in Figure 3.1 c). So $I|_{[0\;0\;4]^T}(\theta_{\mathcal{F}})$ is globally identifiable from $G(s)$.

For this example one can immediately see that $a = 3$ because 3 is the unique pole of $G(s)$. So knowing a priori that $a = 3$ offers no new information, while the other cases do offer new information to specify the true system. In general, knowing A a priori is never sufficient to recover the network structure of the system.

Also note that the first parameterization in Example 6 was not globally identifiable from $G(s)$. However the second two were, and all three of the parameterizations only restricted one element. Therefore, for global identifiability, the value of the a priori information is not necessarily quantified by the amount of information.

Chapter 4

Direct–State Measurement

A special case of this problem is published in [36]. The specific class of systems considered is

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= [\, I_p\ 0\,]x(t),
\end{aligned} \tag{4.1}$$

where a set of the states are directly measured. This chapter focuses on restrictions of the identity parameterization that only fix the elements in the C matrix and fixes them to $C = [I_p\ 0]$, which is denoted by $I|_C$. This class of systems is important for various applications including biochemical reaction networks where scientists are directly measuring some of the chemical species in a mixture [30].

4.1 Global Identifiability of State Measurement Systems

The following theorem statement is changed slightly from [36] to match the parameterization notation.

Theorem 1. [36] *The restriction of the identity parameterization $I|_{C=[I_p\ 0]}$, where no elements of A and B are fixed, is globally identifiable from $G(s)$ if and only if $p = n$.*

Proof. ($\Rightarrow$) We prove necessity by contraposition. By the structure of C in Equation (4.1), $p \leq n$ where if $p = n$ then $C = I$. In other words, $p > n$ is not permissible for this model class.

Assume $p < n$. This implies $C = \begin{bmatrix} I_p & 0_{n-p} \end{bmatrix}$ with $n - p > 0$, thus $C \neq I$. Moreover, for any system there exists a minimal pair of matrices (A, B), not necessarily unique, that realize $G(s)$:

$$G(s) = C(sI - A)^{-1}B. \tag{4.2}$$

Consider the state transformation X and its inverse:

$$X = \begin{bmatrix} I_n & 0 \\ K & I_{n-p} \end{bmatrix}, \quad X^{-1} = \begin{bmatrix} I_n & 0 \\ -K & I_{n-p} \end{bmatrix}, \tag{4.3}$$

where K is an arbitrary $(n - p) \times n$ nonzero matrix.

The resulting transformed system, $\hat{A} = XAX^{-1}$, $\hat{B} = XB$, and $\hat{C} = CX^{-1}$, is still of the form (4.1), since $CX^{-1} = C$, and clearly has the same transfer function $G(s)$. Nevertheless, different values of K will clearly result in various dynamics matrices, $\hat{A}$, and control matrices, $\hat{B}$, thus showing that $p < n$ implies that $I|_C$ is consistent with multiple minimal realizations that give $G(s)$. Therefore $I|_C$ is not globally identifiable from $G(s)$.

($\Leftarrow$) Assume $n = p$, $C = I$ and that $G(s)$ is realized by (A_1, B_1, C_1) and (A_2, B_2, C_2) in the range of $I|_{C=I}$. This implies the existence of an invertible matrix X such that $A_2 = XA_1X^{-1}$, $B_2 = XB_1$, and $C_2 = C_1X^{-1}$. Since $C_1 = C_2 = I$ and $C_2 = C_1X^{-1}$ the only acceptable X matrix is clearly the identity matrix. Therefore $A_1 = A_2$ and $B_1 = B_2$ and only one set of matrices (A, B) realizes $G(s)$ in the range of $I|_{C=I}$. So $I|_{C=I}$ is globally identifiable from $G(s)$.

□

This theorem states that network structure cannot be recovered for the class of models depicted in Equation (4.1), assuming nothing is known about A and B, unless $C = I$. This assumption is unrealistic for *in vivo* biochemical reaction networks because it is impossible to measure all chemical species in a mixture for an uncontrolled environment.
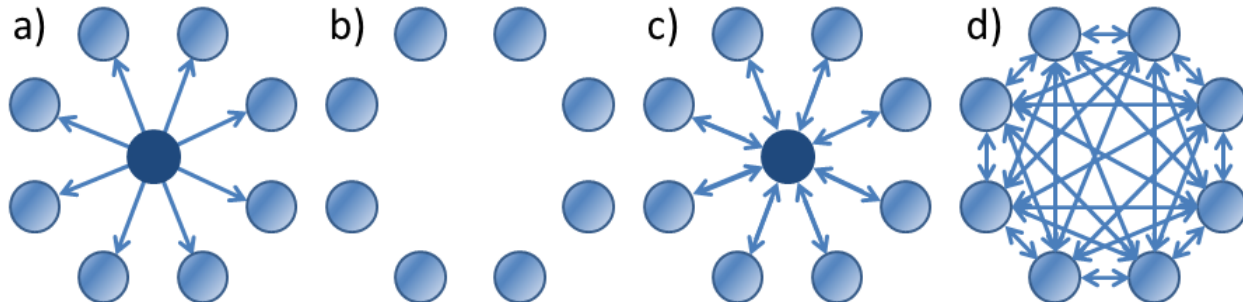
Figure 4.1: In Example 7, if $n = 9$, the actual system can be a nine-node network ranging from a) to c). The hidden node is the darker node and can affect every node. Network reconstruction on a) would yield the network in b) and network reconstruction on c) would yield d).

4.2 Example of Severity

The following example illustrates the severity of the implications of Theorem 1. Even if only one state in a network is not measured, then the structure of the smaller system yielded from a reconstruction algorithm could be completely different from that of the original network. This is shown by presenting a pair of $n \times n$ A matrices with $n - 1$ measured states.

Example 7. [36] *Consider a hub-spoke network, such as in Figure 4.1 a), with $n-1$ nodes on the outside driven by the nth node in the center. Systems with this structure occur commonly in transportation systems, communication networks, and social networks, just to name a few applications. The dynamics matrix for such a system is of the form*

$$A_1 = \begin{bmatrix} D & \Gamma \\ 0 & 1 \end{bmatrix}, \tag{4.4}$$

where D is a nonzero diagonal matrix and Γ is a column vector with no zero entries. This structure is depicted in Figure (4.1A). Moreover, let $B = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}^T$, which is controllable when Γ has no zero entries.

Note that the only observable partial–state measurement configuration for this system, with only one hidden state, is $C = [I_{n-1} \ 0]$. Nevertheless, the set of minimal realizations

clearly preserves the same partial state configuration (i.e. preserve $C = [I_{n-1} \ 0]$) span all possible Boolean structures for the dynamics matrix.

To see this, consider the state transformation $T = \begin{bmatrix} I_{n-1} & 0 \\ \Gamma^T & 1 \end{bmatrix}$. The new, transformed dynamics matrix then becomes:

$$
\begin{aligned}
A_2 = T A_1 T^{-1} &= \begin{bmatrix} I_{n-1} & 0 \\ \Gamma^T & 1 \end{bmatrix} \begin{bmatrix} D & \Gamma \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I_{n-1} & 0 \\ -\Gamma & 1 \end{bmatrix} \\
&= \begin{bmatrix} D & \Gamma \\ \Gamma^T & * \end{bmatrix} \begin{bmatrix} I_{n-1} & 0 \\ -\Gamma^T & 1 \end{bmatrix} = \begin{bmatrix} \hat{\Gamma} & * \\ * & * \end{bmatrix}
\end{aligned}
\tag{4.5}
$$

where $*$ indicates a nonzero entry of the appropriate dimension. The non-diagonal entries of $\hat{\Gamma}$ are equal to $\hat{\gamma}_{ij} = d_i - \gamma_i \gamma_j = d_i - ij$, which is nonzero by the construction of Γ. This results in the structure shown in Figure 4.1 c). Notice that $C = CT^{-1}$, demonstrating that this state transformation preserves the partial state configuration of the system.

Clearly the transfer functions for both systems are equal, since they differ only by a state transformation. Nevertheless, notice that the portion of the underlying realization corresponding to the measured states, i.e. the sub matrix $\hat{\Gamma}$ in the transformed system, can have any Boolean structure, from diagonal, as in Equation (4.4), to completely full, as in Equation (4.5) (or anything in between). Thus, any network reconstruction technique that attempts to recover this system–even knowing a priori that the system is both controllable and observable–cannot distinguish even the Boolean structure relating the measured nodes to each other without more information than the input-output behavior of the system.

From Example 7 we can see that even if a network had 1000 states, and we missed measuring only a single hidden state, then a reconstruction method may reasonably recover a full network, like Figure 4.1 d), where it could actually have been completely disconnected, like Figure 4.1 b), or vice versa.

Chapter 5

General System State-Space Network Realization

In Chapter 4 we solved the Network Realization Problem for the specific state-space system where $C = [I\ 0]$. However, this result did not give a general solution to Problem 3.1. This chapter expands the solution from Chapter 4 to the general state-space system given in Equation (1.1), solving the Network Realization Problem (3.1).

The key idea for the solution technique is the fact, that for a minimal system, every possible vector of parameters in the domain of a restriction of the parameterization, $I|_{\theta_{\mathcal{R}}^*}$, maps to a system that can be related to any other realization of $G(s)$ by a non-singular state transformation X. In Chapter 4, this fact was exploited to provide necessary and sufficient conditions for direct–state measurement systems. We now generalize this idea for generic systems.

5.1   Solution to the Network Realization Problem

Definition 7. *A restriction of the identity parameterization, $I|_{\theta_{\mathcal{R}}^*}$, and a realization, $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$, are compatible if there exists a $\theta = I|_{\theta_{\mathcal{R}}^*}(\theta_{\mathcal{F}}) \in R(I|_{\theta_{\mathcal{R}}^*})$ (i.e. $I|_{\theta_{\mathcal{R}}^*}$ consistent with θ) such that $\hat{C}(sI - \hat{A})^{-1}\hat{B} + \hat{D} = C(\theta_{\mathcal{F}})(sI - A(\theta_{\mathcal{F}}))^{-1}B(\theta_{\mathcal{F}}) + D(\theta_{\mathcal{F}}).$*

Definition 6 explains that a particular state-space model, (A, B, C, D), is consistent with a given restriction of the identity parameterization, $I|_{\theta_{\mathcal{R}}^*}$, if the elements of (A, B, C, D) corresponding to the restricted elements in the indicator function $\Theta_{\mathcal{R}}$ are set to the values in
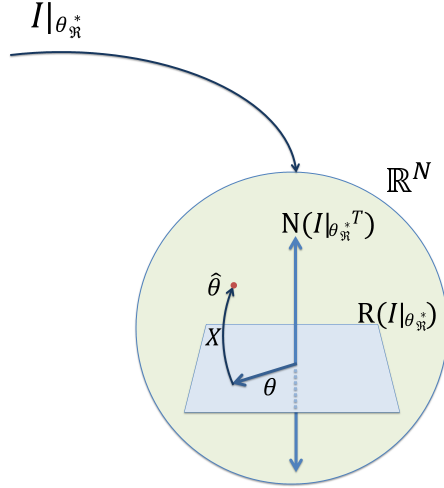
Figure 5.1: The co-domain of $I|_{\theta_{\mathcal{R}}^*}$: A restriction of the identity parameterization, $I|_{\theta_{\mathcal{R}}^*}$, is consistent with all the realizations that are in its range, such as θ. The realizations represented by $\hat{\theta}$ and θ are related by a state transformation X. The realization represented by $\hat{\theta}$ is compatible with $I|_{\theta_{\mathcal{R}}^*}$ because it has the same transfer function as θ, which is consistent with $I|_{\theta_{\mathcal{R}}^*}$, but $I|_{\theta_{\mathcal{R}}^*}$ is not consistent with $\hat{\theta}$ because $\hat{\theta} \notin R(I|_{\theta_{\mathcal{R}}^*})$.

$\theta_{\mathcal{R}}^*$. That is to say, choosing $\theta_{\mathcal{F}}$ to be the values of the non-restricted elements of (A, B, C, D) will yield $I|_{\theta_{\mathcal{R}}^*}(\theta_{\mathcal{F}}) = (A, B, C, D)$.

Definition 7 explains that a particular state-space model only needs to share a transfer function with another state-space model that is consistent with $I|_{\theta_{\mathcal{R}}^*}$ to be compatible with it. Thus, the restricted elements of (A, B, C, D) do not need to be equal to $\theta_{\mathcal{R}}^*$, but there must be a transformation of the system that does have restricted elements equal to $\theta_{\mathcal{R}}^*$. So every realization that $I|_{\theta_{\mathcal{R}}^*}$ is consistent with, is also compatible with $I|_{\theta_{\mathcal{R}}^*}$ via the identity transformation. However there can be many realizations that are compatible with $I|_{\theta_{\mathcal{R}}^*}$ but that are not consistent with $I|_{\theta_{\mathcal{R}}^*}$. Figure 5.1 illustrates the difference between compatibility and consistency.

Given $I|_{\theta_{\mathcal{R}}^*}$ and a realization $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ that are compatible implies there exists a state transformation, X, that satisfies (1.3). Therefore, by vectorization, where $x = vec(X)$,

$$
\begin{bmatrix} A^T \oplus -\hat{A} \\ B^T \otimes I_n \\ I_n \otimes \hat{C} \end{bmatrix} x = \begin{bmatrix} 0 \\ vec(\hat{B}) \\ vec(C) \end{bmatrix}
$$
$$
\mathbb{A} \quad x = \quad b,
$$

(5.1)

where we delete rows of $\mathbb{A}$ that contain an unknown due to the elements in $\theta_{\mathcal{R}}^*$ denoted by the zeros in the indicator function $\Theta_{\mathcal{R}}$.

Example 8. *Consider the system*

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + \begin{bmatrix} I_3 \\ 0_{2\times3} \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} I_2 & 0_{2\times3} \end{bmatrix} x(t),
\end{aligned}
$$

(5.2)

where

$$
A = \begin{bmatrix}
-3 & 0 & 1 & 1 & 1 \\
0 & -2 & 1 & 0 & 1 \\
1 & 0 & -2 & 1 & 0 \\
1 & 1 & 0 & -2 & 0 \\
0 & 1 & 1 & 0 & -3
\end{bmatrix}.
$$

(5.3)

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Let B, C, and the first two columns of A be the fixed parameters defining the restriction of the identity parameterization. Let $\hat{B} = B$, $\hat{C} = C$, and

$$\hat{A} = \begin{bmatrix} -3 & 0 & 1 & -.70 & -.16 \\ 0 & -2 & 1 & -.93 & -.11 \\ 6 & 9 & 2 & -2.9 & -1.4 \\ 5 & 7 & 2 & -3.6 & -.68 \\ 3 & 13 & 10 & -7.9 & -5.4 \end{bmatrix}. \tag{5.4}$$

We can create the equivalent to Equation (5.1),

$$\begin{bmatrix} (A^T \oplus -\hat{A})(1:2n,:) \\ B^T \otimes I_n \\ I_n \otimes \hat{C} \end{bmatrix} x = \begin{bmatrix} 0 \\ vec(\hat{B}) \\ vec(C) \end{bmatrix}, \tag{5.5}$$

which is full column rank so it can be used to solve for the unique state transformation X. Therefore A can be solved for, $A = X^{-1}\hat{A}X$. So Equation (5.1) is sufficient to solve the Network Realization Problem for this restriction of the identity parameterization.

Another sufficient condition, similar to Equation (5.1), with $X^{-1} = \bar{X}$ where $\bar{x} = vec(\bar{X})$, is

$$\begin{aligned} \begin{bmatrix} \hat{A}^T \oplus -A \\ \hat{B}^T \otimes I_n \\ I_n \otimes C \end{bmatrix} \bar{x} &= \begin{bmatrix} 0 \\ vec(B) \\ vec(\hat{C}) \end{bmatrix} \\ \bar{\mathbb{A}} \qquad \bar{x} &= \qquad \bar{b}, \end{aligned} \tag{5.6}$$

again, deleting the rows of $\bar{\mathbb{A}}$ that contain an unknown due to the elements in $\theta_{\mathcal{R}}^*$ denoted by the zeros in the indicator function $\Theta_{\mathcal{R}}$.

Let $x^\star = \mathbb{A}^\dagger b$, $\bar{x}^\star = \bar{\mathbb{A}}^\dagger \bar{b}$, $dim(N(\mathbb{A})) = k$, and $dim(N(\bar{\mathbb{A}})) = l$. Then $N(\mathbb{A}) = span\{x_{n_1}, \ldots, x_{n_k}\}$, and $N(\bar{\mathbb{A}}) = span\{\bar{x}_{n_1}, \ldots, \bar{x}_{n_l}\}$.

Equations (5.1) and (5.6) are sufficient but are not necessary. The following example does not have full column rank for either equation but still can be solved when the two are combined.

Example 9. *Consider the system*

$$\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} -3 & 2 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)
\end{aligned} \tag{5.7}$$

and the state transformation

$$X = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}. \tag{5.8}$$

If we assume a_{11}, a_{12}, a_{21}, b_2, and c_1 are the fixed parameters then they define the restriction of the identity parameterization. Following Equations (5.1) and (5.6) using the restriction of the identity parameterization and system, $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$,

$$\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} 3 & -4 \\ 4 & -6 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} -1 & 2 \end{bmatrix} x(t),
\end{aligned} \tag{5.9}$$

gives

$$\begin{bmatrix} -6 & 4 & 1 & 0 \\ -4 & 3 & 0 & 1 \\ -1 & 2 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{5.10}$$

and

$$\begin{bmatrix} 6 & -2 & 4 & 0 \\ -4 & 0 & -3 & -2 \\ 0 & 1 & 0 & 1 \end{bmatrix} \bar{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{5.11}$$

Clearly neither are full column rank. If we choose to ignore this and try to use $X^\star$, where $vec(X^\star) = x^\star = \mathbb{A}^\dagger b$, clearly it satisfies Equation (5.10). However, it does not satisfy Equation (5.11). So ignoring this and using $X^\star$, yields the system

$$
\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} -3 & 2 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 2.26 \\ 2.46 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 1 & -0.51 \end{bmatrix} x(t).
\end{aligned}
\tag{5.12}
$$

This realization is close to the original however it is not consistent with the restriction of the identity parameterization because $b_2 = 2.46 \neq 0$. Therefore this system is not in the $R(I|_{\theta_{\mathcal{R}}^*})$.

Since $\bar{b} = 0$, $\bar{x}^\star = 0$. Due to the inverse relationship between X and $\bar{X}$, there exist w and $\bar{w}$ such that

$$
\begin{aligned}
X^\star + wX_n &= (\bar{w}\bar{X}_n)^{-1} \\
\begin{bmatrix} 0.4894 & -0.0426 \\ 0.7447 & -0.2766 \end{bmatrix} + w \begin{bmatrix} 0.2063 & 0.8251 \\ 0.1031 & 0.5157 \end{bmatrix} &= \left(\bar{w} \begin{bmatrix} 0.3780 & -0.7559 \\ -0.3780 & 0.3780 \end{bmatrix} \right)^{-1}.
\end{aligned}
\tag{5.13}
$$

This is easily solvable and gives $w = 2.4754$ and $\bar{w} = -2.6458$, which yields X from Equation (5.8), which yields the original system in (5.7).

The idea of combining Equation (5.1) and (5.6) from Example 9 can be generalized. Due to the inverse relationship between X and $\bar{X}$, there exist $\{w_1, \ldots, w_k, \bar{w}_1, \ldots, \bar{w}_l\}$ such that

$$
(X^\star + w_1 X_{n_1} + \cdots + w_k X_{n_k}) = (\bar{X}^\star + \bar{w}_1 \bar{X}_{n_1} + \cdots + \bar{w}_l \bar{X}_{n_l})^{-1}.
\tag{5.14}
$$

Figure 5.2 explains the relationship between Equations (5.1), (5.6), and (5.14).
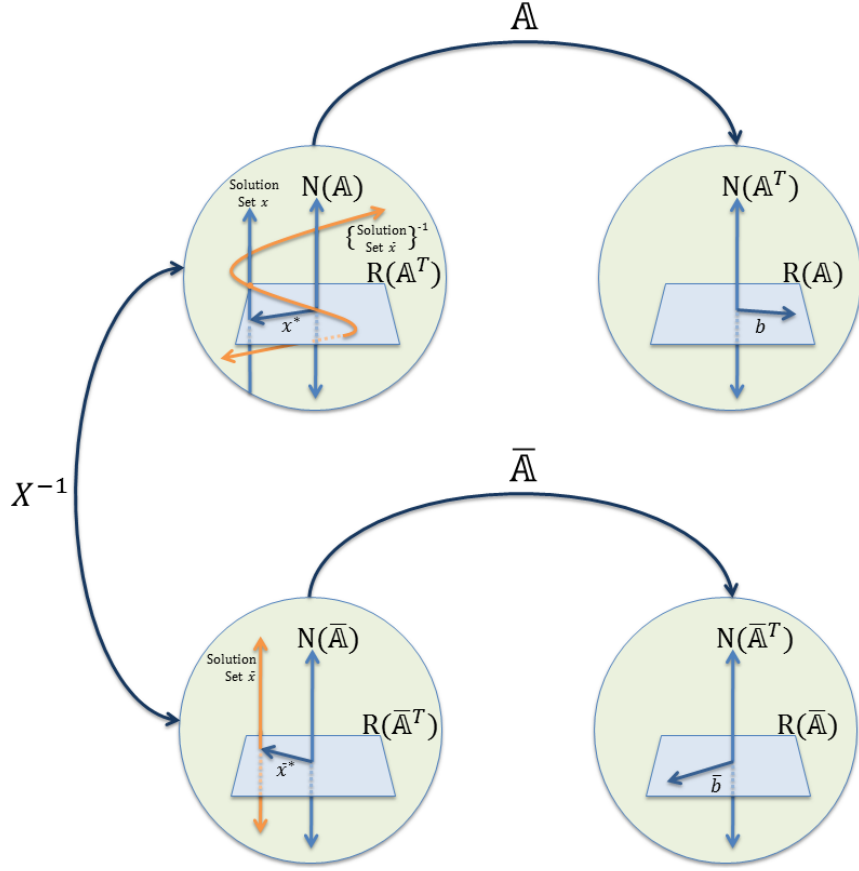
Figure 5.2: Two distinct systems of equations, $\mathbb{A}x = b$ (Equation (5.1)), depicted in the top figure, and $\bar{\mathbb{A}}\bar{x} = \bar{b}$ (Equation (5.6)), depicted in the bottom figure, are linked by a bijective relation between subsets of the domains of these operators (left arrow).

[Top Figure] The four fundamental subspaces of $\mathbb{A}$ are illustrated. Note that b is always in the range of $\mathbb{A}$ by construction. When $\mathbb{A}$ has a nontrivial nullspace, not only does the unique vector x^* in the row space of $\mathbb{A}$ satisfy $Ax = b$, but so does every vector in the affine set $\{\, x \in \mathbb{R}^{n^2} \mid x = x^* + w_1 x_{n_1} + ... + w_k x_{n_k}, w_i \in \mathbb{R}, i = 1, ..., k \,\}$ (Solution Set x).

[Bottom Figure] Similarly, the four fundamental subspaces of $\bar{\mathbb{A}}$ are illustrated and $\bar{b}$ is in the range of $\bar{\mathbb{A}}$ by construction. When $\bar{\mathbb{A}}$ has a nontrivial nullspace, not only does the unique vector $\bar{x}^*$ in the row space of $\bar{\mathbb{A}}$ satisfy $\bar{\mathbb{A}}\bar{x} = \bar{b}$, but so does every vector in the affine set $\{\, \bar{x} \in \mathbb{R}^{n^2} \mid \bar{x} = \bar{x}^* + \bar{w}_1 \bar{x}_{n_1} + ... + \bar{w}_k \bar{x}_{n_k}, \bar{w}_i \in \mathbb{R}, i = 1, ..., k \,\}$ (Solution Set $\bar{x}$).

[Left Arrow] Elements of the domain of $\mathbb{A}$, x, are associated with matrices X as $x = vec(X)$. Likewise, elements of the domain of $\bar{\mathbb{A}}$, $\bar{x}$, are associated with matrices $\bar{X}$ as $\bar{x} = vec(\bar{X})$. The bijective relation represented by the left arrow illustrates that for any x such that X is invertible, $\bar{x} = vec(X^{-1})$.

We focus on the restriction of this bijection to the solutions of $\mathbb{A}x = b$ and $\bar{\mathbb{A}}\bar{x} = \bar{b}$ (the affine sets in the domains of the top and bottom figures marked as "Solution Set x" and "Solution Set $\bar{x}$", respectively). Equation (5.14) characterizes when the image of one of these sets (through the bijection) intersects the other affine set. This intersection is non-empty by construction, but the central question is whether this intersection is a singleton (as illustrated) or contains multiple points.

5.1.1 Main Result

The previous machinery leads to the following theorem statement.

Theorem 2. (Main Result)

1. *Given a proper transfer function, $G(s)$, with two minimal realizations, $S = (A, B, C, D)$ and $\hat{S} = (\hat{A}, \hat{B}, \hat{C}, \hat{D})$, each as in Equation (1.1).*

2. *Suppose part of S is known, specified by the entries of $\theta_{\mathcal{R}}^* \in \mathbb{R}^N$ and the indicator function $\Theta_{\mathcal{R}}$, characterizing the affine restriction of the identity parameterization $I|_{\theta_{\mathcal{R}}^*}$.*

3. *Suppose all of $\hat{S}$ is known.*

4. *Note that $I|_{\theta_{\mathcal{R}}^*}$ and $\hat{S}$ are compatible, thereby specifying $\mathbb{A}$, b, $\bar{\mathbb{A}}$, $\bar{b}$, X^*, X_{n_i}, $\bar{X}^*$, $\bar{X}_{n_j}$, $i = 1, ..., k$, $j = 1, ..., l$ according to Equations (5.1), (5.6), and (5.14).*

Then $I|_{\theta_{\mathcal{R}}^*}$ is globally identifiable from $G(s)$ if and only if Equation (5.14) has a unique solution.

Proof. Since $I|_{\theta_{\mathcal{R}}^*}$ is consistent with S, and compatible with $\hat{S}$ there is at least one solution to (5.14).

($\Rightarrow$) We prove this by contraposition. The negation of the statement, "Equation (5.14) has a unique solution," is, "Equation (5.14) has no solution or it has multiple solutions." However, since there is at least one solution to (5.14), we assume (5.14) has two solutions, $\{w_1, \ldots, w_k, \bar{w}_1, \ldots, \bar{w}_l\} \neq \{v_1, \ldots, v_k, \bar{v}_1, \ldots, \bar{v}_l\}$. Since the set $\{x_{n_1}, \ldots, x_{n_k}\}$ is linearly independent by construction, there exist X_1, X_2, where $X_1 = X^\star + w_1 X_{n_1} + \cdots + w_k X_{n_k} \neq X_2 = X^\star + v_1 X_{n_1} + \cdots + v_k X_{n_k}$. This implies that $I|_{\theta_{\mathcal{R}}^*}$ is consistent with two non-equal realizations, $(A_1, B_1, C_1, D_1) \neq (A_2, B_2, C_2, D_2)$, where $A_1 = X_1^{-1} \hat{A} X_1$ and $A_2 = X_2^{-1} \hat{A} X_2$ and $G(s) = C(sI - A))^{-1} B + D = C_1 (sI - A_1)^{-1} B_1 + D_1 = C_2 (sI - A_2)^{-1} B_2 + D_2$. Therefore, by Lemma 1, $I|_{\theta_{\mathcal{R}}^*}$ is not globally identifiable from $G(s)$.

($\Leftarrow$) Assume (5.14) has a unique solution $\{w_1, \ldots, w_k, \bar{w}_1, \ldots, \bar{w}_l\}$. This implies there is a unique matrix $X = X^\star + w_1 X_{n_1} + \cdots + w_k X_{n_k}$, with X^{-1} well defined, such that X

satisfies Equation (5.1) and X^{-1} satisfies Equation (5.6). The fact that X is invertible implies that X is a valid state transformation from $\hat{S}$ to another system $\tilde{S}$, that is

$$\tilde{A} = X^{-1}\hat{A}X, \ \ \tilde{B} = X^{-1}\hat{B}, \tilde{C} = \hat{C}T, \ \text{and} \ \tilde{D} = \hat{D}.$$

Moreover, since $vec(X)$ satisfies Equation (5.1) and $vec(X^{-1})$ satisfies Equation (5.6), $\tilde{S} \in R(I|_{\theta_{\mathcal{R}}^*})$, that is to say, the entries of $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ that correspond to the fixed elements of $\theta_{\mathcal{R}}^*$ equal these values in $\theta_{\mathcal{R}}^*$ (and thus also equal the values of the corresponding elements in (A, B, C, D)).

Now we need to show that no other solution X to Equation (5.1) (or, likewise, no other solution X^{-1} to Equation (5.6)) yields a system in the range of $I|_{\theta_{\mathcal{R}}^*}$.

1. First, consider solutions to Equation (5.1) (or Equation (5.6)) that do not correspond to invertible matrices; these solutions do not correspond to valid state transformations and thus does not generate a system in the range of $I|_{\theta_{\mathcal{R}}^*}$.

2. Now, consider solutions to Equation (5.1) (or Equation (5.6)) that *are* invertible and thus correspond to valid state transformations. Although these solutions satisfy Equation (5.1) (or Equation (5.6)), their inverses *do not* satisfy Equation (5.6) (or Equation (5.1)) since the solution to Equation (5.14) is unique. Therefore, the realizations that results from these state transformation are not in the range of $I|_{\theta_{\mathcal{R}}^*}$, similar to what Example 9 illustrated.

So there is only one realization $\tilde{S} = S \in R(I|_{\theta_{\mathcal{R}}^*})$. Therefore $I|_{\theta_{\mathcal{R}}^*}$ is globally identifiable from $G(s)$.

$\square$

5.2   Consequences of the Main Result

The next natural question is when does Equation (5.14) have a unique solution. The following are corollaries of Theorem 2 that answer this question.

Corollary 1. *If $k = 0$ or $l = 0$ then Equation (5.14) has a unique solution.*

Proof. If $k = 0$ or $l = 0$ there is no nontrivial nullspace of $\mathbb{A}$ or $\bar{\mathbb{A}}$ so the psuedoinverse gives a unique result. This uniquely solves Equation (5.14) because one side is constant and the other side is made of a linear combination of linearly independent elements. $\square$

This effectively reduces Equation (5.14) back to Equation (5.1) or (5.6), respectively. We now examine cases when this occurs.

Corollary 2. *If C is fixed and invertible then Equation (5.14) has a unique solution.*

Proof. If C is fixed and invertible then $l = 0$. So by Corollary 1 Equation (5.14) has a unique solution. $\square$

Note that this corollary is similar to Theorem 1 because knowing $C = I$ means C is fixed and invertible.

Similarly, we have another corollary.

Corollary 3. *If B is fixed and invertible then Equation (5.14) has a unique solution.*

Proof. If B is fixed and invertible then $k = 0$. So by Corollary 1 Equation (5.14) has a unique solution. $\square$

5.3   Main Result Applied to Examples

Now we revisit each example, that is applicable, and apply the main result to it.

5.3.1 Main Result Applied to Example 2

This was a simple system with two water tanks in series, depicted in Figure 1.3. Plugging in numbers for the flows into Equation 1.8 gives the following system,

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} -1 & 0 \\ 1 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \\
y &= \begin{bmatrix} 0 & 3 \end{bmatrix} x.
\end{aligned}
\tag{5.15}
$$

Assume an arbitrary realization was obtained from input-output data and System Identification and Network Realization or Subspace Identification,

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} -1.5 & 1.5 \\ .5 & -2.5 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \\
y &= \begin{bmatrix} 1.5 & -1.5 \end{bmatrix} x,
\end{aligned}
\tag{5.16}
$$

which is actually the realization if the states were $x_1 + x_2$ and $x_1 - x_2$. As discussed after Example 2, the structure shown in Figure 1.3 is lost in this realization.

If we assume we know B and C, that is how the input flow of water affects the system and how the output is read then we can apply the main result. This defines the restriction of the identity $I|_{B=[1\ 0],C=[0\ 3]}$. Using this parameterization, Equations (5.1) and (5.6) can be built, giving $k = l = 1$ and Equation (5.14),

$$
\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} + w \begin{bmatrix} 0 & .7071 \\ 0 & .7071 \end{bmatrix} = \left(\begin{bmatrix} .5 & .5 \\ .5 & -.5 \end{bmatrix} + \bar{w} \begin{bmatrix} -.7071 & .7071 \\ 0 & 0 \end{bmatrix} \right)^{-1}
\tag{5.17}
$$

which does not have a unique solution. Consider $(w, \bar{w}) = (0, 0)$ and $(w, \bar{w}) = (1.4142, .7071)$. Both are solutions so $I|_{B=[1\ 0],C=[0\ 3]}$ is not globally identifiable from $G(s)$.

5.3.2 Main Result Applied to Example 3

This example was a compartment model based on theory presented in [4] shown in Figure 1.4. Plugging numbers in for the constants, the system becomes

$$
\begin{aligned}
\frac{dx}{dt} &= \begin{bmatrix} .1382 & -.2027 \\ .2005 & -.2005 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x,
\end{aligned}
\tag{5.18}
$$

Assume an arbitrary realization was obtained from input-output data and System Identification and Network Realization or Subspace Identification,

$$
\begin{aligned}
\frac{dx}{dt} &= \begin{bmatrix} .3387 & -.7419 \\ .2005 & -.4010 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & -1 \end{bmatrix} x,
\end{aligned}
\tag{5.19}
$$

First Parametrization

First, assume $C = [1\ 0]$ is known, which is a fair assumption because drug concentration in the blood stream is being measured directly and experimenters should known this. This defines the restriction of the identity $I|_{C=[1\ 0]}$. Using this parameterization, Equations (5.1) and (5.6) can be built, giving $k = l = 2$ and Equation (5.14),

$$
\begin{bmatrix} .5 & 0 \\ -.5 & 0 \end{bmatrix} + w_1 \begin{bmatrix} -.5 & .5 \\ -.5 & .5 \end{bmatrix} + w_2 \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} = \left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + \bar{w}_1 \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} + \bar{w}_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1}
\tag{5.20}
$$

which does not have a unique solution. Consider $(w_1, w_2, \bar{w}_1, \bar{w}_2) = (1, 1, -.5, .5)$ and $(w_1, w_2, \bar{w}_1, \bar{w}_2) = (.5, 1.5, 0, 1)$. Both are solutions to Equation (5.20) and therefore consistent with $I|_{C=[1\ 0]}$. So $I|_{C=[1\ 0]}$ is not globally identifiable from $G(s)$.

Second Parametrization

Now, assume $B = C = [1\ 0]$ is known, which is still a fairly reasonable assumption because the drug affects the bloodstream directly and experimenters could assume this. This defines the restriction of the identity $I|_{B=C=[1\ 0]}$. Using this parameterization, Equations (5.1) and (5.6) can be built, giving $k = l = 1$ and Equation (5.14),

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + w_1 \begin{bmatrix} -.5 & .5 \\ -.5 & .5 \end{bmatrix} = \left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + \bar{w}_1 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \tag{5.21}$$

which does not have a unique solution either. Consider $(w_1, \bar{w}_1) = (1.4142, 1)$ and $(w_1, \bar{w}_1) = (1, 1.4142)$. Both are solutions to Equation (5.21) and consistent with $I|_{B=C=[1\ 0]}$, therefore $I|_{B=C=[1\ 0]}$ is not globally identifiable from $G(s)$.

Third Parametrization

Now if we assume $C = [1\ 0]$ and the first column of A, $A(:, 1) = [.1382\ .2005]^T$, are known, that defines $\theta_{\mathcal{R}}^* = [.1382\ .2005\ 0\ 0\ 0\ 0\ 1\ 0]^T$ with $\Theta_{\mathcal{R}} = [1\ 1\ 0\ 0\ 0\ 0\ 1\ 1]^T$. This gives the parameterization $I|_{\theta_{\mathcal{R}}^*}$, which can be used to construct Equations (5.1) and (5.6), which show that $k = 0$ and $l = 2$. So $I|_{\theta_{\mathcal{R}}^*}$ is globally identifiable from $G(s)$ by Corollary 1, and the unique state transformation is

$$X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \tag{5.22}$$

5.3.3   Main Result Applied to Example 6

Recall that in Example 6 the given transfer function was

$$G(s) = \frac{1}{s - 3}. \tag{5.23}$$

So let our arbitrary realization be $\hat{S} = (3, 1, 1)$. Note that for this example, since $n = 1$ the state transformation X is a scalar. We now examine the three parameterizations.

First Parametrization

The first parameterization was $I|_{[3\,0\,0]^T}(\theta_{\mathcal{F}})$, with indicator function, $\Theta_{\mathcal{R}} = [1\ 0\ 0]^T$, which generated the class of systems parameterized by $\theta = [3\ b\ c]^T$ (or, equivalently, $\theta_{\mathcal{F}} = [b\ c]^T$). Since $A = \hat{A}$ and no information about B and C is available, Equation (5.1) is only

$$[3 - 3]x = 0x = 0. \tag{5.24}$$

Equation (5.6) is the exact same. So $x^\star = \bar{x}^\star = 0$, $k = l = 1$, $x_1 = \bar{x}_1 = 1$. Therefore Equation (5.14) is

$$w_1 x_1 = w_1 = (\bar{w}_1)^{-1} = (\bar{w}_1 \bar{x}_1)^{-1}, \tag{5.25}$$

which clearly has infinite solutions $(w_1, \bar{w}_1)$. So $I|_{[3\,0\,0]^T}(\theta_{\mathcal{F}})$ is not globally identifiable from $G(s)$, which is illustrated in Figure 3.1 a).

Second Parametrization

The second parametrization assumed $b = 3$ was known, yielding $I|_{[0\,3\,0]^T}(\theta_{\mathcal{F}})$, with $\Theta_{\mathcal{R}} = [0\ 1\ 0]^T$, which generated the class of systems parameterized by $\theta = [a\ 3\ c]^T$. For this parametrization Equations (5.1) and (5.6) are

$$3x = 1 \quad \text{and} \quad 1\bar{x} = 3. \tag{5.26}$$

So $x^\star = 1/3$, $\bar{x}^\star = 3$, and $k = l = 0$. Therefore by Corollary 1, Equation (5.14) has only one solution. So $I|_{[0\,3\,0]^T}$ is globally identifiable from the transfer function.

Third Parametrization

The third parametrization assumed $c = 4$ was known, yielding $I|_{[0\ 0\ 4]^T}(\theta_{\mathcal{F}})$, with $\Theta_{\mathcal{R}} = [0\ 0\ 1]^T$, which generated the class of systems parameterized by $\theta = [a\ b\ 4]^T$. Similarly, Equations (5.1) and (5.6) are

$$1x = 4 \quad \text{and} \quad 4\bar{x} = 1. \tag{5.27}$$

So $x^\star = 4$, $\bar{x}^\star = 1/4$, and $k = l = 0$. Therefore by Corollary 1, Equation (5.14) has only one solution. So $I|_{[0\ 0\ 4]^T}$ is globally identifiable from the transfer function.

5.3.4   Main Result Applied to Example 7

For the hub-spoke example let $n = 3$. The network structure is

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{5.28}$$

Let the arbitrary realization be

$$\hat{A} = \begin{bmatrix} -2 & -2 & 1 \\ -2 & -6 & 2 \\ -7 & -16 & 6 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \hat{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{5.29}$$

Now we define the restriction of the identity $I|_{C=[I_2\ 0]}$. Building Equations (5.1) and (5.6) gives $k = l = 3$ and Equation (5.14),

$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + w_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + w_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} + w_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} =
$$
$$
\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \bar{w}_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + \bar{w}_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} + \bar{w}_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^{-1} ,
$$
(5.30)

which does not have a unique solution. Consider $(w_1, w_2, w_3, \bar{w}_1, \bar{w}_2, \bar{w}_3) = (0, 0, 1, 0, 0, 1)$ and $(w_1, w_2, w_3, \bar{w}_1, \bar{w}_2, \bar{w}_3) = (-1, -2, 1, 1, 2, 1)$. Both are solutions to Equation (5.30), therefore $I|_{C=[I_2\ 0]}$ is not globally identifiable from $G(s)$.

5.3.5 Main Result Applied to Examples 8 and 9

It is shown in Example 8 that $k = 0$ therefore by Corollary 1 the parameterization is globally identifiable from the transfer function. Example 9 shows that there is a solution to Equation (5.13), which is Equation (5.14) for this example. Since $\bar{w}$ is a scalar, Equation (5.13) can be manipulated to be

$$
\bar{w} \begin{bmatrix} 0.4894 & -0.0426 \\ 0.7447 & -0.2766 \end{bmatrix} + \bar{w}w \begin{bmatrix} 0.2063 & 0.8251 \\ 0.1031 & 0.5157 \end{bmatrix} = \left(\begin{bmatrix} 0.3780 & -0.7559 \\ -0.3780 & 0.3780 \end{bmatrix} \right)^{-1} .
$$
(5.31)

By evaluating the inverse and then vectorizing we see,

$$
\begin{bmatrix} 0.4894 & 0.2063 \\ 0.7447 & 0.1031 \\ -0.0426 & 0.8251 \\ -0.2766 & 0.5157 \end{bmatrix} \begin{bmatrix} \bar{w} \\ \bar{w}w \end{bmatrix} = \begin{bmatrix} -2.6458 \\ -2.6458 \\ -5.2915 \\ -2.6458 \end{bmatrix} .
$$
(5.32)

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The matrix on the left is full column rank so there is only one solution, which means there is only one solution to Equation (5.13). Therefore the parameterization is globally identifiable from $G(s)$, which we saw in Example 9 when we correctly recovered the network structure.

Chapter 6

Conclusion

The contribution of this work is the formulation and solution of a new problem in systems and control theory, called the Network Realization Problem. The problem is, given a transfer function produced by a set of state-space matrices (A, B, C, D), how many and which elements of the state-space matrices must be known a priori in order to recover all of (A, B, C, D). This work was motivated by the desire to completely quantify the conditions for transitioning between the different mathematical representations of linear time-invariant systems. The solution to this problem, presented in Theorem 2, is useful for theorists because it lays a foundation for quantifying the information cost of identifying a system's complete network structure from the transfer function. Furthermore it also explains that the value of a priori information is not necessarily quantified by the amount of information. For example, knowing the n^2 elements of the A matrix is not sufficient but knowing the n^2 elements of the C matrix, if it is invertible, is sufficient (Corollary 2). This is true because a full rank C matrix completely specifies the states while the A matrix does not.

This problem was also motivated by different prevalent applications that have network structure as shown in Examples 2-4 and in [6, 28–32]. Practitioners in a variety of applications will benefit from this work because it will help them understand the kinds of additional information, beyond input-output data, they must obtain to reconstruct the network structure of their systems, satisfying a unique solution to Equation (5.14). If this cost is too high or unattainable, it will motivate them to seek other mathematical models, possibly such as the DSF [33].

6.1 Place in Systems Theory

This work is shown to be related to the key topics in Systems Theory, namely System Identification, State Realization, Subspace Identification, and Structural Identifiability. The Network Realization Problem was shown to be a sub-problem of Structural Identifiability because it deals with identifiability of a specific kind of parameterization and Structural Identifiability deals with general parameterizations.

The Network Realization Problem was posed as a question of global identifiability from the transfer function because global identifiability from the transfer function was presented in [2] and gave a natural home for this problem of trying to recover the network structure. So it is connected to System Identification because identification produces a transfer function from input-output data. However nothing in the result required the transfer function, only an arbitrary realization $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$, which we assumed came through State Realization of the transfer function. This arbitrary realization could come from System Identification followed by State Realization but since Subspace Identification also produces an arbitrary realization, it could have come from Subspace Identification. Despite where $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ came from, the network structure can be recovered if and only if Equation (5.14) has a unique solution.

6.2 Future Work

The necessary and sufficient conditions are provided therefore the problem is solved. However, as Section 5.2 alludes to, it would be interesting to known exactly when Equation (5.14) has a unique solution and in so doing quantify the information cost of recovering the network structure of a system. Another natural question is what are the necessary and sufficient conditions for obtaining the network structure from other representations of structure such as dynamical structure functions ([33]) or subsystem structure ([35]). These representations have more information built into them, such as for dynamical structural functions how the measured states affect each other, therefore it may be easier to recover the network structure.

Chapter 7

References

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